THE OLYMPIAD CORNER

No. 197

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We begin this number with the 47th Polish Mathematical Olympiad 1995–96 and the problems of the Final Round, written in March 1996. My thanks go to Marcin E. Kuczma, Warszawa, Poland and to Ravi Vakil, Canadian Team Leader to the IMO at Mumbai, for collecting the materials for the *Corner*.

47th POLISH MATHEMATICAL OLYMPIAD
Problems of the Final Round (March 29–30, 1996)
First Day (Time: 5 hours)

1. Find all pairs (n, r), with n a positive integer, r a real number, for which the polynomial $(x+1)^n - r$ is divisible by $2x^2 + 2x + 1$.

2. Given is a triangle ABC and a point P inside it satisfying the conditions: $\angle PBC = \angle PCA < \angle PAB$. Line BP cuts the circumcircle of ABC at B and E. The circumcircle of triangle APE meets line CE at D and F. Show that the points A, P, E, F are consecutive vertices of a quadrilateral. Also show that the ratio of the area of quadrilateral $APEF$ to the area of triangle ABP does not depend on the choice of P.

3. Let $n \geq 2$ be a fixed natural number and let $a_1, a_2, \ldots, a_n$ be positive numbers whose sum equals 1.

(a) Prove the inequality

$$2\sum_{i<j} x_i x_j \leq \frac{n-2}{n-1} + \sum_{i=1}^{n} \frac{a_i x_i^2}{1-a_i}$$

for any positive numbers $x_1, x_2, \ldots, x_n$ summing to 1.

(b) Determine all n–tuples of positive numbers $x_1, x_2, \ldots, x_n$ summing to 1 for which equality holds.

Second Day (Time: 5 hours)

4. Let $ABCD$ be a tetrahedron with

$$\angle BAC = \angle ACD \quad \text{and} \quad \angle ABD = \angle BDC .$$

Show that edges AB and CD have equal lengths.

5. For a natural number $k \geq 1$ let $p(k)$ denote the least prime number which is not a divisor of k. If $p(k) > 2$, define $q(k)$ to be the product of all primes less than $p(k)$, and if $p(k) = 2$, set $q(k) = 1$. Consider the sequence

$$x_0 = 1 , \quad x_{n+1} = \frac{x_n p(x_n)}{q(x_n)} \quad \text{for} \quad n = 0, 1, 2, \dots .$$

Determine all natural numbers n such that $x_n = 111111$.

6. From the set of all permutations f of the set $\{1, 2, \dots, n\}$ that satisfy the condition

$$f(i) \geq i - 1 \quad \text{for} \quad i = 1, 2, \dots, n ,$$

choose one, with an equal probability of each choice. Let p_n be the probability that the permutation (chosen) satisfies the condition

$$f(i) \leq i + 1 \quad \text{for} \quad i = 1, 2, \dots, n .$$

Find all natural numbers n with $p_n > 1/3$.

Continuing our Northern European theme, we give the problems of the 10^{th} Nordic Mathematical Contest of April 1996. My thanks go to Ravi Vakil, Canadian Team Leader to the IMO at Mumbai, for collecting the questions.

10^{th} NORDIC MATHEMATICAL CONTEST
April 11, 1996 (Time: 4 hours)

1. Prove the existence of a positive integer divisible by 1996, the sum of whose decimal digits is 1996.

2. Determine all real x such that $x^n + x^{-n}$ is an integer for any integer n.

3. A circle has the altitude from A in a triangle ABC as a diameter, and intersects AB and AC in the points D and E, respectively, different from A. Prove that the circumcentre of triangle ABC lies on the altitude from A in triangle ADE, or it produced.

4. A real-valued function f is defined for positive integers, and a positive integer a satisfies

$$f(a) = f(1995), \quad f(a+1) = f(1996), \quad f(a+2) = f(1997),$$

$$f(n+a) = \frac{f(n)-1}{f(n)+1} \quad \text{for any positive integer } n\,.$$

(a) Prove that $f(n+4a) = f(n)$ for any positive integer n.

(b) Determine the smallest possible value of a.

As a final contest set for your enjoyment we give the problems of the Dutch Mathematical Olympiad of September 1995. Again my thanks go to Ravi Vakil, Canadian Team Leader to the IMO at Mumbai.

DUTCH MATHEMATICAL OLYMPIAD
September 15, 1995
Second Round

1. A kangaroo jumps from lattice-point to lattice-point in the (x, y)–plane. She can make only two kinds of jumps:

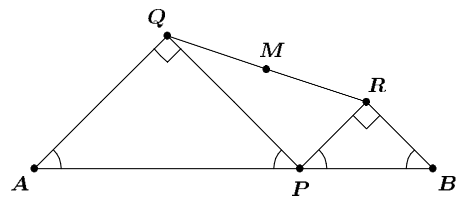
Jump A: 1 to the right (in the positive x–direction) and 3 up (in the positive y–direction).
Jump B: 2 to the left and 4 down.

(a) The start position of the kangaroo is the origin $(0, 0)$. Show that the kangaroo can jump to the point $(19, 95)$, and determine the number of jumps she needs to reach that point.

(b) Take the start position to be the point $(1, 0)$. Show that it is impossible for her to reach the point $(19, 95)$.

(c) The start position of the kangaroo is once more the origin $(0, 0)$. Which points (m, n) with m, $n \geq 0$ can she reach, and which points can she not reach?

2. On a segment AB a point P is chosen. On AP and PB isosceles and right-angled triangles AQP and PRB are constructed with Q and R at the same side of AB. M is the midpoint of QR. Determine the set of all points M for all points P on the segment AB.



3. 101 marbles are numbered from 1 to 101. The marbles are divided over two baskets A and B. The marble numbered 40 is in basket A. This marble is removed from basket A and put in basket B. The average of all the numbers on the marbles in A increases by $\frac{1}{4}$. The average of all the numbers of the marbles in B increases by $\frac{1}{4}$ too. How many marbles were there originally in basket A?

4. A number of spheres, all with radius 1, are being placed in the form of a square pyramid. First, there is a layer in the form of a square with $n \times n$ spheres. On top of that layer comes the next layer with $(n-1) \times (n-1)$ spheres, and so on. The top layer consists of only one sphere. Determine the height of the pyramid.

5. We consider arrays $(a_1, a_2, \ldots, a_{13})$ containing 13 integers. An array is called "tame" when for each $i \in \{1, 2, \ldots, 13\}$ the following condition holds: if you leave a_i out, the remaining twelve integers can be divided into two groups in such a way that the sum of the numbers in one group is equal to the sum of the numbers in the other group. A "tame" array is called "turbo tame" if you can always divide the remaining twelve numbers in two groups of six numbers having the same sum.

(a) Give an example of an array of 13 integers (not all equal!) that is "tame". Show that your array is "tame".

(b) Prove that in a "tame" array all numbers are even or all numbers are odd.

(c) Prove that in a "turbo tame" array all numbers are equal.

An astute reader kept track of the problems proposed to the jury but not used at the International Mathematical Olympiad at Mumbai in 1996 given on [1997: 450], for which we did not publish solutions. Thanks go to Mohammed Aassila, now of the Centre de Recherches Mathématiques, Montréal, Québec, for filling the gaps.

6. Let n be an even positive integer. Prove that there exists a positive integer k such that

$$k = f(x)(x+1)^n + g(x)(x^n+1)$$

for some polynomials $f(x)$, $g(x)$ having integer coefficients. If k_0 denotes the least such k, determine k_0 as a function of n.

Solution by Mohammed Aassila, Centre de Recherches Mathématiques, Montréal, Québec.

The statement of the problem is equivalent to: prove that for any positive integer n, there exist polynomials $f(x)$, $g(x)$ having integer coefficients such that

$$f(x)(x+1)^{2n} + g(x)(x^{2n}+1) \;=\; 2\,,$$

or equivalently to find $f(x)$ and $g(x)$ such that

$$f(x)x^{2n} + g(x)((x-1)^{2n}+1) \;=\; 2\,,$$

or equivalently, to find $g(x)$ such that $g(x)((x-1)^{2n}+1) - 2$ is divisible by x^{2n}.

But, this is not difficult to prove. If

$$g(x) \;=\; a_0 + a_1 x + \cdots + a_{2n-1}x^{2n-1}\,,$$

then we have only to choose a_i, $0 \le i \le 2n-1$ as follows:

$$a_0 = 1 \quad \text{and} \quad 2a_k - \binom{2n}{1}a_{k-1} + \binom{2n}{2}a_{k-2} + \cdots + (-1)^k\binom{2n}{k} \;=\; 0$$

for $1 \le k \le 2n-1$.

8. Let the sequence $a(n)$, $n = 1, 2, 3, \ldots$, be generated as follows: $a(1) = 0$, and for $n > 1$,

$$a(n) \;=\; a(\lfloor n/2 \rfloor) + (-1)^{n(n+1)/2}\,.$$

(Here $\lfloor t \rfloor$ = the greatest integer $\le t$.)

(a) Determine the maximum and minimum value of $a(n)$ over $n \le 1996$, and find all $n \le 1996$ for which these extreme values are attained.

(b) How many terms $a(n)$, $n \le 1996$, are equal to 0?

Solution by Mohammed Aassila, Centre de Recherches Mathématiques, Montréal, Québec.

A simple induction shows that $a(n) = b(n) - c(n)$ where $b(n)$ (resp. $c(n)$) denote the total number of couplings 00, 11 (resp. 01, 10) in the binary representation of n.

(To see this, it suffices to observe that the binary representation of $\lfloor \frac{n}{2} \rfloor$ is obtained from that of n by deleting the last digit.)

(a) The maximum value of $a(n)$ is the largest $n \le 1996$ for which $c(n) = 0$. It is attained for 1111111111 or 1023 where $b(n) = 9$ and hence $a(n) = 9$. The minimum is the largest $n \le 1996$ for which $b(n) = 0$. It is attained for 10101010101 or 1365 where $a(n) = -10$.

(b) $a(n) = 0$ if and only if, in the binary representation of n, the number of the same two consecutive digits is equal to the number of different two consecutive digits. Noting that the first digit has to be 1 and that such representation of n can be formed in $\binom{m}{m/2}$ ways for even m, we deduce that there are

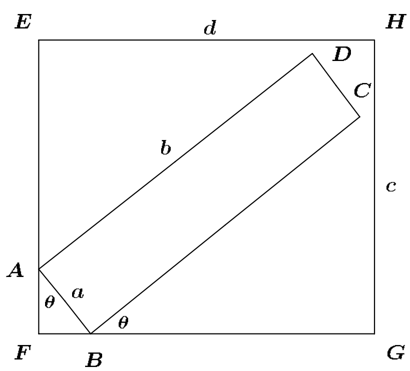
$$\binom{0}{0}+\binom{2}{1}+\binom{4}{2}+\binom{6}{3}+\binom{8}{4}+\binom{10}{5} = 351$$

positive integers $n < 2^{11} = 2048$ with $a(n) = 0$. Finally, since 2002, 2004, 2006, 2010, 2026 are such that $a(n) = 0$ but exceed 1996, there are only 346 numbers ≤ 1996 with $a(n) = 0$.

12. Let the sides of two rectangles be $\{a, b\}$ and $\{c, d\}$ respectively, with $a < c \leq d < b$ and $ab < cd$. Prove that the first rectangle can be placed within the second one if and only if

$$(b^2 - a^2)^2 \leq (bc - ad)^2 + (bd - ac)^2 .$$

Solution by Mohammed Aassila, Centre de Recherches Mathématiques, Montréal, Québec.



Let $ABCD$ and $EFGH$ denote respectively the first and second rectangle.

$ABCD$ can be placed within $EFGH$ if and only if

$$\begin{cases} a\cos\theta + b\sin\theta & \leq & c, \\ b\cos\theta + a\sin\theta & \leq & d. \end{cases}$$

This means that the point of intersection of $a\cos\theta + b\sin\theta = c$ and $b\cos\theta + a\sin\theta = d$, which is $\left(\frac{bd-ac}{b^2-a^2}, \frac{bc-ad}{b^2-a^2}\right)$, lies outside or on the unit circle, or equivalently

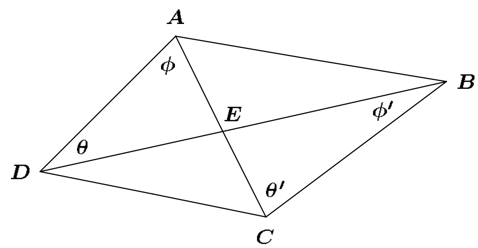
$$(b^2 - a^2)^2 \ \leq\ (bd - ac)^2 + (bc - ad)^2 .$$

Editorial note: The condition, $a < c \leq d < b$, does not appear to have been used. However, it is needed to ensure that the intersection point is in the first quadrant.

14. Let $ABCD$ be a convex quadrilateral, and let R_A, R_B, R_C, R_D denote the circumradii of the triangles DAB, ABC, BCD, CDA respectively. Prove that $R_A + R_C > R_B + R_D$ if and only if $\angle A + \angle C > \angle B + \angle D$.

Solution by Mohammed Aassila, Centre de Recherches Mathématiques, Montréal, Québec.

Let the diagonals meet in E. We may suppose that $\angle AED > 90^\circ$. Then all of $\angle EDA$, $\angle DAE$, $\angle EBC$, $\angle BCE$ are acute. Label them as θ, ϕ, ϕ', θ', respectively. Then $\theta + \phi = \theta' + \phi'$.



Now, if A, B, C and D were concyclic we would have $\phi = \phi'$ and $\theta = \theta'$.

On the other hand, if $\angle A + \angle C > 180^\circ$, then $\theta < \theta'$ and $\phi > \phi'$. This is easily seen by drawing a circumcircle of $\triangle ABC$ and examining the possible locations of B on the extension of DE.

The purpose of isolating acute angles was for a convenient application of the Sine Law. We have

$$R_A = \frac{AB}{2\sin\theta}, \quad R_B = \frac{AB}{2\sin\theta'}, \quad R_C = \frac{DC}{2\sin\phi'}, \quad R_D = \frac{DC}{2\sin\phi}.$$

Thus if $\angle A + \angle C = 180^\circ$ then $\phi' = \phi$ and $\theta = \theta'$ and therefore $R_A + R_C = R_B + R_D$.

More generally,

$$\begin{aligned} \angle A + \angle C > \angle B + \angle D &\iff \angle A + \angle C > 180^\circ \\ &\Longrightarrow \phi' < \phi\,, \quad \theta < \theta' \\ &\iff R_C > R_D\,, \quad R_A > R_B \\ &\Longrightarrow R_A + R_C > R_B + R_D\,. \end{aligned}$$

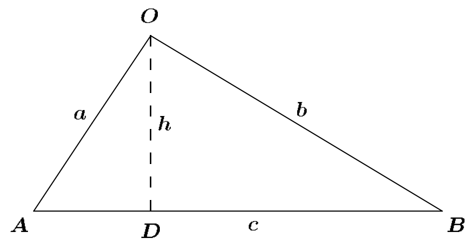
The fact that we have equivalence follows by symmetry.

15. On the plane are given a point O and a polygon $\mathcal{F}$ (not necessarily convex). Let P denote the perimeter of $\mathcal{F}$, D the sum of the distances from O to the vertices of $\mathcal{F}$, and H the sum of the distances from O to the lines containing the sides of $\mathcal{F}$. Prove that $D^2 - H^2 \geq P^2/4$.

Solution by Mohammed Aassila, Centre de Recherches Mathématiques, Montréal, Québec.

We shall prove that $D^2 - H^2 \geq P^2/4$ for any n–gon. We use "induction" on n.

If $n = 2$, the inequality $D^2 - H^2 \geq P^2/4$ is equivalent to $(a+b)^2 - 4h^2 \geq c^2$.



To prove this, consider a line (L) passing through O and parallel to AB, and let B_1 be the image of B under reflection in (L). Then $OA + OB = OA + OB_1 \geq AB_1$; that is $a + b \geq \sqrt{4h^2 + c^2}$, as required.

Now, let the n-gon $\mathcal{F}$ be $P_1P_2 \ldots P_n$ and let $d_i = OP_i$, $p_i = P_iP_{i+1}$, h_i = distance from O to P_iP_{i+1}. Then by the "induction hypothesis" we have for each i

$$d_i + d_{i+1} \;\geq\; \sqrt{4h_i^2 + p_i^2}\,.$$

Summing these inequalities over $i = 1, 2, \ldots, n$ we obtain

$$2D \;\geq\; \sum_i \sqrt{4h_i^2 + p_i^2}\,,$$

or equivalently
$$4D^2 \;\geq\; \left(\sum_i \sqrt{4h_i^2 + p_i^2}\right)^2\,.$$

Now, $4H^2 + P^2 = 4(\sum_i h_i)^2 + (\sum_i p_i)^2$, so if we prove that

$$\sum_i \sqrt{4h_i^2 + p_i^2} \;\geq\; \sqrt{4\left(\sum_i h_i\right)^2 + \left(\sum_i p_i\right)^2}\,,$$

then we are done. But the above inequality follows from the triangle inequality applied to the sum of the vectors $(2h_i, p_i)$.

17. A finite sequence of integers $a_0, a_1, \ldots, a_n$ is called *quadratic* if for each i in the set $\{1, 2, \ldots, n\}$ we have the equality $|a_i - a_{i-1}| = i^2$.

(a) Prove that for any two integers b and c, there exist a natural number n and a quadratic sequence with $a_0 = b$ and $a_n = c$.

(b) Find the smallest natural number n for which there exists a quadratic sequence with $a_0 = 0$ and $a_n = 1996$.

Solution by Mohammed Aassila, Centre de Recherches Mathématiques, Montréal, Québec.

This problem is well known — it is problem 250 from the book of Sierpinski!!

Next, a bit more tidying up. Last issue we gave solutions by the readers to problems of the 17th Austrian-Polish Mathematics Competition [1998: 4–6; 1999: 77–85]. Next we give a solution to problem 6.

6. Let $n > 1$ be an odd positive integer. Assume that the integers x_1, $x_2, \dots, x_n \geq 0$ satisfy the system of equations

$$\begin{aligned} (x_2 - x_1)^2 + 2(x_2 + x_1) + 1 &= n^2\,, \\ (x_3 - x_2)^2 + 2(x_3 + x_2) + 1 &= n^2\,, \\ &\dots\dots\dots \\ (x_1 - x_n)^2 + 2(x_1 + x_n) + 1 &= n^2\,. \end{aligned}$$

Show that either $x_1 = x_n$, or there exists j with $1 \leq j \leq n-1$, such that $x_j = x_{j+1}$.

Solution adapted from one by Pierre Bornsztein, Courdimanche, France.

We show, by induction on m, that, if non-negative integers $x_1, x_2, \dots, x_m \geq 0$, with $m > 1$ being an odd integer, satisfy

$$\begin{aligned} (x_2 - x_1)^2 + 2(x_2 + x_1) + 1 &= n^2\,, \\ (x_3 - x_2)^2 + 2(x_3 + x_1) + 1 &= n^2\,, \\ &\dots\dots\dots \\ (x_1 - x_m)^2 + 2(x_1 + x_m) + 1 &= n^2\,, \end{aligned}$$

then either $x_1 = x_m$, or there is j with $1 \leq j \leq m-1$, such that $x_j = x_{j+1}$. We adopt the convention that subscripts are read modulo m, so that $x_{m+1} = x_1$, etc..

Now notice that for each j, x_{j-1} and x_{j+1} are solutions (possibly equal) to the quadratic equation

$$X^2 + X(2 - 2x_j) + (x_j + 1)^2 - n^2 = 0 \quad (E_j)\,,$$

for which the discriminant $\Delta_j = 4(n^2 - 4x_j) \geq 0$ for there to be a real root. Moreover as n is odd, $\Delta_j \geq 1$.

Also because (E_j) has integral coefficients and at least one integral root, both roots are integers, and they are distinct as $\Delta_j \geq 1$. Denote the roots by α_j and β_j with $\alpha_j < \beta_j$.

Also we have $\alpha_j + \beta_j = 2x_j - 2$, and that $\{x_{j-1}, x_{j+1}\} \subset \{\alpha_j, \beta_j\}$. We claim that *either* $\alpha_j = x_j - 2$ and $\beta_j = x_j$ *or* $\alpha_j < x_j < \beta_j$. (Indeed if $\alpha_j, \beta_j \geq x_j$ then $\alpha_j + \beta_j > 2x_j - 2$, and if $\alpha_j, \beta_j < x_j$ then $\alpha_j + \beta_j \leq 2x_j - 3$.)

Now let j be such that $x_j = \max\{x_1, \dots, x_n\}$. Now x_{j-1} and x_{j+1} are roots of E_j. Unless $x_{j-1} = x_j$ or $x_j = x_{j+1}$ we must have

$$x_{j-1} = x_{j+1}$$

or

$$x_{j-1} < x_j < x_{j+1}$$

or

$$x_{j+1} < x_j < x_{j-1}\,.$$

The last two cases contradict the choice of j, so $x_{j-1} = x_{j+1}$. If $m = 3$ we are done since x_{j-1} and x_{j+1} are cyclically adjacent. Otherwise $m > 3$ and by removing x_{j-1}, x_j we obtain a solution with $m - 2$ values, to which the induction hypothesis applies.

Now we turn our attention to solutions from the readers to five of the problems of the Iranian National Mathematical Olympiad, February 6, 1994, Second Round [1998: 6–7].

IRANIAN NATIONAL MATHEMATICAL OLYMPIAD February 6, 1994 Second Round

1. Suppose that p is a prime number and is greater than 3. Prove that $7^p - 6^p - 1$ is divisible by 43.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Yeo Keng Hee, Hwa Chong Junior College, Singapore; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by Pavlos Maragoudakis, Pireas, Greece; by Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Klamkin's generalization.

More generally we show that $[(x+1)^n - x^n - 1]/[x^2+x+1]$ is an integral polynomial for all natural odd n not divisible by 3. The given problem corresponds to the special case $x = 6$.

By the Remainder Theorem $(x+1)^n - x^n - 1$ will be divisible by x^2+x+1 if it vanishes for the zeros of the latter, which are ω and ω^2, the two complex cube roots of unity. Here,

$$(\omega+1)^n - \omega^n - 1 \;=\; (-1)^n\omega^{2n} - \omega^n - 1\,.$$

As known, $\omega^{2m}+\omega^m+1 = 0$ if and only if m is not divisible by 3. The same also applies to the other zero ω^2. Also we must have $(-1)^n$ to be 1, so that n is even.

Comment. Similarly, $[(x+1)^n + x^n + 1]$ is divisible by $[x^2+x+1]$ for all natural even n not divisible by 3. As a special case, $7^n + 6^n + 1$ is divisible by 43 for any of these values of n.

3. Let n and r be natural numbers. Find the smallest natural number m satisfying this condition: For each partition of the set $\{1, 2, \ldots, m\}$ into r subsets $A_1, A_2, \ldots, A_r$, there exist two numbers a and b in some A_i $(1 \le i \le r)$ such that $1 < \frac{a}{b} \le 1 + \frac{1}{n}$.

Solution by Yeo Keng Hee, Hwa Chong Junior College, Singapore.

I claim the answer is $(nr+r)$. Suppose $m < nr + r$. Then for each $i \le m$, let i be put into the subset A_j, $(j = 1, 2, \ldots, r)$ such that $i \equiv j \pmod r$. Then for any 2 numbers a, b in the same subset with $a > b$, we have $b \le (a - r)$. Thus

$$a/b \;=\; (1+(a-b)/b) \;\ge\; (1+r/b) \;>\; (1+r/nr) \;=\; (1+1/n)\,.$$

Therefore the condition in the question cannot hold. For $m = (nr + r)$, consider the $(r+1)$ numbers $nr, (nr+1), \ldots, (nr+r)$. By the Pigeonhole Principle, among them there exists $a > b$ such that a and b are in the same subset. Then

$$a/b = (1 + (a-b)/b) \leq (1 + r/b) \leq (1 + r/nr) = (1 + 1/n),$$

and $1 < a/b$ and we are done.

4. G is a graph with n vertices $A_1, A_2, \ldots, A_n$ such that for each pair of non-adjacent vertices A_i and A_j there exists another vertex A_k that is adjacent to both A_i and A_j.

(a) Find the minimum number of edges of such a graph.

(b) If $n = 6$ and $A_1, A_2, A_3, A_4, A_5, A_6$ form a cycle of length 6, find out the number of edges that must be added to this cycle such that the above condition holds.

Solution by Yeo Keng Hee, Hwa Chong Junior College, Singapore.

(a) G must be a connected graph. Thus it must have at least $(n-1)$ edges. If the $(n-1)$ edges are the edges that join A_i, $(i = 1, 2, \ldots, n-1)$ to A_n, the condition is indeed satisfied. Thus the minimum number of edges is $(n-1)$.

(b) Without loss of generality, let A_i be joined to A_{i-1} and A_{i+1}, $(i = 1, 2, \ldots, 6)$, $A_0 = A_6$, $A_7 = A_1$. Then consider the three pairs of vertices (A_1, A_4), (A_2, A_5), (A_3, A_6). In each pair of vertices, the degree of at least one vertex needs to be increased by 1, because any path joining the two vertices has length at least 3, so one vertex must be joined to the other vertex or a vertex adjacent to that vertex.

Thus the sum of degrees of all the vertices must increase by at least 3; that is, at least two edges must be added. Adding the edges A_1A_4 and A_2A_6 satisfies the conditions.

5. Show that if D_1 and D_2 are two skew lines, then there are infinitely many straight lines such that their points have equal distance from D_1 and D_2.

Solutions by Michel Bataille, Rouen, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Bataille's solution.

Let H_1 on D_1, H_2 on D_2 be such that: $H_1H_2 \perp D_1$ and $H_1H_2 \perp D_2$ and let O be the midpoint of H_1H_2. We will work in the following system of rectangular axes: we take O as origin, the z–axis along H_1H_2 and, as x–axis and y–axis, we take the bisectors of the angle formed at O by the parallels to D_1 and D_2. Then there exist non-zero real numbers a and m such that:

D_1 is the intersection of the planes $(P_1) : z = a$ and $(Q_1) : y = mx$;
D_2 is the intersection of the planes $(P_2) : z = -a$ and $(Q_2) : y = -mx$.

Now, let $A(\alpha, 0, 0)$ be a point lying on the x–axis and Δ_A be the line through A directed by $\vec{u}(0, 1, t)$. We show that it is possible to choose t (depending on α) such that each point $M(\alpha, k, kt)$, $(k \in \mathbb{R})$ of Δ_A has equal distance from D_1 and D_2.

As the planes (P_1) and (Q_1) are perpendicular, we have:

$$[d(M, D_1)]^2 = [d(M, P_1)]^2 + [d(M, Q_1)]^2 = (kt - a)^2 + \frac{(k - m\alpha)^2}{1 + m^2}.$$

Similarly: $[d(M, D_2)]^2 = (kt + a)^2 + \dfrac{(k + m\alpha)^2}{1 + m^2}$.

Hence, $d(M, D_1) = d(M, D_2)$ is equivalent to $4k\left(at + \dfrac{m\alpha}{1 + m^2}\right) = 0$ so that, by choosing $t = -\dfrac{m\alpha}{a\,(1 + m^2)}$, we have $d(M, D_1) = d(M, D_2)$ for all M on Δ_A.

Since $\Delta_A \neq \Delta_B$ whenever $A \neq B$ on the x–axis (Δ_A and Δ_B are in strictly parallel planes), the family of lines Δ_A, when α takes all real values, answers the question.

6. $f(x)$ and $g(x)$ are polynomials with real coefficients such that for infinitely many rational values x, $\frac{f(x)}{g(x)}$ is rational. Prove that $\frac{f(x)}{g(x)}$ can be written as the ratio of two polynomials with rational coefficients.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

With little change, a solution of this problem is given in [1]. Without loss of generality we can assume that f and g are relatively prime polynomials and let r denote the sum of the degrees of f and g. Also let $R(x) = f(x)/g(x)$, assuming that the degree of f is $\geq$ than the degree of g. If not, we consider $R^{-1}(x)$. For $r = 0$, the result is obvious. Now let a be one of the rational numbers such that $g(a) \neq 0$ and $R(a)$ is rational. Now define $f_1(x)$ by

$$f_1(x)/g(x) = \{R(x) - f(a)/g(a)\}/(x - a).$$

Then $f_1(x)/g(x)$ is also rational for all the rational values that $f(x)/g(x)$ is rational except $x = a$. Since

$$f_1(x) = [g(a)f(x) - f(a)g(x)]/g(a)(x - a),$$

the degree of $f_1(x)$ is less than that of $f(x)$. Thus the sum of the degrees of $f_1(x)$ and $g(x)$ is less than that of $f(x)$ and $g(x)$. The desired result now follows by mathematical induction.

Reference: [1] G. Polya, G. Szego, **Problems and Theorems in Analysis**, Springer-Verlag, II, NY, 1976, pp. 130, 321, Problem 92.

Next we look at readers' solutions to two problems of the Japan Mathematical Olympiad, Final Round, February 1994 given in [1998: 68–69].

1. For a positive integer n, let a_n be the nearest positive integer to $\sqrt{n}$, and let $b_n = n + a_n$. Dropping all b_n $(n = 1, 2, \dots)$ from the set of all positive integers N, we get a sequence of positive integers in ascending order $\{c_n\}$. Represent c_n by n.

Solution by Pierre Bornsztein, Courdimanche, France.

On a

$$\begin{array}{lll} a_1 = 1, & b_1 = 2 \\ a_2 = 1, & b_2 = 3 \\ a_3 = 2, & b_3 = 5 \\ \text{etc..} \end{array}$$

La suite $\{a_n\}$ est croissante, donc $\{b_n\}$ est strictement croissante.

Soit $n \in \mathbb{N}^*$. On a $n < \sqrt{n^2+n} < n + \frac{1}{2}$, donc $a_{n^2+n} = n$, et alors $b_{n^2+n} = (n+1)^2 - 1$. En plus $n + \frac{1}{2} < \sqrt{n^2+n+1} < n+1$, donc $a_{n^2+n+1} = n+1$, et alors $b_{n^2+n+1} = (n+1)^2 + 1$. On en déduit que $\{b_n\}$ ne contient aucun carré.

Pour $n \in \mathbb{N}^*$, la cardinalité de l'ensemble $\{b_1, \dots, b_{n^2+n}\} = n^2+n$, et la cardinalité de $\{p \in \mathbb{N}^* : p \leq (n+1)^2 - 1 \text{ et } p \text{ n'est pas un carré}\} = n^2 + n$.

De plus,

$$\{b_1, \dots, b_{n^2+n}\} \subset \{p \in \mathbb{N}^* : p \leq (n+1)^2 - 1, \ p \text{ non carré}\}$$

d'où

$$\{b_1, \dots, b_{n^2+n}\} = \{p \in \mathbb{N}^* : p \leq (n+1)^2 - 1, \ p \text{ non carré}\},$$

et comme n est arbitraire

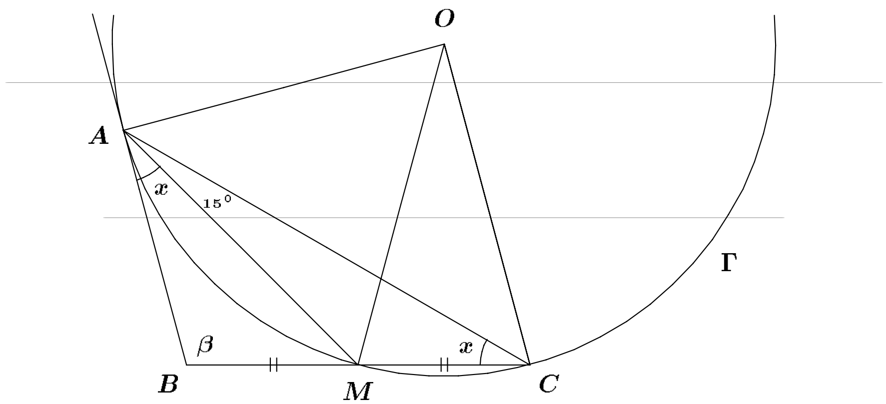
$$\{b_1, b_2, \dots\} = \mathbb{N}^* \backslash \{1, 4, 9, 16, \dots\}.$$

On en obtient pour $n \geq 1$, $c_n = n^2$.

4. We consider a triangle ABC such that $\angle MAC = 15^\circ$ where M is the midpoint of BC. Determine the possible maximum value of $\angle B$.

Solutions by Pierre Bornsztein, Courdimanche, France; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give the solution of Smeenk.

Let $\Gamma(O, R)$ be the circumcircle of $\triangle AMC$, with centre O and radius R. Denote $\angle B = \beta$. Now β has its maximum when AB touches Γ at A. We denote $\angle BAM = \angle ACM = x$.



Since AM is a median in $\triangle ABC$ we have:

$$\begin{aligned} \sin x : \sin 15^\circ &= \sin\beta : \sin x\,, \\ 2\sin^2 x &= 2\sin\beta\sin 15^\circ\,, \\ 1-\cos 2x &= \cos(\beta-15^\circ)-\cos(\beta+15^\circ)\,. \end{aligned}$$

Since $\beta + 15^\circ + 2x = 180^\circ$, we have $2x = 165^\circ - \beta$. Thus $1+\cos(\beta+15^\circ) = \cos(\beta-15^\circ)-\cos(\beta+15^\circ)$, or

$$2\cos(\beta+15^\circ)-\cos(\beta-15^\circ)+1 \;=\; 0\,. \qquad (1)$$

It is easy to verify that this is satisfied by $\beta = 105^\circ$.

Write $\beta = 105^\circ + y$. From (1) we obtain

$$\begin{aligned} 2\cos(120^\circ+y)-\cos(90^\circ+y)+1 &= 0\,, \\ 2\cos 120^\circ\cos y - 2\sin 120^\circ \sin y + \sin y + 1 &= 0\,, \\ 1-\cos y &= \sin y\left(\sqrt{3}-1\right)\,, \\ 2\sin^2\left(\frac{1}{2}\right)y &= 2\sin\left(\frac{1}{2}y\right)\cos\left(\frac{1}{2}y\right)(\sqrt{3}-1)\,. \end{aligned}$$

Therefore, $\sin\left(\frac{1}{2}y\right) = 0 \Longrightarrow y = 0$ or

$$\begin{aligned} \tan\left(\frac{1}{2}y\right) &= \sqrt{3}-1 \\ \Longrightarrow \quad y &= 72.4\ldots^\circ \\ \Longrightarrow \quad \beta &= 177.4\ldots^\circ\,. \end{aligned}$$

Since $x > 15^\circ$, this does not hold. So β has a maximum of 105°.

That completes the *Corner* for this issue. Do not forget to send me Olympiad contests and your nice solutions to the problems we have given.