Euler's Triangle Theorem

G.C. Shephard

1. In 1780, Leonhard Euler proved the following remarkable result [4, p. 96]:

Theorem: Let $[A, B, C]$ be an arbitrary triangle and O any point of the plane which does not lie on a side of the triangle. Let AO, BO, CO meet BC, CA and AB in the points D, E, F respectively. Then

$$\frac{||AO||}{||OD||} + \frac{||BO||}{||OE||} + \frac{||CO||}{||OF||} \;=\; \frac{||AO||}{||OD||} \cdot \frac{||BO||}{||OE||} \cdot \frac{||CO||}{||OF||} + 2\,. \qquad (1)$$

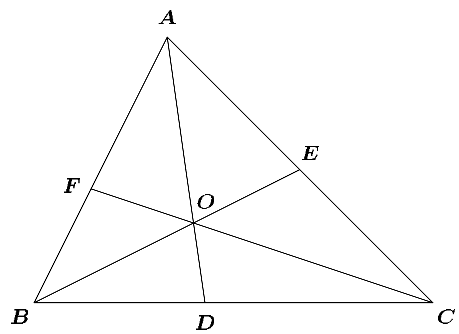

Figure 1

To begin with we shall assume, like Euler, that O lies in the interior of the triangle, see Figure 1. For the present, the notation $\dfrac{||XY||}{||YZ||}$ will simply indicate the ratio of the lengths of the indicated segments. We describe this theorem as remarkable not only because of its early date, but also because it connects the value of a cyclic sum (on the left side of (1)) with the value of a cyclic product (on the right side of (1)). There exist a great number of results concerning cyclic products for triangles and n-gons, the best known of which are the theorems of Ceva and Menelaus. Many more such theorems are given in [6] and [7]. On the other hand there are few papers on cyclic sums, such as [5] and [8]. Apart from [1], Euler's paper [4] is, so far as we are aware, the only one which related these two concepts.

Euler's proof of his theorem is by algebra and trigonometry; he calculates the ratios of the lengths of the line segments in (1) using trigonometrical formulae involving the sines of the angles between the lines at O. The proof takes two and a half pages, and whilst we do not wish to criticize the work of one of the most illustrious of mathematicians, it is worth considering the shortcomings of his proof. Apart from its length and complexity, the proof gives no insight as to *why* relation (1) is true, and therefore does nothing to suggest the existence of other relations of a similar nature. The following approach seems to overcome these criticisms; it depends on what has been called the "area principle" in [6] and the "area method" in [2].

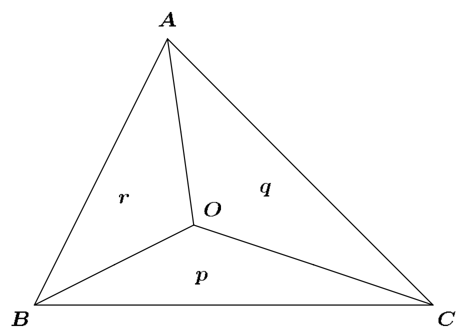

Figure 2

2. Let $p = ||COB||$, $q = ||AOC||$, $r = ||BOA||$ be the areas of the indicated triangles (see Figure 2). Then

$$\frac{||BD||}{||DC||} = \frac{||BOA||}{||AOC||} = \frac{r}{q}.$$

The first equality follows since the triangles $[B, O, A]$ and $[A, O, C]$ have the same base $[A, O]$ and their heights are proportional to $||BD|| : ||DC||$. By cyclic permutation of the letters,

$$\frac{||BD||}{||DC||} = \frac{r}{q}, \frac{||CE||}{||EA||} = \frac{p}{r}, \frac{||AF||}{||FB||} = \frac{q}{p}. \tag{2}$$

Also $\frac{||AD||}{||OD||} = \frac{||ABC||}{||OBC||} = \frac{p+q+r}{p}$. The first equality follows since the triangles $[A, B, C]$ and $[O, B, C]$ have the same base $[B, C]$ and their heights are proportional to $||AD|| : ||OD||$. Thus we obtain three more relations

$$\frac{||AD||}{||OD||} = \frac{p+q+r}{p}, \frac{||BE||}{||OE||} = \frac{p+q+r}{q}, \frac{||CF||}{||OF||} = \frac{p+q+r}{r}. \tag{3}$$

Finally,

$$\frac{||AO||}{||OD||} = \frac{||AD|| - ||OD||}{||OD||} = \frac{||AD||}{||OD||} - 1 = \frac{p+q+r}{p} - 1 = \frac{q+r}{p}$$

leading in a similar fashion to the three relations

$$\frac{||AO||}{||OD||} = \frac{q+r}{p}, \frac{||BO||}{||OE||} = \frac{r+p}{q}, \frac{||CO||}{||OF||} = \frac{p+q}{r}. \tag{4}$$

Relations (4) can also be proved directly (without using (3)) by the area principle. For our proof of (1) we only need relations (4):

$$\begin{aligned}\frac{||AO||}{||OD||}\cdot\frac{||BO||}{||OE||}\cdot\frac{||CO||}{||OF||} &= \frac{q+r}{p}\cdot\frac{r+p}{q}\cdot\frac{p+q}{r}\\ &= \frac{q^2r+r^2q+r^2p+p^2r+p^2q+q^2p+2pqr}{pqr}\\ &= \frac{q+r}{p}+\frac{r+p}{q}+\frac{p+q}{r}+2\\ &= \frac{||AO||}{||OD||}+\frac{||BO||}{||OE||}+\frac{||CO||}{||OF||}+2\,,\end{aligned}$$

as required.

3. We can exploit (2), (3) and (4) to yield many more relations between the ratios of lengths in the triangle. The first two of these are well known.

$$1 = \frac{r}{q}\cdot\frac{q}{p}\cdot\frac{p}{r} = \frac{||BD||}{||DC||}\cdot\frac{||AF||}{||FB||}\cdot\frac{||CE||}{||EA||}\,,$$

which is Ceva's Theorem for the triangle. Also, from (3),

$$\frac{||OD||}{||AD||} = \frac{p}{p+q+r}\,,\quad \frac{||OE||}{||BE||} = \frac{q}{p+q+r}\,,\quad \frac{||OF||}{||CF||} = \frac{r}{p+q+r}\,,$$

and hence

$$\frac{||OD||}{||AD||}+\frac{||OE||}{||BE||}+\frac{||OF||}{||CF||} = \frac{p+q+r}{p+q+r} = 1\,, \tag{5}$$

which is mentioned in [5] and [8]. Euler also gives a proof of this result [4, p. 102].

Almost as easy is

$$\frac{||AO||}{||OD||} = \frac{q+r}{p} = \frac{q}{p}+\frac{r}{p} = \frac{||AF||}{||FB||}+\frac{||AE||}{||EC||}\,,$$

which is not trivial, as one finds if one tries to prove it directly by calculating the lengths of the line segments by trigonometry and not using the area principle.

Also we observe:

$$\frac{||AO||}{||OD||}\cdot\frac{||CE||}{||EA||} = \frac{q+r}{p}\cdot\frac{p}{r} = 1+\frac{q}{r} = 1+\frac{||CD||}{||DB||}\,.$$

Another example is:

$$\begin{aligned}\frac{||AO||}{||OD||}\cdot\frac{||BO||}{||OE||} &= \frac{(q+r)(r+p)}{pq} = 1+\frac{r(p+q+r)}{pq}\\ &= 1+\frac{||AD||}{||OD||}\cdot\frac{||BD||}{||DC||} = 1+\frac{||BE||}{||OE||}\cdot\frac{||AE||}{||EC||}\,.\end{aligned}$$

In fact, every algebraic identity in the variables p, q, r which is homogeneous of degree zero, leads, in at least one way, to an identity between ratios of lengths in the triangle.

4. However, this is not all. We supposed that O was in the interior of the triangle, but this restriction is not necessary; we need only assume that O does not lie on a side (or side extended) of the triangle. Euler does not discuss this case, but the result is true since the above proof still holds with appropriate interpretation.

Firstly, we must use signed or directed lengths; this is the meaning of the doubled modulus lines. We assign a positive direction to every line in the plane and then $||XY||$ is positive if $\overrightarrow{XY}$ lies in the assigned direction and is negative otherwise. Thus $||YX|| = -||XY||$ and if X, Y, Z are distinct collinear points then $\dfrac{||XY||}{||YZ||}$ is positive if and only if Y lies between X and Z. Secondly, we must use signed areas, which means that the area $||XYZ||$ of a triangle $[X, Y, Z]$ is taken to be positive if the vertices are named in a counterclockwise and negative if the vertices are named in a clockwise direction. All of the equations in Sections 2 and 3 have been written in such a way that, with these sign conventions, they hold regardless of the relative positions of the points. In particular they remain true whether the point O is in the interior of the triangle or not.

5. The primary purpose of Euler's paper was not to prove the "remarkable theorem" discussed above, but to solve the following problem:

Given positive numbers x, y, z, is it possible to construct a triangle $[A, B, C]$ so that, with the notation of Figure 1,

$$x = \frac{||AO||}{||OD||}, \; y = \frac{||BO||}{||OE||}, \; z = \frac{||CO||}{||OF||}?$$

The theorem shows that this is impossible unless $xyz = x + y + z + 2$, but the question remains as to whether this condition is sufficient. Euler proves that it is so, but again his proof takes two and a half pages of algebraic and trigonometrical calculation.

Our approach, using the area principle, also gives an easy solution to this problem. If a triangle satisfying the given condition exists, then by (4),

$$x = \frac{q+r}{p}, \; y = \frac{r+p}{q}, \; z = \frac{p+q}{r},$$

from which we obtain,

$$p = \frac{\triangle}{x+1}, \; q = \frac{\triangle}{y+1}, \; r = \frac{\triangle}{z+1},$$

where $\triangle = p + q + r$ is the total area of the triangle, and this may be chosen arbitrarily.

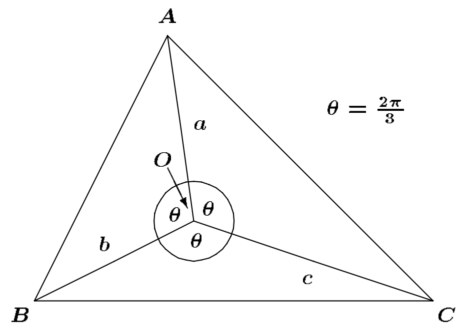

Figure 3

Now consider the triangle shown in Figure 3. Here the three line segments $[AO]$, $[BO]$, $[CO]$ are equally inclined at angles $\dfrac{2\pi}{3}$ to each other, and the lengths of the segments are denoted by a, b, c respectively. Then, by the well-known formula for the area of a triangle,

$$p = bc\sin\frac{2\pi}{3} = \frac{\sqrt{3}bc}{2}, \; q = \frac{\sqrt{3}ca}{2}, \; r = \frac{\sqrt{3}ab}{2},$$

and so we can solve for a, b, c in terms of the areas of the triangles p, q, r obtained above. Clearly,

$$a^2 = \frac{2qr}{\sqrt{3}p}, \; b^2 = \frac{2rp}{\sqrt{3}q}, \; c^2 = \frac{2pq}{\sqrt{3}r},$$

from which the (positive) values of a, b, c can be determined. With these values the triangle in Figure 3 has the required properties.

The reader may ask why we were justified in taking the lines OA, OB and OC as equally inclined to each other. The reason is that the theorem, our proof, and the problem are *affine-invariant*. This means that, in particular, if any triangle can be found which solves the problem, then the result of applying a non-singular affine transformation (that is, a change of scale possibly followed by a shear) yields another triangle which is also a solution to the problem. It is well known that an affine transformation can be found which will transform any three intersecting lines into three such lines which are equally inclined (at angle $\frac{2\pi}{3}$) to each other.

6. The study of early mathematical papers is fraught with difficulty. Sometimes the notation is strange and the language is often obscure. In some cases (as with Euler's paper) it is well worth the effort to overcome these difficulties.

The study of such papers also gives an insight into the knowledge and thought processes of early authors. Clearly Euler, in 1780, did not know of the area principle, though this was certainly known to A.L. Crelle in 1816 since he uses it in his book [3]. Euler knew about affine transformations, for he mentions them in his *Introd. in Analysin Infinitorum II* of 1748. It is therefore surprising that he did not use them to simplify his solution of the problem stated in Section 5. Perhaps he was not aware that it is possible, using affine transformations, to convert any three concurrent lines into three such lines which are equally inclined to each other.

These trivial observations do not, of course, detract from the fact that Euler's contributions to very many branches of mathematics, are both profound and extensive.

Acknowledgements: My thanks are due to Miss Lucy Parker for her help in the translation of Euler's paper and to the referees for their helpful comments and suggestions.

References

[1] C.J. Bradley, Solution to Problem 1997, *Crux Math.* **21** (1995), 283-285.

[2] S.-C. Chou, X.-S. Gao, J.-Z. Zhang, *Machine Proofs in Geometry*, World-Scientific, Singapore-New Jersey-London-Hong Kong, 1994.

[3] A.L. Crelle, *Über einige Eigenschaften des ebenen geradlinigen Dreiecks rücksichtlish dreier durch die Winkel-Spitzen gezogenen geraden Linien*, Berlin 1816.

[4] L. Euler, Geometrica et sphaerica quaedam, *Mémoires de l'Académie des Sciences de St. Petersbourg* **5** (1812), 96-114 (Received 1 May 1780) = L. Euleri, *Opera Omnis*, Series 1, Volume **26** (1953), 344-358.

[5] Branko Grünbaum, Cyclic ratio sums and products, *Crux Math.* **24** (1998), 20-25.

[6] Branko Grünbaum, and G.C. Shephard, Ceva, Menelaus and the area principle, *Math. Magazine* **68** (1995), 254-268.

[7] Branko Grünbaum and G.C. Shephard, Some new transversality properties, *Geom. Dedicata* **71** (1998), 179-208.

[8] G.C. Shephard, Cyclic sums for polygons, *Math. Magazine* (to appear).

G.C. Shephard
School of Mathematics
University of East Anglia
Norwich NR4 7TJ
England, U.K.
g.shephard@uea.ac.uk