PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (⋆) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}$*"×11" or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 November 1999**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2388 [1998, 503] Correction. *Proposed by Daniel Kupper, Büllingen, Belgium.*

Suppose that $n \geq 1 \in \mathbb{N}$ is given and that, for each integer $k \in \{0, 1, \ldots, n-1\}$, the numbers a_k, b_k, $z_k \in \mathbb{C}$ are given, with the z_k^2 distinct. Suppose that the polynomials

$$A_n(z) \;=\; z^n + \sum_{k=0}^{n-1} a_k\, z^k \quad \text{and} \quad B_n(z) \;=\; z^n + \sum_{k=0}^{n-1} b_k\, z^k$$

satisfy $A_n(z_j) = B_n(z_j^2) = 0$ for all $j \in \{0, 1, \ldots, n-1\}$.

Find an expression for b_0, b_1, …, b_{n-1} in terms of a_0, a_1, …, a_{n-1}.

2389 [1998, 503] Correction. *Proposed by Nikolaos Dergiades, Thessaloniki, Greece.*

Suppose that f is continuous on $\mathbb{R}^n$ and satisfies the condition that when any two of its variables are replaced by their arithmetic mean, the value of the function increases; for example:

$$f(a_1, a_2, a_3, \ldots, a_n) \;\leq\; f\left(\frac{a_1+a_3}{2}, a_2, \frac{a_1+a_3}{2}, a_4, \ldots, a_n\right).$$

Let $m = \dfrac{a_1 + a_2 + \ldots + a_n}{n}$. Prove that

$$f(a_1, a_2, a_3, \ldots, a_n) \;\leq\; f(m, m, m, \ldots, m).$$

2426. *Proposed by Mohammed Aassila, Strasbourg, France.*

(a) Show that there are two polynomials, $p(x)$ and $q(x)$, both having three integer roots and such that $p(x) - q(x)$ is a non-zero constant.

(b)* Do there exist two polynomials, $p(x)$ and $q(x)$, both having $n > 3$ integer roots and such that $p(x) - q(x)$ is a non-zero constant?

2427. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Suppose that G is the centroid of triangle ABC, and that

$$\angle GAB + \angle GBC + \angle GCA = 90^\circ .$$

Characterize triangle ABC.

2428. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Given triangle ABC with $\angle BAC = 90^\circ$. The incircle of triangle ABC touches AB and AC at D and E respectively. Let M be the mid-point of BC, and let P and Q be the incentres of triangles ABM and ACM respectively. Prove that

1. $PD \parallel QE$;
2. $PD^2 + QE^2 = PQ^2$.

2429. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*
Suppose that D, E and F are points on the side AB (or its production) of triangle ABC. Suppose further that CD is a median, that CE is the bisector of $\angle ACB$, and that CF is its external bisector.

The circumcircle, Γ, of triangle EFC intersects CD again at P. Suppose that Γ_A and Γ_B are the circumcircles of triangles CPA and CPB respectively.

Show that Γ_A and Γ_B are tangent to AB at A and B respectively.

2430. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*
Points A and B lie outside circle Γ. Find a point C on Γ with the following property:

AC and BC intersect Γ again at D and E respectively, with $DE \parallel AB$.

2431. *Proposed by Jill Taylor, student, Mount Allison University, Sackville, New Brunswick.*
Let $n \in \mathbb{N}$. Prove that there exist triangles with integer area, integer side lengths, one side n and perimeter $4n$, where n is not necessarily prime.

For a given n, are such triangles uniquely determined?

[Compare problem **2331**.]

2432. *Proposed by K.R.S. Sastry, Bangalore, India.*

In $\triangle ABC$, we use the standard notation: O is the circumcentre, H is the orthocentre. Let M be the mid-point of BC, $OH = m$, $OM = n$ ($m, n \in \mathbb{N}$), and suppose that $OH \parallel BC$.

How many sides of $\triangle ABC$ can have integer lengths?

2433. *Proposed by K.R.S. Sastry, Bangalore, India.*

In $\triangle ABC$, let e denote the length of the segment of the Euler line between the orthocentre and the circumcentre.

Prove or disprove that $\triangle ABC$ is right angled if and only if e equals one half of the length of one of the sides of $\triangle ABC$.

2434. *Proposed by K.R.S. Sastry, Bangalore, India.*

In $\triangle ABC$, let $\angle ABC = 60^\circ$. Point P is on the line segment AC such that $\angle CBP = \angle BAC$. Point Q is on the line segment BP such that $BQ = BC$.

Prove that Q lies on the altitude through A of $\triangle ABC$ if and only if $\angle BAC = 40^\circ$.

2435. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Show that, for $x > 0$, the following functions are increasing:

$$f(x) = \frac{\left(1+\frac{1}{x}\right)^x}{(1+x)^{\frac{1}{x}}} \quad \text{and} \quad g(x) = \left(1+\frac{1}{x}\right)^x - (1+x)^{\frac{1}{x}}.$$

2436. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Find all real solutions of

$$2\cosh(xy) + 2^y - \left[(2\cosh(x))^y + 2\right] = 0\,.$$

2437. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

Let P be a point in the plane of triangle ABC. If the mid-points of the line segments AP, BP, CP all lie on the nine-point circle of triangle ABC, prove that P must be the orthocentre of triangle ABC.

2438. *Proposed by Peter Hurthig, Columbia College, Burnaby, BC.*

Show how to tile an equilateral triangle with congruent pentagons. Reflections are allowed. (Compare problem **1988**.)