MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section (except for the problems section, which have their own editors) should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Haven, CT 06520–8283 USA. The electronic address is still

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Toronto). The rest of the staff consists of Adrian Chan (Upper Canada College), Jimmy Chui (Earl Haig Secondary School), David Savitt (Harvard University) and Wai Ling Yee (University of Waterloo).

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the new schedule of eight issues per year, we request that solutions from this issue be submitted in time for issue 4 of 2000.

High School Problems

Editor: Adrian Chan, 229 Old Yonge Street, Toronto, Ontario, Canada. M2P 1R5 <all238@ipoline.com>

H253. Find all real solutions to the equation

$$\sqrt{3x^2 - 18x + 52} + \sqrt{2x^2 - 12x + 162} \;=\; \sqrt{-x^2 + 6x + 160}\,.$$

H254. *Proposed by Alexandre Trichtchenko, 1st year, Carleton University.*

Let p and q be relatively prime positive integers, and n a multiple of pq. Find all ordered pairs (a, b) of non-negative integers that satisfy the diophantine equation $n = ap + bq$.

H255. We have a set of tiles which contains an infinite number of regular n-gons, for each $n = 3, 4, \ldots$. Which subsets of tiles can be chosen, so that they fit around a common vertex? For example, we can choose four squares, or four triangles and a hexagon.

H256. Let $A = 2^a p_1^b p_2^c$, where p_1 and p_2 are primes, possibly equal to each other and to 2, and a, b, and c are positive integers. It is known that $p_1 \equiv p_2 \pmod 4$, $b \equiv c \pmod 2$, and that 2^a, p_1^b, p_2^c are three consecutive terms of an arithmetic sequence, not necessarily in that order. Find all possible values for A.

Advanced Problems

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A229. In tetrahedron $SABC$, the medians of the faces SAB, SBC, and SCA, taken from the vertex S, make equal angles with the edges that they lead to. Prove that $|SA| = |SB| = |SC|$.

(Polish Olympiad)

A230. *Proposed by Naoki Sato.*

For non-negative integers n and k, let $P_{n,k}(x)$ denote the rational function

$$\frac{(x^n - 1)(x^n - x)\cdots(x^n - x^{k-1})}{(x^k - 1)(x^k - x)\cdots(x^k - x^{k-1})}.$$

Show that $P_{n,k}(x)$ is actually a polynomial for all n, k.

A231. *Proposed by Mohammed Aassila, Centre de Recherches Mathématiques, Montréal, Québec.*

For the sides of a triangle a, b, and c, prove that

$$\frac{13}{27} \le \frac{(a+b+c)(a^2+b^2+c^2)+4abc}{(a+b+c)^3} \le \frac{1}{2}.$$

A232. Five distinct points A, B, C, D, and E lie on a line (in this order) and $|AB| = |BC| = |CD| = |DE|$. The point F lies outside the line. Let G be the circumcentre of triangle ADF and H the circumcentre of triangle BEF. Show that the lines GH and FC are perpendicular.

(1997 Baltic Way)

Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C85. *Proposed by Christopher Long, graduate student, Rutgers University.* (*From a set of course notes on analytic number theory.*)

Let $C = (c_{i,j})$ be an $m \times n$ matrix with complex entries, and let D be a real number. Show that the following statements are equivalent:

(i) For any complex numbers a_j, $j = 1, \ldots, n$,

$$\sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_j c_{i,j} \right|^2 \leq D \sum_{j=1}^{n} |a_j|^2 .$$

(ii) For any complex numbers b_i, $i = 1, \ldots, m$,

$$\sum_{j=1}^{n} \left| \sum_{i=1}^{m} b_i c_{i,j} \right|^2 \leq D \sum_{i=1}^{m} |b_i|^2 .$$

C86. Let K_n denote the complete graph on n vertices; that is, the graph on n vertices with all possible edges present. Show that K_n can be decomposed into $n-1$ disjoint paths of length $1, 2, \ldots, n-1$. For example, for $n = 4$, the graph K_4, with vertices A, B, C, and D, decomposes into the paths $\{AC\}$, $\{BD, DA\}$, and $\{AB, BC, CD\}$.

Can we require that the paths in the decomposition be simple? (We say that a path is simple if it passes through each vertex at most once; that is, if no vertex is the end-point of more than two edges along the path.)

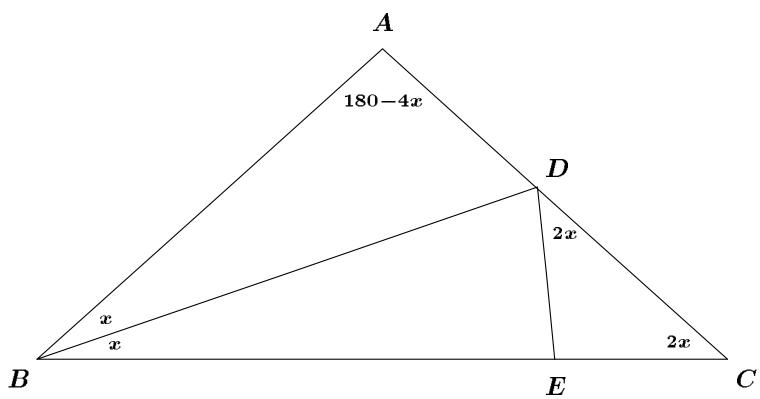
Problem of the Month

Jimmy Chui, student, Earl Haig S.S.

Problem. Let ABC be an isosceles triangle with $AB = AC$. Suppose that the angle bisector of $\angle B$ meets AC at D and that $BC = BD + AD$. Determine $\angle A$.

(1996 CMO, Problem 4)

Solution.



Construct E on BC such that $DE = EC$. Let $x = \angle ABD = \angle DBC$. Then, $\angle DCE = \angle ACB = 2x$. Hence, $\angle CDE = 2x$, and further $\angle ADE = 180° - 2x$. Thus, quadrilateral $ABED$ is cyclic.

Drawing AE, we see that $\angle DAE = \angle DBE = x$, and $\angle DEA = \angle DBA = x$. Therefore, triangle ADE is isosceles, and so $AD = DE$.

Now, $BC = BD + AD = BD + DE = BD + EC$. Also, $BC = BE + EC$. So, $BD + EC = BE + EC$. Thus, $BD = BE$.

Now, $\angle BED = 180° - \angle DEC = 4x$, and $\angle BDE = 180° - 5x$, so $4x = 180° - 5x$.

Solving for x, we obtain $x = 20°$.

Thus, $\angle A = 180° - 4x = 100°$.

Self-Centred Triangles

Cyrus Hsia

student, University of Toronto

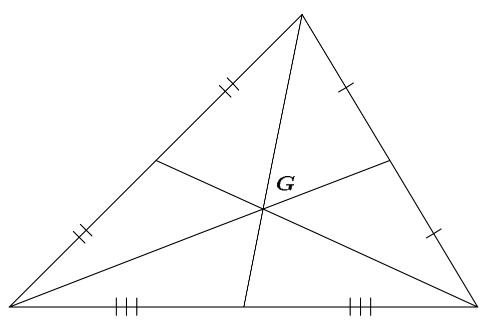
There are four famous triangular centres that all students of classical geometry should be familiar with. Readers already familiar with these should go on to the next section to see how the information on one will be useful to another. Otherwise, the centres are as follows:

Central Definitions

The following definitions all pertain to an arbitrary triangle in Euclidean space.

Centroid

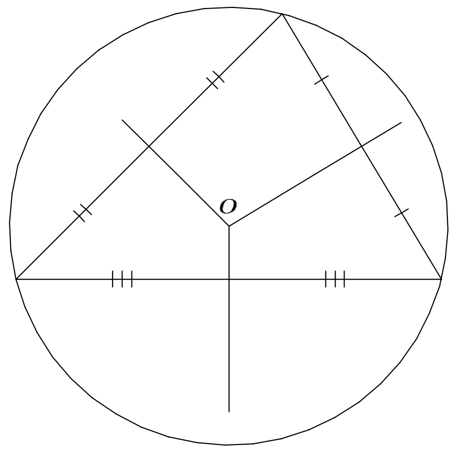
The centroid, denoted by G, is the point of intersection of the medians of a triangle. A median is a line from a vertex of a triangle to the mid-point of the opposite side.



Circumcentre

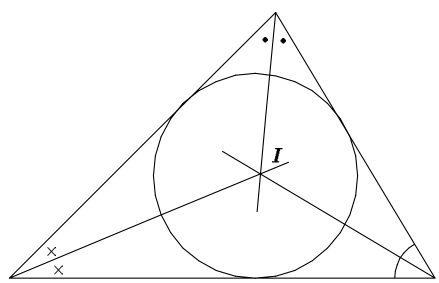
The circumcentre, denoted by O, is the point of intersection of the right bisectors of the three sides. A right bisector of a line segment is a line perpendicular to it and divides it into two equal segments. It should be noted that O is also the center of the circle that passes through the vertices of the triangle. This circle is called the circumcircle and hence O is referred to as the circumcentre. The confirmation that these two definitions are equivalent is left to the reader.

Copyright © 1999 Canadian Mathematical Society



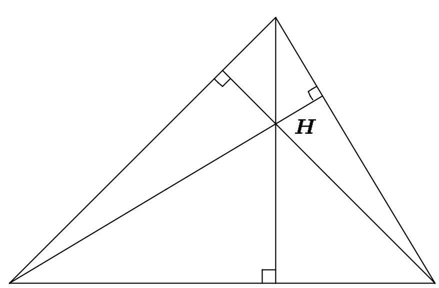
Incentre

The incentre, denoted by I, is the point of intersection of the three angle bisectors of the triangle. An angle bisector is a line which divides a given angle into two equal angles. It should be noted that in this case, I is the centre of the circle inside the triangle tangent to all three sides. These definitions are equivalent and we leave the work to the reader.



Orthocentre

The orthocentre, denoted by H, is the point of intersection of the three altitudes of the triangle. An altitude is a line through a vertex that is perpendicular to the side opposite this vertex.



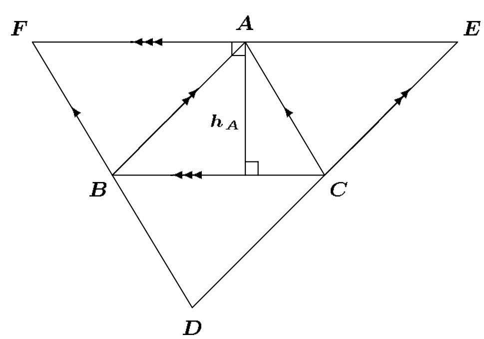
The Central Problem

Although it was taken for granted in the definitions, it is not trivial to show that the points above exist. After all, why do three line segments have to intersect at a common point? We say the lines concur if they do. In fact, draw three lines from the vertices of a triangle to their opposite sides. These lines are called *cevians*. It is easy to see that these cevians need not concur.

There are many ways to show that the points described above exist and to show that certain cevians always concur for any triangle (See [1] and [2]). Here we will consider something of purely mathematical interest. There is a certain similarity between the definitions of the four centres and so it may be possible that the existence of one centre is enough to prove the existence of another. For example, if it has been shown that the centroid exists, then is it possible to show that the incentre exists from this fact? We present two such scenarios here.

Circumcentre implies Orthocentre

Given triangle ABC, let h_A, h_B, h_C be the altitudes from vertices A, B, C respectively. We wish to show that these three altitudes (cevians) concur. At vertex A, draw a line perpendicular to h_A and similarly draw lines for the other two sides as shown.



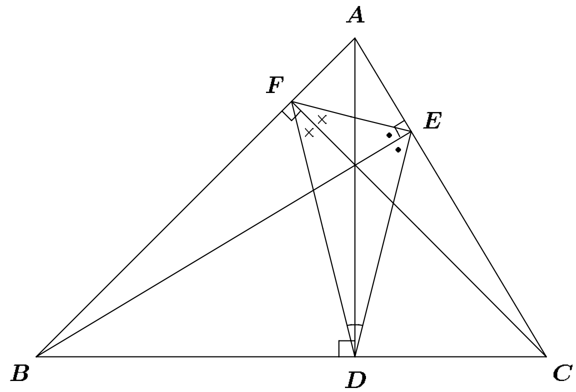
Let the lines intersect at points D, E, F opposite A, B, C respectively as shown. Now FE is perpendicular to h_A and BC is also perpendicular to h_A so FE and BC are parallel. Likewise, FD and AC are parallel and ED and AB are parallel.

Thus $FA = BC$, since $FACB$ is a parallelogram, and $AE = BC$, since $AECB$ is also a parallelogram. Thus $FA = AE$, so A is the mid-point of FE and h_A is perpendicular to FE. In other words, h_A is the perpendicular bisector of FE. Likewise, h_B is the perpendicular bisector of FD and h_C is that of DE.

Now we see that if the circumcentre of a triangle exists, in this case h_A, h_B, and h_C of triangle DEF concur, means that the altitudes of triangle ABC concur. Thus the orthocentre exists by definition.

Incentre implies Orthocentre

Given a triangle ABC, let the altitudes from vertices A, B, C, be h_A, h_B, h_C respectively. Let the feet of the altitudes be D, E, F as shown.



Let α, β, γ be the angles of triangle ABC at vertices A, B, C respectively. Now $\angle FEB = \angle FCB$ since $FECB$ is a cyclic quadrilateral. $\angle FEB = 90° - \beta$. Also, $\angle BED = \angle BAD = 90° - \beta$. In other words, $h_B = BE$ is the angle bisector of $\angle FED$. Likewise, h_A and h_C are angle bisectors of vertices D and F respectively in triangle FED.

As the reader has suspected, if the incentre exists, the three angle bisectors h_A, h_B, h_C concur in triangle DEF, and since these are the altitudes of triangle ABC, then the orthocentre exists.

Epilogue

Here, we have shown that the existence of the incentre or the circumcentre implies the existence of the orthocentre. These and the other implications are listed in the table below. We welcome the reader to explore the other possibilities and to send us feedback on any results they find. A further exploration, for the brave at heart, would be to investigate other famous centres of triangles as well.

implies	centroid	circumcentre	incentre	orthocentre
centroid	clear			
circumcentre		clear		proved
incentre			clear	proved
orthocentre				clear

Exercises

1. Provide a proof that the existence of the orthocentre implies the existence of the incentre and the circumcentre by a similar method to the ones given.

2. Are there other proofs to the implications above?

3. For exploration, are other well-known centres of triangles related as nicely to the orthocentre, circumcentre and incentre? Try the following:

4. (a) The nine-point circle of a triangle is defined as the circle through the three mid-points of the three sides, the three feet of the altitudes, and the three mid-points of the line segments joining the vertices to the orthocentre. The centre of the nine-point circle is denoted by the letter N. By definition, the existence of N depends on the existence of the orthocentre. How about the other centres?

   (b) The Fermat point, F, of a triangle is the point in a triangle that minimizes the sum of the lengths from this point to each of the three vertices. Is it possible to determine the existence of this point by the existence of the other centres?

Acknowledgements

The proof of the first result was modified from the lecture notes in MAT325, by Professor J.W. Lorimer, University of Toronto, Toronto, Ontario, and was the inspiration for this article.

References

1. Grossman, J.P. *Ye Olde Geometry Shoppe - Part I*, **Mathematical Mayhem**, Volume 6, Issue 2. November/December 1993. pp. 13-17.

2. Grossman, J.P. *Ye Olde Geometry Shoppe - Part II*, **Mathematical Mayhem**, Volume 6, Issue 3. January/February 1994. pp. 7-12.

J.I.R. McKnight Problems Contest 1988

TIME: $2\frac{1}{2}$ HOURS

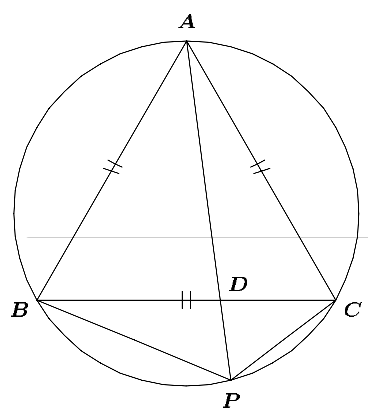
1. Calculators are permitted.
2. Complete solutions are necessary to all problems for full marks.
3. Do all questions in Part A and Part B.

PART A
(6 questions $\times$ 5 marks = 30 marks)

1. Find all ordered pairs of real numbers (x, y) such that $\log(\frac{x^2}{y^3}) = 1$ and $\log(x^2y^3) = 7$.
2. (a) Determine the difference between the number of zeroes at the end of each of the numbers $(10^3)!$ and $(10^3)^{100}$.
   (b) Determine n, if $n!$ ends in 17 zeroes.
3. A particle's position at time t seconds, from a point P, is given in metres (s is distance) by
$$s = t^3 - 3t^2 + 4\,, \qquad t \geq 0\,.$$
Find the time(s) when the particle is speeding up.
4. Determine all integer solutions to the following system of equations:
$$\begin{aligned} a+b+c &= 0\,, \\ ab+ac+bc &= -19\,, \\ abc &= -30\,. \end{aligned}$$
5. In triangle ABC, the angle at B is obtuse and $AB > BC$. An angle bisector of an exterior angle at A meets CB at D, and an angle bisector of an exterior angle at B meets AC at E. If $AD = AB = BE$, find $\angle BAC$.
6. Find the measure of the acute angle θ for which it is true that
$$\left(\frac{16}{81}\right)^{\sin^2\theta} + \left(\frac{16}{81}\right)^{\cos^2\theta} = \frac{26}{27}\,.$$

PART B
(5 questions $\times$ 10 marks = 50 marks)

1. Find the sum of the first 100 terms of the following double arithmetic series:
$$1 \times 4 + 5 \times 7 + 9 \times 10 + 13 \times 13 + 17 \times 16 + \cdots\,.$$

2. 

Points A, B, C, and P are located on a circle as shown. Prove that if triangle ABC is equilateral and AP intersects BC at D, then

$$\frac{1}{PD} = \frac{1}{PB} + \frac{1}{PC} .$$

3. Determine all triangles ABC which satisfy the condition

$$\tan(A - B) + \tan(B - C) + \tan(C - A) = 0 .$$

4. Find all integers a such that $(n - a)(n - 10) + 1$ can be written as a product $(n + b)(n + c)$, where b and c are integers.

5. A transformation is given by

$$P(x, y) \to P'(y + 246, -x - 430) .$$

Giving reasons, describe this transformation in simplest possible terms.

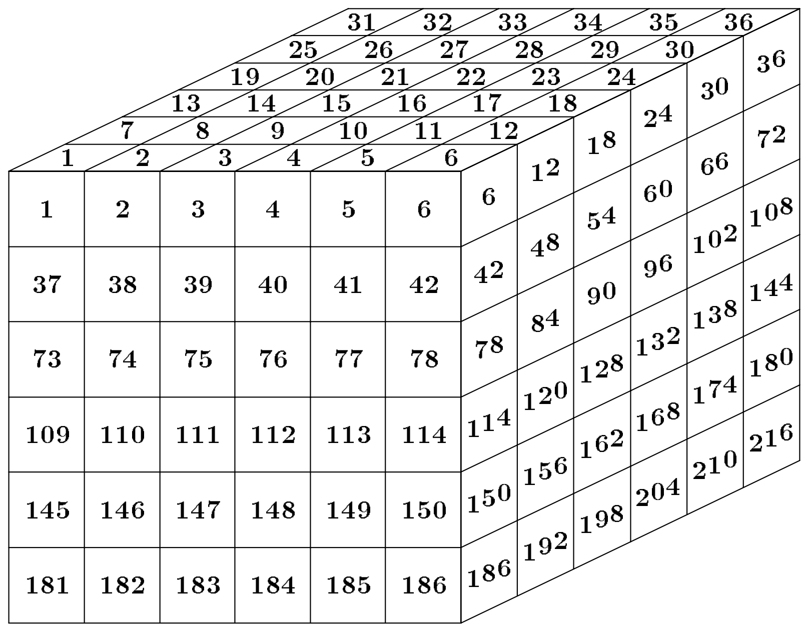
Swedish Mathematics Olympiad

1990 Qualifying Round

1. A boy has got two-thirds of the way over a railroad bridge, when he catches sight of a train coming towards him. He can just get off the bridge, and so escape the train, by running as fast as he can either towards the train or away from it. The train is approaching at a speed of 60 km/h. How fast can the boy run?

2. Which six-digit numbers of the form $abcabc$ are divisible by 33?

3. A large cube, $6 \times 6 \times 6$, is constructed of 216 unit cubes. These are numbered from 1 to 216 as shown in the figure on the next page. The first layer is numbered from 1 to 36, the first row in that layer from 1 to 6, the second from 7 to 12, and so on, always from left to right. The next layer is similarly numbered from 37 to 72, and so on. Choose 36 unit cubes so that no two of the chosen cubes come from the same row, column, or rank (parallel to one of the edges of the cube). Let S denote the sum of the numbers assigned to the 36 cubes. What are the possible values of S?



4. A is one of the angles in a given triangle. The corresponding side has length a, and the other two sides have lengths b and c. Show that

$$\sin A \leq \frac{a}{2\sqrt{bc}}.$$

5. Find all real solutions of the system of equations

$$\begin{aligned} x_1|x_1| &= x_2|x_2| + (x_1 - a)|x_1 - a| \\ x_2|x_2| &= x_3|x_3| + (x_2 - a)|x_2 - a| \\ &\cdots \\ x_n|x_n| &= x_1|x_1| + (x_n - a)|x_n - a| \end{aligned}$$

where a is a given positive integer.

6. Four houses are situated at the corners of a rectangle $ABCD$ with sides 3000 metres and 500 metres. The owners of the houses intend to sink a common well and run water mains from the well to each house. They have access to 1020 metres of pipe and a device that can join together two or more pieces of pipe. Where should the well be placed so that the length of pipe is sufficient?

1990 Final Round

1. The positive divisors of $n = 1900!$ are $d_1, d_2, \ldots, d_k$. Show that

$$\frac{d_1}{\sqrt{n}} + \frac{d_2}{\sqrt{n}} + \cdots + \frac{d_k}{\sqrt{n}} = \frac{\sqrt{n}}{d_1} + \frac{\sqrt{n}}{d_2} + \cdots + \frac{\sqrt{n}}{d_k}.$$

2. The points $A_1, A_2, \ldots, A_{2n}$ lie, in this order, on a straight line, and $|A_k A_{k+1}| = k$ for $k = 1, 2, \ldots, 2n-1$. The point P is situated on the line so that the sum $\sum_{k=1}^{2n} |PA_k|$ is as small as possible.

   Find this sum.

3. The numbers a and b are such that $\sin x + \sin a \geq b \cos x$ for all x. Find a and b.

4. A quadrilateral $ABCD$ is inscribed in a circle. The angle bisectors of A and B meet at a point E. Draw a line through E parallel to CD which meets AD in L and BC in M. Show that $|LA| + |MB| = |LM|$.

5. Find all (not necessarily strict) monotonic, positive functions f which are defined on the positive reals, and which satisfy

$$f(xy) \cdot f\left(\frac{f(y)}{x}\right) = 1$$

   for all $x, y > 0$.

6. Find all positive integers x and y such that $y \leq 500$ and

$$\frac{117}{158} > \frac{x}{y} > \frac{97}{131}.$$