THE ACADEMY CORNER

No. 24

Bruce Shawyer

All communications about this column should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7

Here is the second of four articles from the 1998 Canadian Undergraduate Mathematics Conference, held at the University of British Columbia in July 1998.

Abstracts • Résumés

Canadian Undergraduate Mathematics Conference 1998 — Part 2

Topological Censorship: A Theorem in Global General Relativity

Joel K. Erickson
University of British Columbia

Since the 1960's, modern differential geometry and algebraic topology have been used to study the global properties of space-times. In particular, these tools have been used to study the questions "what is the topology of our universe?" and "why don't we observe any non-trivial topology?" Neither the Einstein equations nor physical considerations provide grounds for restricting the range of possible spatial topologies; there is no reason to assume that spatial slices of our universe have one of the trivial topologies S^3 or $\mathbb{R}^3$. Friedman, Schleich, and Witt have proved that if a number of physically reasonable conditions are satisfied by space-time, then non-trivial topological structures cannot be probed; that is, a *topological censorship principle* holds. Following a brief introduction to causal structure, the theorem and its proof will be described. Current efforts to generalize the theorem to cosmological models will then be discussed.

Modern Answers to an Old Number Theory Question

Alexandru E. Ghîtză
McGill University

What integers can be expressed as a sum of two cubes of rational numbers? This simply stated problem, more than 350 years old, has been investigated by some of the

big names in mathematics: Fermat, Lagrange, Euler, Legendre, Dirichlet. Nonetheless, the question has only been solved in a few particular cases. We shall present some of these from two completely different points of view: that of classical number theory, illustrated by a 19th century theorem of Sylvester, and that of modern number theory, represented by the Heegner point construction which has been used recently to attack the problem.

Les partages d'entiers

Philippe Girard
Université du Québec à Montréal

Le problème général de la théorie additive des nombres consiste à déterminer le nombre de façons d'exprimer un entier naturel en une somme de ces derniers. Le partages d'entiers (en anglais "integer partitions") s'intéressent à ces décompositions dans la mesure où l'ordre des termes est sans importance. Le premier mathématicien à se pencher sur la question et qui apporta des résultats intéressant est Euler dans les années 1740. Dans cet exposé, une légère introduction à la théorie des partages d'entiers sera présentée à l'aide de quelques définitions, d'exemples, de théorèmes proposés par Euler et de démonstrations.

Variations sur quelques généralisations en analyse

Alexandre Girouard
Université de Montréal

L'analyse c'est vue, au cours du dernier siècle, completement transformée. Nous présentons ici quelques une des généralisations lui ayant donnée sa puissance actuelle ! Le matériel couvert ici est trés facilement disponible dans la littérature. Pour cette raison, ce document n'est presque rien de plus qu'une liste de référence.

Finite-Difference Approximations for Partial Differential Equations

Jacinthe Granger-Piché
Université de Montréal

Partial differential equations are used to represent a wide range of phenomena. In particular, they are well used to describe cloud formation. Often, analytical solutions of such equations cannot be found, so numerical techniques are needed. In this paper, we will present different finite-difference approximations applied to the advection and the heat equations, as well as their application to cloud formation.

Clifford Algebras and Spinors

Marco Gualtieri
McGill University

Clifford algebras are ideally suited for the study of rotations (for any signature metric). In fact, Clifford called his invention "Geometric Algebra."

I will describe the Real and Complex Clifford algebras, and the marvellous Bott 8-periodicity. Then I will give a general description of a *spinor*, and how it is used in physics to represent "spin - 1/2 particles" like the electron. I may finally discuss the mysterious coincidence called "Triality" which can be used to understand the *octonions*, the exceptional Lie groups, and other exotic animals.

For maximum enjoyment, you should know some linear algebra, (especially the Sylvester Inertia theorem for real symmetric bilinear forms).

Witten's Formulas for Symplectic Volumes of Moduli Spaces

Patrick Hayden
McGill University

This paper develops the geometric and algebraic tools necessary to understand Witten's calculation of the symplectic volumes of moduli spaces of flat connections over Riemann surfaces and then goes on to supply a detailed account of his argument. We summarize the basic constructions and theorems from differential geometry that are required for our investigation of moduli spaces before defining the Reidemeister torsion and proving some useful results for evaluating the torsion of complexes defined over surfaces. Next, we introduce the moduli space of flat connections on a Riemann surface, giving both the geometric construction and the equivalent representation-theoretic one. Our account ends with a proof of the equivalence of the symplectic volume and Reidemeister torsion and a calculation for the symplectic volume of the moduli space of flat connections over compact oriented surfaces of genus $g \geq 2$.

Reduced Decompositions of Permutations

Sylvie Hébert
Université du Québec à Montréal

After having recalled that the symmetric group is generated by adjacent transpositions, we will define reduced expressions (or decompositions) of permutations. The length of such a decomposition is the number of inversions of the permutation. We will show how to pass from a reduced decomposition to another, and how one can draw such a decomposition by a configuration of lines in the plane. Finally we will deduce the Coxeter presentation of the symmetric group.

A New Characterization of Topology?

Patrick Ingram
Simon Fraser University

In most classical texts on topology we are given several equivalent methods of representing topological spaces : by open sets, by closed sets, by an interior function, and by a closure function. We are also given methods of representing a topological space in terms of another topological space and a function between them that we wish to be continuous. This author, however, was not able to find a characterization from sequence convergence and/or clustering, and as such this paper explores these possibilities. This paper contains elementary definitions to make it possible for a student with a minimal pure math background to read it.

THE OLYMPIAD CORNER

No. 197

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

We begin this number with the 47^{th} Polish Mathematical Olympiad 1995–96 and the problems of the Final Round, written in March 1996. My thanks go to Marcin E. Kuczma, Warszawa, Poland and to Ravi Vakil, Canadian Team Leader to the IMO at Mumbai, for collecting the materials for the *Corner*.

47^{th} POLISH MATHEMATICAL OLYMPIAD
Problems of the Final Round (March 29–30, 1996)
First Day (Time: 5 hours)

1. Find all pairs (n, r), with n a positive integer, r a real number, for which the polynomial $(x+1)^n - r$ is divisible by $2x^2 + 2x + 1$.

2. Given is a triangle ABC and a point P inside it satisfying the conditions: $\angle PBC = \angle PCA < \angle PAB$. Line BP cuts the circumcircle of ABC at B and E. The circumcircle of triangle APE meets line CE at D and F. Show that the points A, P, E, F are consecutive vertices of a quadrilateral. Also show that the ratio of the area of quadrilateral $APEF$ to the area of triangle ABP does not depend on the choice of P.

3. Let $n \geq 2$ be a fixed natural number and let a_1, a_2, ..., a_n be positive numbers whose sum equals 1.

(a) Prove the inequality

$$2\sum_{i<j} x_i x_j \;\leq\; \frac{n-2}{n-1} + \sum_{i=1}^{n} \frac{a_i x_i^2}{1-a_i}$$

for any positive numbers $x_1, x_2, \ldots, x_n$ summing to 1.

(b) Determine all n–tuples of positive numbers $x_1, x_2, \ldots, x_n$ summing to 1 for which equality holds.

Second Day (Time: 5 hours)

4. Let $ABCD$ be a tetrahedron with

$$\angle BAC = \angle ACD \quad \text{and} \quad \angle ABD = \angle BDC .$$

Show that edges AB and CD have equal lengths.

5. For a natural number $k \geq 1$ let $p(k)$ denote the least prime number which is not a divisor of k. If $p(k) > 2$, define $q(k)$ to be the product of all primes less than $p(k)$, and if $p(k) = 2$, set $q(k) = 1$. Consider the sequence

$$x_0 = 1 , \quad x_{n+1} = \frac{x_n p(x_n)}{q(x_n)} \quad \text{for} \quad n = 0, 1, 2, \dots .$$

Determine all natural numbers n such that $x_n = 111111$.

6. From the set of all permutations f of the set $\{1, 2, \dots, n\}$ that satisfy the condition

$$f(i) \geq i - 1 \quad \text{for} \quad i = 1, 2, \dots, n ,$$

choose one, with an equal probability of each choice. Let p_n be the probability that the permutation (chosen) satisfies the condition

$$f(i) \leq i + 1 \quad \text{for} \quad i = 1, 2, \dots, n .$$

Find all natural numbers n with $p_n > 1/3$.

Continuing our Northern European theme, we give the problems of the 10^{th} Nordic Mathematical Contest of April 1996. My thanks go to Ravi Vakil, Canadian Team Leader to the IMO at Mumbai, for collecting the questions.

10^{th} NORDIC MATHEMATICAL CONTEST
April 11, 1996 (Time: 4 hours)

1. Prove the existence of a positive integer divisible by 1996, the sum of whose decimal digits is 1996.

2. Determine all real x such that $x^n + x^{-n}$ is an integer for any integer n.

3. A circle has the altitude from A in a triangle ABC as a diameter, and intersects AB and AC in the points D and E, respectively, different from A. Prove that the circumcentre of triangle ABC lies on the altitude from A in triangle ADE, or it produced.

4. A real-valued function f is defined for positive integers, and a positive integer a satisfies

$$f(a) = f(1995), \quad f(a+1) = f(1996), \quad f(a+2) = f(1997),$$

$$f(n+a) = \frac{f(n)-1}{f(n)+1} \quad \text{for any positive integer } n.$$

(a) Prove that $f(n+4a) = f(n)$ for any positive integer n.

(b) Determine the smallest possible value of a.

As a final contest set for your enjoyment we give the problems of the Dutch Mathematical Olympiad of September 1995. Again my thanks go to Ravi Vakil, Canadian Team Leader to the IMO at Mumbai.

DUTCH MATHEMATICAL OLYMPIAD
September 15, 1995
Second Round

1. A kangaroo jumps from lattice-point to lattice-point in the (x, y)–plane. She can make only two kinds of jumps:

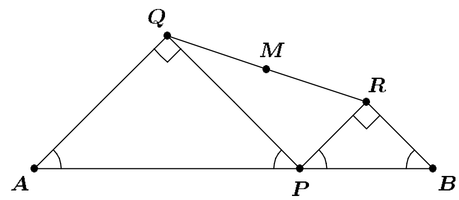
Jump A: 1 to the right (in the positive x–direction) and 3 up (in the positive y–direction).
Jump B: 2 to the left and 4 down.

(a) The start position of the kangaroo is the origin $(0, 0)$. Show that the kangaroo can jump to the point $(19, 95)$, and determine the number of jumps she needs to reach that point.

(b) Take the start position to be the point $(1, 0)$. Show that it is impossible for her to reach the point $(19, 95)$.

(c) The start position of the kangaroo is once more the origin $(0, 0)$. Which points (m, n) with m, $n \geq 0$ can she reach, and which points can she not reach?

2. On a segment AB a point P is chosen. On AP and PB isosceles and right-angled triangles AQP and PRB are constructed with Q and R at the same side of AB. M is the midpoint of QR. Determine the set of all points M for all points P on the segment AB.



3. 101 marbles are numbered from 1 to 101. The marbles are divided over two baskets A and B. The marble numbered 40 is in basket A. This marble is removed from basket A and put in basket B. The average of all the numbers on the marbles in A increases by $\frac{1}{4}$. The average of all the numbers of the marbles in B increases by $\frac{1}{4}$ too. How many marbles were there originally in basket A?

4. A number of spheres, all with radius 1, are being placed in the form of a square pyramid. First, there is a layer in the form of a square with $n \times n$ spheres. On top of that layer comes the next layer with $(n-1) \times (n-1)$ spheres, and so on. The top layer consists of only one sphere. Determine the height of the pyramid.

5. We consider arrays $(a_1, a_2, \ldots, a_{13})$ containing 13 integers. An array is called "tame" when for each $i \in \{1, 2, \ldots, 13\}$ the following condition holds: if you leave a_i out, the remaining twelve integers can be divided into two groups in such a way that the sum of the numbers in one group is equal to the sum of the numbers in the other group. A "tame" array is called "turbo tame" if you can always divide the remaining twelve numbers in two groups of six numbers having the same sum.

(a) Give an example of an array of 13 integers (not all equal!) that is "tame". Show that your array is "tame".

(b) Prove that in a "tame" array all numbers are even or all numbers are odd.

(c) Prove that in a "turbo tame" array all numbers are equal.

An astute reader kept track of the problems proposed to the jury but not used at the International Mathematical Olympiad at Mumbai in 1996 given on [1997: 450], for which we did not publish solutions. Thanks go to Mohammed Aassila, now of the Centre de Recherches Mathématiques, Montréal, Québec, for filling the gaps.

6. Let n be an even positive integer. Prove that there exists a positive integer k such that

$$k = f(x)(x+1)^n + g(x)(x^n+1)$$

for some polynomials $f(x)$, $g(x)$ having integer coefficients. If k_0 denotes the least such k, determine k_0 as a function of n.

Solution by Mohammed Aassila, Centre de Recherches Mathématiques, Montréal, Québec.

The statement of the problem is equivalent to: prove that for any positive integer n, there exist polynomials $f(x)$, $g(x)$ having integer coefficients such that

$$f(x)(x+1)^{2n} + g(x)(x^{2n}+1) \;=\; 2\,,$$

or equivalently to find $f(x)$ and $g(x)$ such that

$$f(x)x^{2n} + g(x)((x-1)^{2n}+1) \;=\; 2\,,$$

or equivalently, to find $g(x)$ such that $g(x)((x-1)^{2n}+1) - 2$ is divisible by x^{2n}.

But, this is not difficult to prove. If

$$g(x) \;=\; a_0 + a_1 x + \cdots + a_{2n-1}x^{2n-1}\,,$$

then we have only to choose a_i, $0 \le i \le 2n-1$ as follows:

$$a_0 = 1 \quad \text{and} \quad 2a_k - \binom{2n}{1}a_{k-1} + \binom{2n}{2}a_{k-2} + \cdots + (-1)^k\binom{2n}{k} \;=\; 0$$

for $1 \le k \le 2n-1$.

8. Let the sequence $a(n)$, $n = 1, 2, 3, \ldots$, be generated as follows: $a(1) = 0$, and for $n > 1$,

$$a(n) \;=\; a(\lfloor n/2 \rfloor) + (-1)^{n(n+1)/2}\,.$$

(Here $\lfloor t \rfloor$ = the greatest integer $\le t$.)

(a) Determine the maximum and minimum value of $a(n)$ over $n \le 1996$, and find all $n \le 1996$ for which these extreme values are attained.

(b) How many terms $a(n)$, $n \le 1996$, are equal to 0?

Solution by Mohammed Aassila, Centre de Recherches Mathématiques, Montréal, Québec.

A simple induction shows that $a(n) = b(n) - c(n)$ where $b(n)$ (resp. $c(n)$) denote the total number of couplings 00, 11 (resp. 01, 10) in the binary representation of n.

(To see this, it suffices to observe that the binary representation of $\lfloor \frac{n}{2} \rfloor$ is obtained from that of n by deleting the last digit.)

(a) The maximum value of $a(n)$ is the largest $n \le 1996$ for which $c(n) = 0$. It is attained for 1111111111 or 1023 where $b(n) = 9$ and hence $a(n) = 9$. The minimum is the largest $n \le 1996$ for which $b(n) = 0$. It is attained for 10101010101 or 1365 where $a(n) = -10$.

(b) $a(n) = 0$ if and only if, in the binary representation of n, the number of the same two consecutive digits is equal to the number of different two consecutive digits. Noting that the first digit has to be 1 and that such representation of n can be formed in $\binom{m}{m/2}$ ways for even m, we deduce that there are

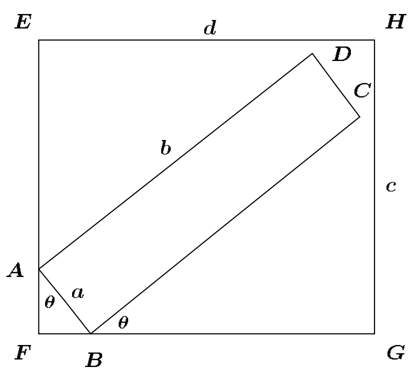
$$\binom{0}{0}+\binom{2}{1}+\binom{4}{2}+\binom{6}{3}+\binom{8}{4}+\binom{10}{5} \;=\; 351$$

positive integers $n < 2^{11} = 2048$ with $a(n) = 0$. Finally, since 2002, 2004, 2006, 2010, 2026 are such that $a(n) = 0$ but exceed 1996, there are only 346 numbers ≤ 1996 with $a(n) = 0$.

12. Let the sides of two rectangles be $\{a, b\}$ and $\{c, d\}$ respectively, with $a < c \leq d < b$ and $ab < cd$. Prove that the first rectangle can be placed within the second one if and only if

$$(b^2 - a^2)^2 \leq (bc - ad)^2 + (bd - ac)^2 .$$

Solution by Mohammed Aassila, Centre de Recherches Mathématiques, Montréal, Québec.



Let $ABCD$ and $EFGH$ denote respectively the first and second rectangle.

$ABCD$ can be placed within $EFGH$ if and only if

$$\begin{cases} a\cos\theta + b\sin\theta & \leq & c\,, \\ b\cos\theta + a\sin\theta & \leq & d\,. \end{cases}$$

This means that the point of intersection of $a\cos\theta + b\sin\theta = c$ and $b\cos\theta + a\sin\theta = d$, which is $\left(\frac{bd-ac}{b^2-a^2}, \frac{bc-ad}{b^2-a^2}\right)$, lies outside or on the unit circle, or equivalently

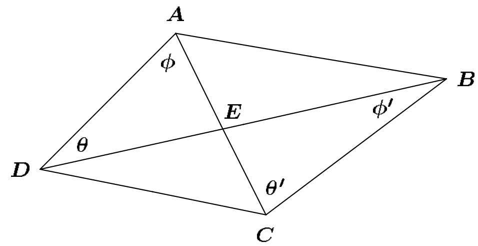
$$(b^2 - a^2)^2 \;\leq\; (bd - ac)^2 + (bc - ad)^2 .$$

Editorial note: The condition, $a < c \leq d < b$, does not appear to have been used. However, it is needed to ensure that the intersection point is in the first quadrant.

14. Let $ABCD$ be a convex quadrilateral, and let R_A, R_B, R_C, R_D denote the circumradii of the triangles DAB, ABC, BCD, CDA respectively. Prove that $R_A + R_C > R_B + R_D$ if and only if $\angle A + \angle C > \angle B + \angle D$.

Solution by Mohammed Aassila, Centre de Recherches Mathématiques, Montréal, Québec.

Let the diagonals meet in E. We may suppose that $\angle AED > 90^\circ$. Then all of $\angle EDA$, $\angle DAE$, $\angle EBC$, $\angle BCE$ are acute. Label them as θ, ϕ, ϕ', θ', respectively. Then $\theta + \phi = \theta' + \phi'$.



Now, if A, B, C and D were concyclic we would have $\phi = \phi'$ and $\theta = \theta'$.

On the other hand, if $\angle A + \angle C > 180^\circ$, then $\theta < \theta'$ and $\phi > \phi'$. This is easily seen by drawing a circumcircle of $\triangle ABC$ and examining the possible locations of B on the extension of DE.

The purpose of isolating acute angles was for a convenient application of the Sine Law. We have

$$R_A = \frac{AB}{2\sin\theta}, \quad R_B = \frac{AB}{2\sin\theta'}, \quad R_C = \frac{DC}{2\sin\phi'}, \quad R_D = \frac{DC}{2\sin\phi}.$$

Thus if $\angle A + \angle C = 180^\circ$ then $\phi' = \phi$ and $\theta = \theta'$ and therefore $R_A + R_C = R_B + R_D$.

More generally,

$$\begin{aligned}\angle A + \angle C > \angle B + \angle D &\iff \angle A + \angle C > 180^\circ \\ &\Longrightarrow \phi' < \phi, \quad \theta < \theta' \\ &\iff R_C > R_D, \quad R_A > R_B \\ &\Longrightarrow R_A + R_C > R_B + R_D.\end{aligned}$$

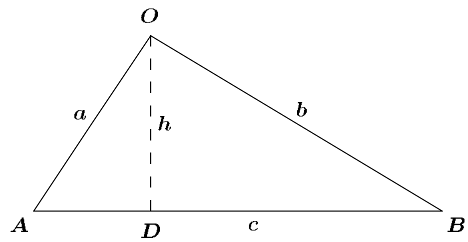
The fact that we have equivalence follows by symmetry.

15. On the plane are given a point O and a polygon $\mathcal{F}$ (not necessarily convex). Let P denote the perimeter of $\mathcal{F}$, D the sum of the distances from O to the vertices of $\mathcal{F}$, and H the sum of the distances from O to the lines containing the sides of $\mathcal{F}$. Prove that $D^2 - H^2 \geq P^2/4$.

Solution by Mohammed Aassila, Centre de Recherches Mathématiques, Montréal, Québec.

We shall prove that $D^2 - H^2 \geq P^2/4$ for any n–gon. We use "induction" on n.

If $n = 2$, the inequality $D^2 - H^2 \geq P^2/4$ is equivalent to $(a+b)^2 - 4h^2 \geq c^2$.



To prove this, consider a line (L) passing through O and parallel to AB, and let B_1 be the image of B under reflection in (L). Then $OA + OB = OA + OB_1 \geq AB_1$; that is $a + b \geq \sqrt{4h^2 + c^2}$, as required.

Now, let the n-gon $\mathcal{F}$ be $P_1P_2 \dots P_n$ and let $d_i = OP_i$, $p_i = P_iP_{i+1}$, h_i = distance from O to P_iP_{i+1}. Then by the "induction hypothesis" we have for each i

$$d_i + d_{i+1} \ \geq \ \sqrt{4h_i^2 + p_i^2}\,.$$

Summing these inequalities over $i = 1, 2, \dots, n$ we obtain

$$2D \ \geq \ \sum_i \sqrt{4h_i^2 + p_i^2}\,,$$

or equivalently

$$4D^2 \ \geq \ \left(\sum_i \sqrt{4h_i^2 + p_i^2}\right)^2 .$$

Now, $4H^2 + P^2 = 4(\sum_i h_i)^2 + (\sum_i p_i)^2$, so if we prove that

$$\sum_i \sqrt{4h_i^2 + p_i^2} \ \geq \ \sqrt{4\left(\sum_i h_i\right)^2 + \left(\sum_i p_i\right)^2}\,,$$

then we are done. But the above inequality follows from the triangle inequality applied to the sum of the vectors $(2h_i, p_i)$.

17. A finite sequence of integers $a_0, a_1, \dots, a_n$ is called *quadratic* if for each i in the set $\{1, 2, \dots, n\}$ we have the equality $|a_i - a_{i-1}| = i^2$.

(a) Prove that for any two integers b and c, there exist a natural number n and a quadratic sequence with $a_0 = b$ and $a_n = c$.

(b) Find the smallest natural number n for which there exists a quadratic sequence with $a_0 = 0$ and $a_n = 1996$.

Solution by Mohammed Aassila, Centre de Recherches Mathématiques, Montréal, Québec.

This problem is well known — it is problem 250 from the book of Sierpinski!!

Next, a bit more tidying up. Last issue we gave solutions by the readers to problems of the 17th Austrian-Polish Mathematics Competition [1998: 4–6; 1999: 77–85]. Next we give a solution to problem 6.

6. Let $n > 1$ be an odd positive integer. Assume that the integers x_1, $x_2, \ldots, x_n \geq 0$ satisfy the system of equations

$$\begin{aligned} (x_2 - x_1)^2 + 2(x_2 + x_1) + 1 &= n^2\,, \\ (x_3 - x_2)^2 + 2(x_3 + x_2) + 1 &= n^2\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (x_1 - x_n)^2 + 2(x_1 + x_n) + 1 &= n^2\,. \end{aligned}$$

Show that either $x_1 = x_n$, or there exists j with $1 \leq j \leq n-1$, such that $x_j = x_{j+1}$.

Solution adapted from one by Pierre Bornsztein, Courdimanche, France.

We show, by induction on m, that, if non-negative integers $x_1, x_2, \ldots$, $x_m \geq 0$, with $m > 1$ being an odd integer, satisfy

$$\begin{aligned} (x_2 - x_1)^2 + 2(x_2 + x_1) + 1 &= n^2\,, \\ (x_3 - x_2)^2 + 2(x_3 + x_1) + 1 &= n^2\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (x_1 - x_m)^2 + 2(x_1 + x_m) + 1 &= n^2\,, \end{aligned}$$

then either $x_1 = x_m$, or there is j with $1 \leq j \leq m-1$, such that $x_j = x_{j+1}$. We adopt the convention that subscripts are read modulo m, so that $x_{m+1} = x_1$, etc..

Now notice that for each j, x_{j-1} and x_{j+1} are solutions (possibly equal) to the quadratic equation

$$X^2 + X(2 - 2x_j) + (x_j + 1)^2 - n^2 \;=\; 0 \quad (E_j)\,,$$

for which the discriminant $\Delta_j = 4(n^2 - 4x_j) \geq 0$ for there to be a real root. Moreover as n is odd, $\Delta_j \geq 1$.

Also because (E_j) has integral coefficients and at least one integral root, both roots are integers, and they are distinct as $\Delta_j \geq 1$. Denote the roots by α_j and β_j with $\alpha_j < \beta_j$.

Also we have $\alpha_j + \beta_j = 2x_j - 2$, and that $\{x_{j-1}, x_{j+1}\} \subset \{\alpha_j, \beta_j\}$. We claim that *either* $\alpha_j = x_j - 2$ and $\beta_j = x_j$ *or* $\alpha_j < x_j < \beta_j$. (Indeed if $\alpha_j, \beta_j \geq x_j$ then $\alpha_j + \beta_j > 2x_j - 2$, and if $\alpha_j, \beta_j < x_j$ then $\alpha_j + \beta_j \leq 2x_j - 3$.)

Now let j be such that $x_j = \max\{x_1, \ldots, x_n\}$. Now x_{j-1} and x_{j+1} are roots of E_j. Unless $x_{j-1} = x_j$ or $x_j = x_{j+1}$ we must have

$$x_{j-1} \;=\; x_{j+1}$$

or

$$x_{j-1} \;<\; x_j \;<\; x_{j+1}$$

or

$$x_{j+1} \;<\; x_j \;<\; x_{j-1}\,.$$

The last two cases contradict the choice of j, so $x_{j-1} = x_{j+1}$. If $m = 3$ we are done since x_{j-1} and x_{j+1} are cyclically adjacent. Otherwise $m > 3$ and by removing x_{j-1}, x_j we obtain a solution with $m - 2$ values, to which the induction hypothesis applies.

Now we turn our attention to solutions from the readers to five of the problems of the Iranian National Mathematical Olympiad, February 6, 1994, Second Round [1998: 6–7].

IRANIAN NATIONAL MATHEMATICAL OLYMPIAD
February 6, 1994
Second Round

1. Suppose that p is a prime number and is greater than 3. Prove that $7^p - 6^p - 1$ is divisible by 43.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Michel Bataille, Rouen, France; by Pierre Bornsztein, Courdimanche, France; by Yeo Keng Hee, Hwa Chong Junior College, Singapore; by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; by Pavlos Maragoudakis, Pireas, Greece; by Bob Prielipp, University of Wisconsin–Oshkosh, Wisconsin, USA; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Klamkin's generalization.

More generally we show that $[(x+1)^n - x^n - 1]/[x^2+x+1]$ is an integral polynomial for all natural odd n not divisible by 3. The given problem corresponds to the special case $x = 6$.

By the Remainder Theorem $(x+1)^n - x^n - 1$ will be divisible by x^2+x+1 if it vanishes for the zeros of the latter, which are ω and ω^2, the two complex cube roots of unity. Here,

$$(\omega+1)^n - \omega^n - 1 \;=\; (-1)^n\omega^{2n} - \omega^n - 1\,.$$

As known, $\omega^{2m}+\omega^m+1 = 0$ if and only if m is not divisible by 3. The same also applies to the other zero ω^2. Also we must have $(-1)^n$ to be 1, so that n is even.

Comment. Similarly, $[(x+1)^n + x^n + 1]$ is divisible by $[x^2+x+1]$ for all natural even n not divisible by 3. As a special case, $7^n + 6^n + 1$ is divisible by 43 for any of these values of n.

3. Let n and r be natural numbers. Find the smallest natural number m satisfying this condition: For each partition of the set $\{1, 2, \ldots, m\}$ into r subsets $A_1, A_2, \ldots, A_r$, there exist two numbers a and b in some A_i $(1 \le i \le r)$ such that $1 < \frac{a}{b} \le 1 + \frac{1}{n}$.

Solution by Yeo Keng Hee, Hwa Chong Junior College, Singapore.

I claim the answer is $(nr+r)$. Suppose $m < nr + r$. Then for each $i \le m$, let i be put into the subset A_j, $(j = 1, 2, \ldots, r)$ such that $i \equiv j \pmod r$. Then for any 2 numbers a, b in the same subset with $a > b$, we have $b \le (a-r)$. Thus

$$a/b \;=\; (1+(a-b)/b) \;\ge\; (1+r/b) \;>\; (1+r/nr) \;=\; (1+1/n)\,.$$

Therefore the condition in the question cannot hold. For $m = (nr + r)$, consider the $(r+1)$ numbers $nr, (nr+1), \ldots, (nr+r)$. By the Pigeonhole Principle, among them there exists $a > b$ such that a and b are in the same subset. Then

$$a/b \;=\; (1 + (a-b)/b) \;\leq\; (1 + r/b) \;\leq\; (1 + r/nr) \;=\; (1 + 1/n)\,,$$

and $1 < a/b$ and we are done.

4. G is a graph with n vertices A_1, A_2, ..., A_n such that for each pair of non-adjacent vertices A_i and A_j there exists another vertex A_k that is adjacent to both A_i and A_j.

(a) Find the minimum number of edges of such a graph.

(b) If $n = 6$ and A_1, A_2, A_3, A_4, A_5, A_6 form a cycle of length 6, find out the number of edges that must be added to this cycle such that the above condition holds.

Solution by Yeo Keng Hee, Hwa Chong Junior College, Singapore.

(a) G must be a connected graph. Thus it must have at least $(n-1)$ edges. If the $(n-1)$ edges are the edges that join A_i, $(i = 1, 2, \ldots, n-1)$ to A_n, the condition is indeed satisfied. Thus the minimum number of edges is $(n-1)$.

(b) Without loss of generality, let A_i be joined to A_{i-1} and A_{i+1}, $(i = 1, 2, \ldots, 6)$, $A_0 = A_6$, $A_7 = A_1$. Then consider the three pairs of vertices (A_1, A_4), (A_2, A_5), (A_3, A_6). In each pair of vertices, the degree of at least one vertex needs to be increased by 1, because any path joining the two vertices has length at least 3, so one vertex must be joined to the other vertex or a vertex adjacent to that vertex.

Thus the sum of degrees of all the vertices must increase by at least 3; that is, at least two edges must be added. Adding the edges A_1A_4 and A_2A_6 satisfies the conditions.

5. Show that if D_1 and D_2 are two skew lines, then there are infinitely many straight lines such that their points have equal distance from D_1 and D_2.

Solutions by Michel Bataille, Rouen, France; and by Murray S. Klamkin, University of Alberta, Edmonton, Alberta. We give Bataille's solution.

Let H_1 on D_1, H_2 on D_2 be such that: $H_1H_2 \perp D_1$ and $H_1H_2 \perp D_2$ and let O be the midpoint of H_1H_2. We will work in the following system of rectangular axes: we take O as origin, the z–axis along H_1H_2 and, as x–axis and y–axis, we take the bisectors of the angle formed at O by the parallels to D_1 and D_2. Then there exist non-zero real numbers a and m such that:

D_1 is the intersection of the planes $(P_1) : z = a$ and $(Q_1) : y = mx$;
D_2 is the intersection of the planes $(P_2) : z = -a$ and $(Q_2) : y = -mx$.

Now, let $A(\alpha, 0, 0)$ be a point lying on the x–axis and Δ_A be the line through A directed by $\vec{u}(0, 1, t)$. We show that it is possible to choose t (depending on α) such that each point $M(\alpha, k, kt)$, $(k \in \mathbb{R})$ of Δ_A has equal distance from D_1 and D_2.

As the planes (P_1) and (Q_1) are perpendicular, we have:

$$[d(M, D_1)]^2 \;=\; [d(M, P_1)]^2 + [d(M, Q_1)]^2 \;=\; (kt - a)^2 + \frac{(k - m\alpha)^2}{1 + m^2}\,.$$

Similarly: $[d(M, D_2)]^2 = (kt + a)^2 + \dfrac{(k + m\alpha)^2}{1 + m^2}$.

Hence, $d(M, D_1) = d(M, D_2)$ is equivalent to $4k\left(at + \dfrac{m\alpha}{1 + m^2}\right) = 0$ so that, by choosing $t = -\dfrac{m\alpha}{a\,(1 + m^2)}$, we have $d(M, D_1) = d(M, D_2)$ for all M on Δ_A.

Since $\Delta_A \neq \Delta_B$ whenever $A \neq B$ on the x–axis (Δ_A and Δ_B are in strictly parallel planes), the family of lines Δ_A, when α takes all real values, answers the question.

6. $f(x)$ and $g(x)$ are polynomials with real coefficients such that for infinitely many rational values x, $\frac{f(x)}{g(x)}$ is rational. Prove that $\frac{f(x)}{g(x)}$ can be written as the ratio of two polynomials with rational coefficients.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.

With little change, a solution of this problem is given in [1]. Without loss of generality we can assume that f and g are relatively prime polynomials and let r denote the sum of the degrees of f and g. Also let $R(x) = f(x)/g(x)$, assuming that the degree of f is $\geq$ than the degree of g. If not, we consider $R^{-1}(x)$. For $r = 0$, the result is obvious. Now let a be one of the rational numbers such that $g(a) \neq 0$ and $R(a)$ is rational. Now define $f_1(x)$ by

$$f_1(x)/g(x) \;=\; \{R(x) - f(a)/g(a)\}/(x - a)\,.$$

Then $f_1(x)/g(x)$ is also rational for all the rational values that $f(x)/g(x)$ is rational except $x = a$. Since

$$f_1(x) \;=\; [g(a)f(x) - f(a)g(x)]/g(a)(x - a)\,,$$

the degree of $f_1(x)$ is less than that of $f(x)$. Thus the sum of the degrees of $f_1(x)$ and $g(x)$ is less than that of $f(x)$ and $g(x)$. The desired result now follows by mathematical induction.

Reference: [1] G. Polya, G. Szego, **Problems and Theorems in Analysis**, Springer-Verlag, II, NY, 1976, pp. 130, 321, Problem 92.

Next we look at readers' solutions to two problems of the Japan Mathematical Olympiad, Final Round, February 1994 given in [1998: 68–69].

1. For a positive integer n, let a_n be the nearest positive integer to $\sqrt{n}$, and let $b_n = n + a_n$. Dropping all b_n $(n = 1, 2, \ldots)$ from the set of all positive integers N, we get a sequence of positive integers in ascending order $\{c_n\}$. Represent c_n by n.

Solution by Pierre Bornsztein, Courdimanche, France.

On a

$$\begin{array}{lll} a_1 = 1, & b_1 = 2 \\ a_2 = 1, & b_2 = 3 \\ a_3 = 2, & b_3 = 5 \\ \text{etc.}. \end{array}$$

La suite $\{a_n\}$ est croissante, donc $\{b_n\}$ est strictement croissante.

Soit $n \in \mathbb{N}^*$. On a $n < \sqrt{n^2+n} < n + \frac{1}{2}$, donc $a_{n^2+n} = n$, et alors $b_{n^2+n} = (n+1)^2 - 1$. En plus $n + \frac{1}{2} < \sqrt{n^2+n+1} < n+1$, donc $a_{n^2+n+1} = n+1$, et alors $b_{n^2+n+1} = (n+1)^2+1$. On en déduit que $\{b_n\}$ ne contient aucun carré.

Pour $n \in \mathbb{N}^*$, la cardinalité de l'ensemble $\{b_1, \ldots, b_{n^2+n}\} = n^2+n$, et la cardinalité de $\{p \in \mathbb{N}^* : p \leq (n+1)^2 - 1$ et p n'est pas un carré$\} = n^2+n$.

De plus,

$$\{b_1, \ldots, b_{n^2+n}\} \subset \{p \in \mathbb{N}^* : p \leq (n+1)^2 - 1\,,\, p \text{ non carré}\}$$

d'où

$$\{b_1, \ldots, b_{n^2+n}\} = \{p \in \mathbb{N}^* : p \leq (n+1)^2 - 1\,,\, p \text{ non carré}\}\,,$$

et comme n est arbitraire

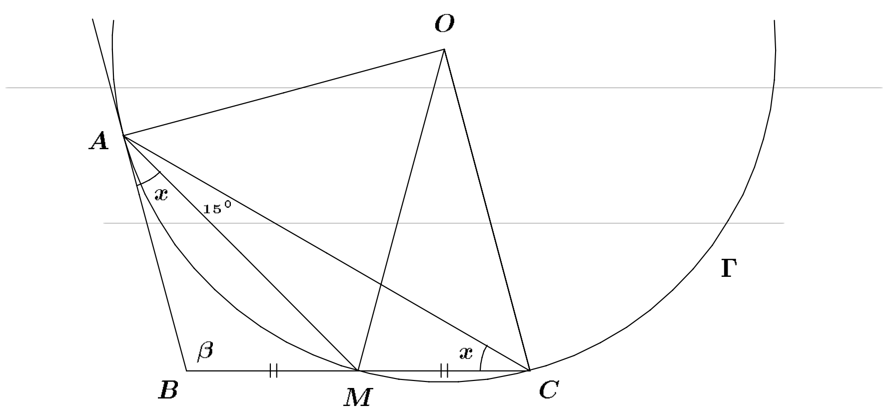
$$\{b_1, b_2, \ldots\} = \mathbb{N}^* \backslash \{1, 4, 9, 16, \ldots\}\,.$$

On en obtient pour $n \geq 1$, $c_n = n^2$.

4. We consider a triangle ABC such that $\angle MAC = 15°$ where M is the midpoint of BC. Determine the possible maximum value of $\angle B$.

Solutions by Pierre Bornsztein, Courdimanche, France; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give the solution of Smeenk.

Let $\Gamma(O, R)$ be the circumcircle of $\triangle AMC$, with centre O and radius R. Denote $\angle B = \beta$. Now β has its maximum when AB touches Γ at A. We denote $\angle BAM = \angle ACM = x$.



Since AM is a median in $\triangle ABC$ we have:

$$\begin{aligned} \sin x : \sin 15^\circ &= \sin\beta : \sin x\,, \\ 2\sin^2 x &= 2\sin\beta\sin 15^\circ\,, \\ 1-\cos 2x &= \cos(\beta-15^\circ)-\cos(\beta+15^\circ)\,. \end{aligned}$$

Since $\beta + 15^\circ + 2x = 180^\circ$, we have $2x = 165^\circ - \beta$. Thus $1+\cos(\beta+15^\circ) = \cos(\beta-15^\circ)-\cos(\beta+15^\circ)$, or

$$2\cos(\beta+15^\circ)-\cos(\beta-15^\circ)+1 \;=\; 0\,. \qquad (1)$$

It is easy to verify that this is satisfied by $\beta = 105^\circ$.

Write $\beta = 105^\circ + y$. From (1) we obtain

$$\begin{aligned} 2\cos(120^\circ+y)-\cos(90^\circ+y)+1 &= 0\,, \\ 2\cos 120^\circ\cos y - 2\sin 120^\circ \sin y + \sin y + 1 &= 0\,, \\ 1-\cos y &= \sin y\left(\sqrt{3}-1\right)\,, \\ 2\sin^2\left(\frac{1}{2}\right)y &= 2\sin\left(\frac{1}{2}y\right)\cos\left(\frac{1}{2}y\right)(\sqrt{3}-1)\,. \end{aligned}$$

Therefore, $\sin\left(\frac{1}{2}y\right) = 0 \Longrightarrow y = 0$ or

$$\begin{aligned} \tan\left(\frac{1}{2}y\right) &= \sqrt{3}-1 \\ \Longrightarrow\quad y &= 72.4\ldots^\circ \\ \Longrightarrow\quad \beta &= 177.4\ldots^\circ\,. \end{aligned}$$

Since $x > 15^\circ$, this does not hold. So β has a maximum of 105°.

That completes the *Corner* for this issue. Do not forget to send me Olympiad contests and your nice solutions to the problems we have given.

BOOK REVIEWS

ALAN LAW

Interdisciplinary Lively Application Projects (ILAPs), edited by David C. Amey, Published by The Mathematical Association of America, USA, 1997, ISBN# 0-88385-706-5, softcover, 222+ pages.
Reviewed by **T. W. Leung**, *Hong Kong Polytechnic University.*

This book is a collection of eight modelling projects (interdisciplinary lively application projects) developed by academics from a consortium of twelve schools led by the United States Military Academy. It is not a book on modelling theories and techniques [1], nor a collection of many modelling cases [3]. Rather, every project is a step-by-step guide to how a situation arises that requires mathematical modelling, about complications involved when trying to set up a model, and means to solve the model. In spirit, the projects are akin to the UMAP modules.

Mathematics backgrounds employed in these projects are calculus, linear algebra and basic statistics. For instance, in the project "Getting Fit with Mathematics", simple regression analysis is used to compare several variables, whereas in "Parachute Panic", the basic ingredient is a first order ordinary differential equation, while in "Flying with Differential Equation", it is in essence a demonstration of oscillation and resonance. Making use of vector calculus, Stokes's theorem and the like, the project "Contaminant Transport" is mathematically the most sophisticated.

Nevertheless because these projects were produced as "interdisciplinary" efforts, new elements were added. We learn about oxygen consumption, ATP, VO_2 and others in "Getting Fit with Mathematics", and it is interesting to observe so many variables playing a part. In the "Decked Out", only simple algebraic manipulations are used to solve the model, but we also learn about Bill of Materials (BOM). In "Parachute Panic", besides the free fall, we see also "severity index" and the interplay of different forces. A novelty of these projects is that they were produced by cooperative efforts from different disciplines, which makes the projects more realistic, instead of merely exercises in mathematics (for comparison, see the review [4]).

The authors take a "no nonsense" approach, analyze the situation, set up a model using simple mathematics, solve the model, freely using computer packages if necessary. The hard part of each project is the formulation of the model, and the problem of verifications of a model is often unanswered. They demonstrated using mathematics as a "tool", and a tool should be simple to use (user-friendly), and serves few special functions, I believe. Time and again I was told by my colleagues from engineering departments that we should teach mathematics as a tool, and that it may help them to solve problems. This is a point often neglected by mathematicians, and our sense of importance in mathematics may differ from the engineers or the scientists. Concerning using computer packages, it seems the authors concede that students may not understand the theories involved, and it is OK. A similar view was echoed in [2].

Another point seemingly to be advocated by the authors is that students may learn the necessary mathematics during the modelling process. This is in line with the recent movement of curriculum reform (see p.205, also for example [5]). In these days the students' backgrounds are so weak and varied, perhaps it is better that students take basic courses in calculus and linear algebra first, before entering into this kind of endeavour.

In short it is nice to try these projects in a modelling course, asking students to look for variations and verifications. Therefore this book serves as a good reference for investigation.

References

1. D.E. Edwards and M. Hamson, *Guide to Mathematical Modelling*, Macmillan Education Limited, London, 1989.
2. M.S. Jones, Teaching Mathematical Modelling, Int. J. Math. Edn. in Sci. & Tech., Vol. 28, # 4, July/August, 1997, pp. 553–560.
3. M.S. Klamkin (Editor), *Mathematical Modelling, Classroom Notes in Applied Mathematics*, SIAM, 1987.
4. D. Lawson, Review of *Industrial Mathematics: A Course in Solving Real World Problems*, Bull. IMA, Vol. 31, # 7/8, July/August, 1995, pp. 124–125.
5. D. O'Shea and H. Pollatsek, Do we need prerequisites? Notices of the AMS, Vol. 44, # 5, 1997, pp. 564–570.

Euler's Triangle Theorem

G.C. Shephard

1. In 1780, Leonhard Euler proved the following remarkable result [4, p. 96]:

Theorem: Let $[A, B, C]$ be an arbitrary triangle and O any point of the plane which does not lie on a side of the triangle. Let AO, BO, CO meet BC, CA and AB in the points D, E, F respectively. Then

$$\frac{||AO||}{||OD||} + \frac{||BO||}{||OE||} + \frac{||CO||}{||OF||} = \frac{||AO||}{||OD||} \cdot \frac{||BO||}{||OE||} \cdot \frac{||CO||}{||OF||} + 2\,. \qquad (1)$$

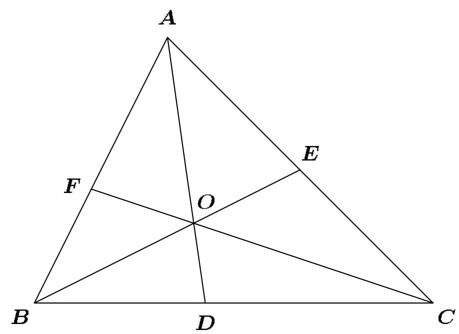


Figure 1

To begin with we shall assume, like Euler, that O lies in the interior of the triangle, see Figure 1. For the present, the notation $\frac{||XY||}{||YZ||}$ will simply indicate the ratio of the lengths of the indicated segments. We describe this theorem as remarkable not only because of its early date, but also because it connects the value of a cyclic sum (on the left side of (1)) with the value of a cyclic product (on the right side of (1)). There exist a great number of results concerning cyclic products for triangles and n-gons, the best known of which are the theorems of Ceva and Menelaus. Many more such theorems are given in [6] and [7]. On the other hand there are few papers on cyclic sums, such as [5] and [8]. Apart from [1], Euler's paper [4] is, so far as we are aware, the only one which related these two concepts.

Euler's proof of his theorem is by algebra and trigonometry; he calculates the ratios of the lengths of the line segments in (1) using trigonometrical formulae involving the sines of the angles between the lines at O. The proof takes two and a half pages, and whilst we do not wish to criticize the work of one of the most illustrious of mathematicians, it is worth considering the shortcomings of his proof. Apart from its length and complexity, the proof gives no insight as to *why* relation (1) is true, and therefore does nothing to suggest the existence of other relations of a similar nature. The following approach seems to overcome these criticisms; it depends on what has been called the "area principle" in [6] and the "area method" in [2].

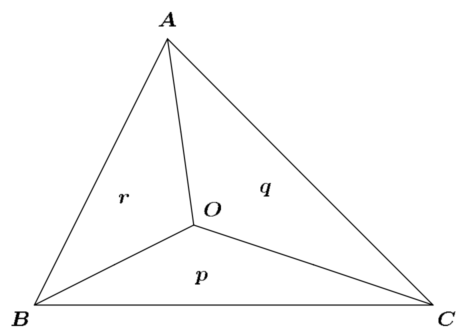


Figure 2

2. Let $p = ||COB||$, $q = ||AOC||$, $r = ||BOA||$ be the areas of the indicated triangles (see Figure 2). Then

$$\frac{||BD||}{||DC||} = \frac{||BOA||}{||AOC||} = \frac{r}{q}.$$

The first equality follows since the triangles $[B, O, A]$ and $[A, O, C]$ have the same base $[A, O]$ and their heights are proportional to $||BD|| : ||DC||$. By cyclic permutation of the letters,

$$\frac{||BD||}{||DC||} = \frac{r}{q}, \quad \frac{||CE||}{||EA||} = \frac{p}{r}, \quad \frac{||AF||}{||FB||} = \frac{q}{p}. \tag{2}$$

Also $\frac{||AD||}{||OD||} = \frac{||ABC||}{||OBC||} = \frac{p+q+r}{p}$. The first equality follows since the triangles $[A, B, C]$ and $[O, B, C]$ have the same base $[B, C]$ and their heights are proportional to $||AD|| : ||OD||$. Thus we obtain three more relations

$$\frac{||AD||}{||OD||} = \frac{p+q+r}{p}, \quad \frac{||BE||}{||OE||} = \frac{p+q+r}{q}, \quad \frac{||CF||}{||OF||} = \frac{p+q+r}{r}. \tag{3}$$

Finally,

$$\frac{||AO||}{||OD||} = \frac{||AD|| - ||OD||}{||OD||} = \frac{||AD||}{||OD||} - 1 = \frac{p+q+r}{p} - 1 = \frac{q+r}{p}$$

leading in a similar fashion to the three relations

$$\frac{||AO||}{||OD||} = \frac{q+r}{p}, \quad \frac{||BO||}{||OE||} = \frac{r+p}{q}, \quad \frac{||CO||}{||OF||} = \frac{p+q}{r}. \tag{4}$$

Relations (4) can also be proved directly (without using (3)) by the area principle. For our proof of (1) we only need relations (4):

$$\begin{aligned}
\frac{||AO||}{||OD||}\cdot\frac{||BO||}{||OE||}\cdot\frac{||CO||}{||OF||} &= \frac{q+r}{p}\cdot\frac{r+p}{q}\cdot\frac{p+q}{r} \\
&= \frac{q^2r+r^2q+r^2p+p^2r+p^2q+q^2p+2pqr}{pqr} \\
&= \frac{q+r}{p}+\frac{r+p}{q}+\frac{p+q}{r}+2 \\
&= \frac{||AO||}{||OD||}+\frac{||BO||}{||OE||}+\frac{||CO||}{||OF||}+2\,,
\end{aligned}$$

as required.

3. We can exploit (2), (3) and (4) to yield many more relations between the ratios of lengths in the triangle. The first two of these are well known.

$$1 \;=\; \frac{r}{q}\cdot\frac{q}{p}\cdot\frac{p}{r} \;=\; \frac{||BD||}{||DC||}\cdot\frac{||AF||}{||FB||}\cdot\frac{||CE||}{||EA||}\,,$$

which is Ceva's Theorem for the triangle. Also, from (3),

$$\frac{||OD||}{||AD||} \;=\; \frac{p}{p+q+r}\,,\quad \frac{||OE||}{||BE||} \;=\; \frac{q}{p+q+r}\,,\quad \frac{||OF||}{||CF||} \;=\; \frac{r}{p+q+r}\,,$$

and hence

$$\frac{||OD||}{||AD||}+\frac{||OE||}{||BE||}+\frac{||OF||}{||CF||} \;=\; \frac{p+q+r}{p+q+r} \;=\; 1\,, \qquad (5)$$

which is mentioned in [5] and [8]. Euler also gives a proof of this result [4, p. 102].

Almost as easy is

$$\frac{||AO||}{||OD||} \;=\; \frac{q+r}{p} \;=\; \frac{q}{p}+\frac{r}{p} \;=\; \frac{||AF||}{||FB||}+\frac{||AE||}{||EC||}\,,$$

which is not trivial, as one finds if one tries to prove it directly by calculating the lengths of the line segments by trigonometry and not using the area principle.

Also we observe:

$$\frac{||AO||}{||OD||}\cdot\frac{||CE||}{||EA||} \;=\; \frac{q+r}{p}\cdot\frac{p}{r} \;=\; 1+\frac{q}{r} \;=\; 1+\frac{||CD||}{||DB||}\,.$$

Another example is:

$$\begin{aligned}
\frac{||AO||}{||OD||}\cdot\frac{||BO||}{||OE||} &= \frac{(q+r)(r+p)}{pq} = 1+\frac{r(p+q+r)}{pq} \\
&= 1+\frac{||AD||}{||OD||}\cdot\frac{||BD||}{||DC||} \;=\; 1+\frac{||BE||}{||OE||}\cdot\frac{||AE||}{||EC||}\,.
\end{aligned}$$

In fact, every algebraic identity in the variables p, q, r which is homogeneous of degree zero, leads, in at least one way, to an identity between ratios of lengths in the triangle.

4. However, this is not all. We supposed that O was in the interior of the triangle, but this restriction is not necessary; we need only assume that O does not lie on a side (or side extended) of the triangle. Euler does not discuss this case, but the result is true since the above proof still holds with appropriate interpretation.

Firstly, we must use signed or directed lengths; this is the meaning of the doubled modulus lines. We assign a positive direction to every line in the plane and then $||XY||$ is positive if $\overrightarrow{XY}$ lies in the assigned direction and is negative otherwise. Thus $||YX|| = -||XY||$ and if X, Y, Z are distinct collinear points then $\dfrac{||XY||}{||YZ||}$ is positive if and only if Y lies between X and Z. Secondly, we must use signed areas, which means that the area $||XYZ||$ of a triangle $[X, Y, Z]$ is taken to be positive if the vertices are named in a counterclockwise and negative if the vertices are named in a clockwise direction. All of the equations in Sections 2 and 3 have been written in such a way that, with these sign conventions, they hold regardless of the relative positions of the points. In particular they remain true whether the point O is in the interior of the triangle or not.

5. The primary purpose of Euler's paper was not to prove the "remarkable theorem" discussed above, but to solve the following problem:

Given positive numbers x, y, z, is it possible to construct a triangle $[A, B, C]$ so that, with the notation of Figure 1,

$$x = \frac{||AO||}{||OD||}, \; y = \frac{||BO||}{||OE||}, \; z = \frac{||CO||}{||OF||}?$$

The theorem shows that this is impossible unless $xyz = x + y + z + 2$, but the question remains as to whether this condition is sufficient. Euler proves that it is so, but again his proof takes two and a half pages of algebraic and trigonometrical calculation.

Our approach, using the area principle, also gives an easy solution to this problem. If a triangle satisfying the given condition exists, then by (4),

$$x = \frac{q+r}{p}, \; y = \frac{r+p}{q}, \; z = \frac{p+q}{r},$$

from which we obtain,

$$p = \frac{\triangle}{x+1}, \; q = \frac{\triangle}{y+1}, \; r = \frac{\triangle}{z+1},$$

where $\triangle = p + q + r$ is the total area of the triangle, and this may be chosen arbitrarily.

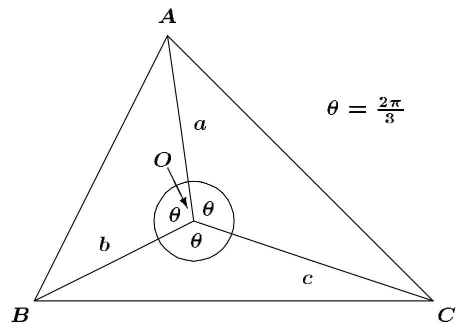


Figure 3

Now consider the triangle shown in Figure 3. Here the three line segments $[AO]$, $[BO]$, $[CO]$ are equally inclined at angles $\dfrac{2\pi}{3}$ to each other, and the lengths of the segments are denoted by a, b, c respectively. Then, by the well-known formula for the area of a triangle,

$$p = bc\sin\frac{2\pi}{3} = \frac{\sqrt{3}bc}{2}, \quad q = \frac{\sqrt{3}ca}{2}, \quad r = \frac{\sqrt{3}ab}{2},$$

and so we can solve for a, b, c in terms of the areas of the triangles p, q, r obtained above. Clearly,

$$a^2 = \frac{2qr}{\sqrt{3}p}, \quad b^2 = \frac{2rp}{\sqrt{3}q}, \quad c^2 = \frac{2pq}{\sqrt{3}r},$$

from which the (positive) values of a, b, c can be determined. With these values the triangle in Figure 3 has the required properties.

The reader may ask why we were justified in taking the lines OA, OB and OC as equally inclined to each other. The reason is that the theorem, our proof, and the problem are *affine-invariant*. This means that, in particular, if any triangle can be found which solves the problem, then the result of applying a non-singular affine transformation (that is, a change of scale possibly followed by a shear) yields another triangle which is also a solution to the problem. It is well known that an affine transformation can be found which will transform any three intersecting lines into three such lines which are equally inclined (at angle $\frac{2\pi}{3}$) to each other.

6. The study of early mathematical papers is fraught with difficulty. Sometimes the notation is strange and the language is often obscure. In some cases (as with Euler's paper) it is well worth the effort to overcome these difficulties.

The study of such papers also gives an insight into the knowledge and thought processes of early authors. Clearly Euler, in 1780, did not know of the area principle, though this was certainly known to A.L. Crelle in 1816 since he uses it in his book [3]. Euler knew about affine transformations, for he mentions them in his *Introd. in Analysin Infinitorum II* of 1748. It is therefore surprising that he did not use them to simplify his solution of the problem stated in Section 5. Perhaps he was not aware that it is possible, using affine transformations, to convert any three concurrent lines into three such lines which are equally inclined to each other.

These trivial observations do not, of course, detract from the fact that Euler's contributions to very many branches of mathematics, are both profound and extensive.

Acknowledgements: My thanks are due to Miss Lucy Parker for her help in the translation of Euler's paper and to the referees for their helpful comments and suggestions.

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G.C. Shephard
School of Mathematics
University of East Anglia
Norwich NR4 7TJ
England, U.K.
g.shephard@uea.ac.uk

THE SKOLIAD CORNER

No. 37

R.E. Woodrow

We begin this number with the problems of Part I of the Alberta High School Prize Exam, written November 1998. My thanks go to the organizing committee, chaired by Ted Lewis, University of Alberta, Edmonton, for supplying the contest and its problems.

THE ALBERTA HIGH SCHOOL MATHEMATICS COMPETITION
Part I — November 1998

1. A restaurant usually sells its bottles of wine for 100% more than it pays for them. Recently it managed to buy some bottles of its most popular wine for half of what it usually pays for them, but still charged its customers what it would normally charge. For these bottles of wine, the selling price was what percent more than the purchase price?

(a) 50 (b) 200 (c) 300 (d) 400 (e) not enough information

2. How many integer solutions n are there to the inequality $34n \geq n^2 + 289$?

(a) 0 (b) 1 (c) 2 (d) 3 (e) more than 3

3. A university evaluates five magazines. Last year, the rankings were MacLuck with a rating of 150, followed by MacLock with 120, MacLick with 100, MacLeck with 90, and MacLack with 80. This year, the ratings of these magazines are down 50%, 40%, 20%, 10% and 5% respectively. How does the ranking change for MacLeck?

(a) up 3 places (b) up 2 places (c) up 1 place (d) unchanged (e) down 1 place

4. Parallel lines are drawn on a rectangular piece of paper. The paper is then cut along each of the lines, forming n identical rectangular strips. If the strips have the same length to width ratio as the original, what is this ratio?

(a) $\sqrt{n} : 1$ (b) $n : 1$ (c) $n : \frac{\sqrt{5}+1}{2}$ (d) $n : 2$ (e) $n^2 : 1$

5. "The smallest integer which is at least $a\%$ of 20 is 10." For how many integers a is this statement true?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

6. Let $S = 1 + 2 + 3 + \cdots + 10^n$. How many factors of 2 appear in the prime factorization of S?

(a) 0 (b) 1 (c) $n-1$ (d) n (e) $n+1$

7. When $1 + x + x^2 + x^3 + x^4 + x^5$ is factored as far as possible into polynomials with integral coefficients, what is the number of such factors, not counting trivial factors consisting of the constant polynomial 1?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

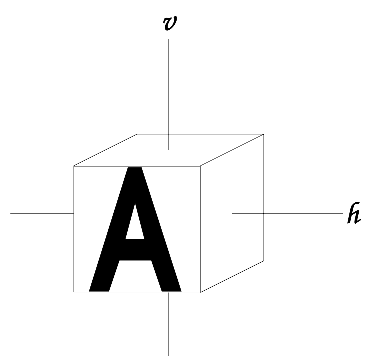
8. In triangle ABC, $AB = AC$. The perpendicular bisector of AB passes through the midpoint of BC. If the length of AC is $10\sqrt{2}$ cm, what is the area of ABC in cm^2 ?

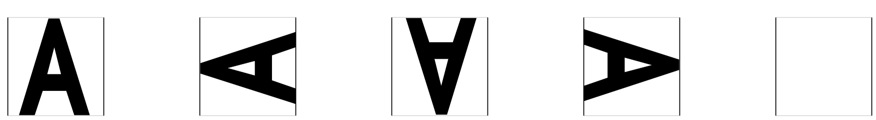
(a) 25 (b) $25\sqrt{2}$ (c) 50 (d) $50\sqrt{2}$ (e) none of these

9. If $f(x) = x^x$, what is $f(f(x))$ equal to?

(a) x^{2x} (b) $x^{(x^2)}$ (c) $x^{(x^x)}$ (d) $x^{(x^{(x+1)})}$ (e) $x^{(x^{(x^x)})}$

10. A certain TV station has a logo which is a rotating cube in which one face has an A on it and the other five faces are blank. Originally the A-face is at the front of the cube as shown on the right. Then you perform the following sequence of three moves over and over: rotate the cube 90° around the vertical axis v, so that the front face moves to the left; then rotate the cube 90° around a horizontal axis h, so that the new front face moves down; then rotate the cube 90° around the vertical axis again, so that the new front face moves to the left. Suppose you perform this sequence of three moves a total of 1998 times. What will the front face look like when you have finished?



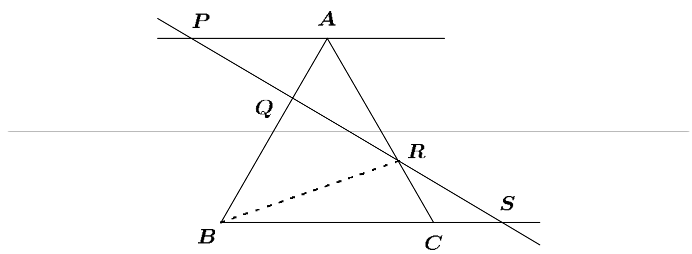


(a) (b) (c) (d) (e)

11. How many triples (x, y, z) of real numbers satisfy the simultaneous equations $x + y = 2$ and $xy - z^2 = 1$?

(a) 1 (b) 2 (c) 3 (d) 4 (e) infinitely many

12. In the diagram, ABC is an equilateral triangle of side length 3 and PA is parallel to BS. If $PQ = QR = RS$, what is the length of BR?



(a) $\sqrt{6}$ (b) $\sqrt{6}$ (c) $\frac{3\sqrt{3}}{2}$ (d) $\sqrt{7}$ (e) none of these

13. Let a, b, c and d be the roots of $x^4 - 8x^3 - 21x^2 + 148x - 160 = 0$. What is the value of $\dfrac{1}{abc} + \dfrac{1}{abd} + \dfrac{1}{acd} + \dfrac{1}{bcd}$?

(a) $-\frac{4}{37}$ (b) $-\frac{1}{20}$ (c) $\frac{1}{20}$ (d) $\frac{4}{37}$ (e) none of these

14. Wei writes down, in order of size, all positive integers b with the property that b and 2^b end in the same digit when they are written in base 10. What is the 1998^{th} number in Wei's list?

(a) 19974 (b) 19976 (c) 19994 (d) 19996 (e) none of these

15. Suppose that $x = 3^{1998}$. How many integers are there between $\sqrt{x^2 + 2x + 4}$ and $\sqrt{4x^2 + 2x + 1}$?

(a) $3^{1998} - 2$ (b) $3^{1998} - 1$ (c) 3^{1998} (d) $3^{1998} + 1$ (e) $3^{1998} + 2$

16. The lengths of all three sides of a right triangle are positive integers. The area of the triangle is 120. What is the length of the hypotenuse?

(a) 13 (b) 17 (c) 26 (d) 34 (e) none of these

Last number we gave the problems of the Mini demi-finale 1996 of the Olympiade Mathematique Belge. Somehow the last three problems were overlooked.

28. Lors d'un championnat de poursuite, deux cyclistes partent en même temps de deux points diamétralement opposés d'un vélodrome de 250 m de tours. Le vainqueur a rattrapé son rival après avoir parcouru 8 tours. Quel est le rapport de la vitesse moyenne du vainqueur à celle du perdant ?

(a) $\frac{9}{8}$ (b) $\frac{8}{7}$ (c) $\frac{17}{16}$ (d) $\frac{16}{15}$ (e) $\frac{7}{8}$

29. Il était une fois ... deux fontaines et une citerne. La première fontaine mettait un jour entier à remplir la citerne, la seconde quatre. En combien de temps la citerne peut-elle être remplie par les deux fontaines coulant ensemble ?

(a) 1 h 15 min **(b)** 8 h **(c)** 18 h **(d)** 19 h 12 min **(e)** 30 h

30. Laquelle des propositions suivantes est vraie dans tout carré ?

(a) La longueur d'un côté est égale à la moitié de la longueur d'une diagonale.
(b) La longueur d'un côté est égale à la racine carrée du périmètre.
(c) La longueur d'un côté est supérieure aux deux tiers de la longueur d'une diagonale.
(d) La longueur d'un côté est inférieure aux deux tiers de la longueur d'une diagonale.
(e) La longueur d'un côté est égale à la longueur d'une diagonale multipliée par $\sqrt{2}$.

And now the answers.

1.	101	2.	e	3.	c	4.	e	5.	c
6.	c	7.	b	8.	64	9.	b	10.	e
11.	e	12.	b	13.	b	14.	c	15.	a
16.	c	17.	d	18.	d	19.	e	20.	315
21.	d	22.	c	23.	d	24.	7	25.	b
26.	e	27.	b	28.	d	29.	d	30.	c

That completes the *Skoliad Corner* for this issue. Send me your suitable contest materials, as well as suggestions for features for this *Corner*.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section (except for the problems section, which have their own editors) should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Haven, CT 06520–8283 USA. The electronic address is still

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Toronto). The rest of the staff consists of Adrian Chan (Upper Canada College), Jimmy Chui (Earl Haig Secondary School), David Savitt (Harvard University) and Wai Ling Yee (University of Waterloo).

Mayhem Problems

The Mayhem Problems editors are:

Adrian Chan	*Mayhem High School Problems Editor,*
Donny Cheung	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the new schedule of eight issues per year, we request that solutions from this issue be submitted in time for issue 4 of 2000.

High School Problems

Editor: Adrian Chan, 229 Old Yonge Street, Toronto, Ontario, Canada. M2P 1R5 <all238@ipoline.com>

H253. Find all real solutions to the equation

$$\sqrt{3x^2-18x+52}+\sqrt{2x^2-12x+162} \;=\; \sqrt{-x^2+6x+160}\,.$$

H254. *Proposed by Alexandre Trichtchenko, 1st year, Carleton University.*

Let p and q be relatively prime positive integers, and n a multiple of pq. Find all ordered pairs (a, b) of non-negative integers that satisfy the diophantine equation $n = ap + bq$.

H255. We have a set of tiles which contains an infinite number of regular n-gons, for each $n = 3, 4, \ldots$. Which subsets of tiles can be chosen, so that they fit around a common vertex? For example, we can choose four squares, or four triangles and a hexagon.

H256. Let $A = 2^a p_1^b p_2^c$, where p_1 and p_2 are primes, possibly equal to each other and to 2, and a, b, and c are positive integers. It is known that $p_1 \equiv p_2 \pmod 4$, $b \equiv c \pmod 2$, and that 2^a, p_1^b, p_2^c are three consecutive terms of an arithmetic sequence, not necessarily in that order. Find all possible values for A.

Advanced Problems

Editor: Donny Cheung, c/o Conrad Grebel College, University of Waterloo, Waterloo, Ontario, Canada. N2L 3G6 <dccheung@uwaterloo.ca>

A229. In tetrahedron $SABC$, the medians of the faces SAB, SBC, and SCA, taken from the vertex S, make equal angles with the edges that they lead to. Prove that $|SA| = |SB| = |SC|$.

(Polish Olympiad)

A230. *Proposed by Naoki Sato.*

For non-negative integers n and k, let $P_{n,k}(x)$ denote the rational function

$$\frac{(x^n-1)(x^n-x)\cdots(x^n-x^{k-1})}{(x^k-1)(x^k-x)\cdots(x^k-x^{k-1})}.$$

Show that $P_{n,k}(x)$ is actually a polynomial for all n, k.

A231. *Proposed by Mohammed Aassila, Centre de Recherches Mathématiques, Montréal, Québec.*

For the sides of a triangle a, b, and c, prove that

$$\frac{13}{27} \le \frac{(a+b+c)(a^2+b^2+c^2)+4abc}{(a+b+c)^3} \le \frac{1}{2}.$$

A232. Five distinct points A, B, C, D, and E lie on a line (in this order) and $|AB| = |BC| = |CD| = |DE|$. The point F lies outside the line. Let G be the circumcentre of triangle ADF and H the circumcentre of triangle BEF. Show that the lines GH and FC are perpendicular.

(1997 Baltic Way)

Challenge Board Problems

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C85. *Proposed by Christopher Long, graduate student, Rutgers University.* (*From a set of course notes on analytic number theory.*)

Let $C = (c_{i,j})$ be an $m \times n$ matrix with complex entries, and let D be a real number. Show that the following statements are equivalent:

(i) For any complex numbers a_j, $j = 1, \ldots, n$,

$$\sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_j c_{i,j} \right|^2 \leq D \sum_{j=1}^{n} |a_j|^2 .$$

(ii) For any complex numbers b_i, $i = 1, \ldots, m$,

$$\sum_{j=1}^{n} \left| \sum_{i=1}^{m} b_i c_{i,j} \right|^2 \leq D \sum_{i=1}^{m} |b_i|^2 .$$

C86. Let K_n denote the complete graph on n vertices; that is, the graph on n vertices with all possible edges present. Show that K_n can be decomposed into $n-1$ disjoint paths of length $1, 2, \ldots, n-1$. For example, for $n = 4$, the graph K_4, with vertices A, B, C, and D, decomposes into the paths $\{AC\}$, $\{BD, DA\}$, and $\{AB, BC, CD\}$.

Can we require that the paths in the decomposition be simple? (We say that a path is simple if it passes through each vertex at most once; that is, if no vertex is the end-point of more than two edges along the path.)

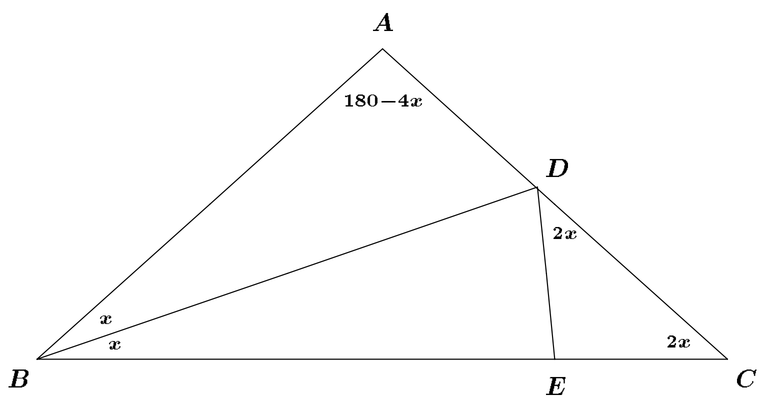
Problem of the Month

Jimmy Chui, student, Earl Haig S.S.

Problem. Let ABC be an isosceles triangle with $AB = AC$. Suppose that the angle bisector of $\angle B$ meets AC at D and that $BC = BD + AD$. Determine $\angle A$.

(1996 CMO, Problem 4)

Solution.



Construct E on BC such that $DE = EC$. Let $x = \angle ABD = \angle DBC$. Then, $\angle DCE = \angle ACB = 2x$. Hence, $\angle CDE = 2x$, and further $\angle ADE = 180° - 2x$. Thus, quadrilateral $ABED$ is cyclic.

Drawing AE, we see that $\angle DAE = \angle DBE = x$, and $\angle DEA = \angle DBA = x$. Therefore, triangle ADE is isosceles, and so $AD = DE$.

Now, $BC = BD + AD = BD + DE = BD + EC$. Also, $BC = BE + EC$. So, $BD + EC = BE + EC$. Thus, $BD = BE$.

Now, $\angle BED = 180° - \angle DEC = 4x$, and $\angle BDE = 180° - 5x$, so $4x = 180° - 5x$.

Solving for x, we obtain $x = 20°$.

Thus, $\angle A = 180° - 4x = 100°$.

Self-Centred Triangles

Cyrus Hsia
student, University of Toronto

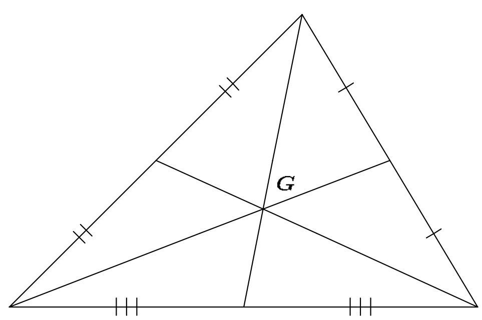
There are four famous triangular centres that all students of classical geometry should be familiar with. Readers already familiar with these should go on to the next section to see how the information on one will be useful to another. Otherwise, the centres are as follows:

Central Definitions

The following definitions all pertain to an arbitrary triangle in Euclidean space.

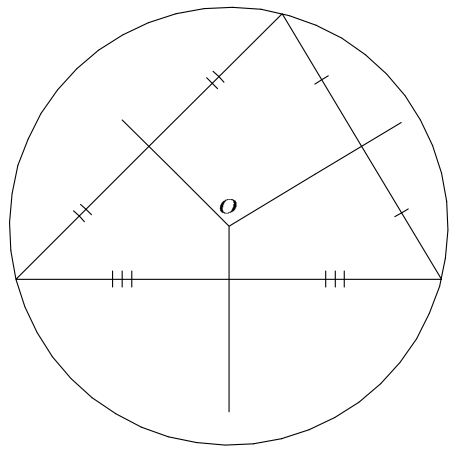
Centroid

The centroid, denoted by G, is the point of intersection of the medians of a triangle. A median is a line from a vertex of a triangle to the mid-point of the opposite side.



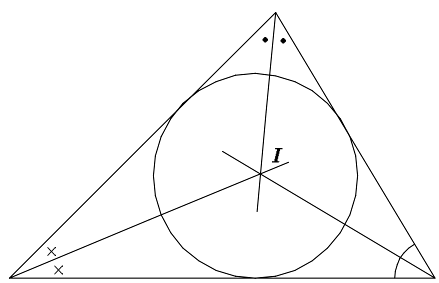
Circumcentre

The circumcentre, denoted by O, is the point of intersection of the right bisectors of the three sides. A right bisector of a line segment is a line perpendicular to it and divides it into two equal segments. It should be noted that O is also the center of the circle that passes through the vertices of the triangle. This circle is called the circumcircle and hence O is referred to as the circumcentre. The confirmation that these two definitions are equivalent is left to the reader.



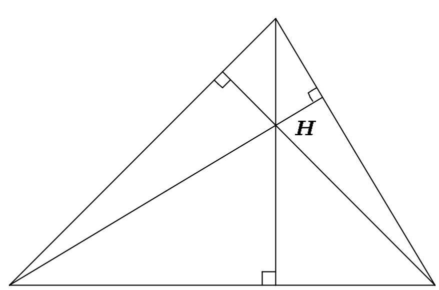
Incentre

The incentre, denoted by I, is the point of intersection of the three angle bisectors of the triangle. An angle bisector is a line which divides a given angle into two equal angles. It should be noted that in this case, I is the centre of the circle inside the triangle tangent to all three sides. These definitions are equivalent and we leave the work to the reader.



Orthocentre

The orthocentre, denoted by H, is the point of intersection of the three altitudes of the triangle. An altitude is a line through a vertex that is perpendicular to the side opposite this vertex.



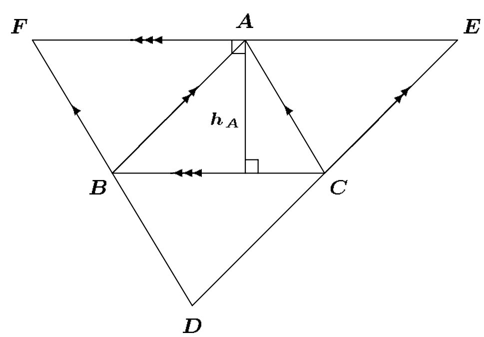
The Central Problem

Although it was taken for granted in the definitions, it is not trivial to show that the points above exist. After all, why do three line segments have to intersect at a common point? We say the lines concur if they do. In fact, draw three lines from the vertices of a triangle to their opposite sides. These lines are called *cevians*. It is easy to see that these cevians need not concur.

There are many ways to show that the points described above exist and to show that certain cevians always concur for any triangle (See [1] and [2]). Here we will consider something of purely mathematical interest. There is a certain similarity between the definitions of the four centres and so it may be possible that the existence of one centre is enough to prove the existence of another. For example, if it has been shown that the centroid exists, then is it possible to show that the incentre exists from this fact? We present two such scenarios here.

Circumcentre implies Orthocentre

Given triangle ABC, let h_A, h_B, h_C be the altitudes from vertices A, B, C respectively. We wish to show that these three altitudes (cevians) concur. At vertex A, draw a line perpendicular to h_A and similarly draw lines for the other two sides as shown.



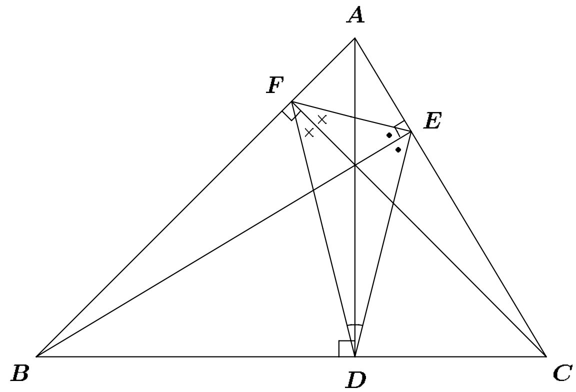
Let the lines intersect at points D, E, F opposite A, B, C respectively as shown. Now FE is perpendicular to h_A and BC is also perpendicular to h_A so FE and BC are parallel. Likewise, FD and AC are parallel and ED and AB are parallel.

Thus $FA = BC$, since $FACB$ is a parallelogram, and $AE = BC$, since $AECB$ is also a parallelogram. Thus $FA = AE$, so A is the mid-point of FE and h_A is perpendicular to FE. In other words, h_A is the perpendicular bisector of FE. Likewise, h_B is the perpendicular bisector of FD and h_C is that of DE.

Now we see that if the circumcentre of a triangle exists, in this case h_A, h_B, and h_C of triangle DEF concur, means that the altitudes of triangle ABC concur. Thus the orthocentre exists by definition.

Incentre implies Orthocentre

Given a triangle ABC, let the altitudes from vertices A, B, C, be h_A, h_B, h_C respectively. Let the feet of the altitudes be D, E, F as shown.



Let α, β, γ be the angles of triangle ABC at vertices A, B, C respectively. Now $\angle FEB = \angle FCB$ since $FECB$ is a cyclic quadrilateral. $\angle FEB = 90° - \beta$. Also, $\angle BED = \angle BAD = 90° - \beta$. In other words, $h_B = BE$ is the angle bisector of $\angle FED$. Likewise, h_A and h_C are angle bisectors of vertices D and F respectively in triangle FED.

As the reader has suspected, if the incentre exists, the three angle bisectors h_A, h_B, h_C concur in triangle DEF, and since these are the altitudes of triangle ABC, then the orthocentre exists.

Epilogue

Here, we have shown that the existence of the incentre or the circumcentre implies the existence of the orthocentre. These and the other implications are listed in the table below. We welcome the reader to explore the other possibilities and to send us feedback on any results they find. A further exploration, for the brave at heart, would be to investigate other famous centres of triangles as well.

implies	centroid	circumcentre	incentre	orthocentre
centroid	clear			
circumcentre		clear		proved
incentre			clear	proved
orthocentre				clear

Exercises

1. Provide a proof that the existence of the orthocentre implies the existence of the incentre and the circumcentre by a similar method to the ones given.
2. Are there other proofs to the implications above?

3. For exploration, are other well-known centres of triangles related as nicely to the orthocentre, circumcentre and incentre? Try the following:

4. (a) The nine-point circle of a triangle is defined as the circle through the three mid-points of the three sides, the three feet of the altitudes, and the three mid-points of the line segments joining the vertices to the orthocentre. The centre of the nine-point circle is denoted by the letter N. By definition, the existence of N depends on the existence of the orthocentre. How about the other centres?

   (b) The Fermat point, F, of a triangle is the point in a triangle that minimizes the sum of the lengths from this point to each of the three vertices. Is it possible to determine the existence of this point by the existence of the other centres?

Acknowledgements

The proof of the first result was modified from the lecture notes in MAT325, by Professor J.W. Lorimer, University of Toronto, Toronto, Ontario, and was the inspiration for this article.

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J.I.R. McKnight Problems Contest 1988

TIME: $2\frac{1}{2}$ HOURS

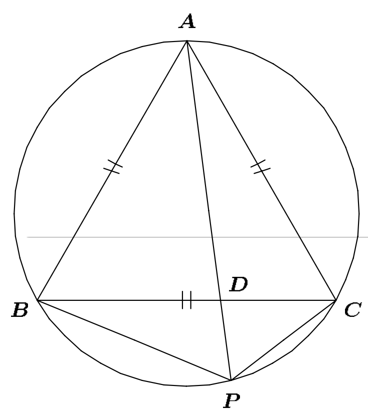
1. Calculators are permitted.
2. Complete solutions are necessary to all problems for full marks.
3. Do all questions in Part A and Part B.

PART A
(6 questions × 5 marks = 30 marks)

1. Find all ordered pairs of real numbers (x, y) such that $\log(\frac{x^2}{y^3}) = 1$ and $\log(x^2y^3) = 7$.
2. (a) Determine the difference between the number of zeroes at the end of each of the numbers $(10^3)!$ and $(10^3)^{100}$.
   (b) Determine n, if $n!$ ends in 17 zeroes.
3. A particle's position at time t seconds, from a point P, is given in metres (s is distance) by
$$s = t^3 - 3t^2 + 4\,, \qquad t \geq 0\,.$$
   Find the time(s) when the particle is speeding up.
4. Determine all integer solutions to the following system of equations:
$$\begin{aligned} a + b + c &= 0\,, \\ ab + ac + bc &= -19\,, \\ abc &= -30\,. \end{aligned}$$
5. In triangle ABC, the angle at B is obtuse and $AB > BC$. An angle bisector of an exterior angle at A meets CB at D, and an angle bisector of an exterior angle at B meets AC at E. If $AD = AB = BE$, find $\angle BAC$.
6. Find the measure of the acute angle θ for which it is true that
$$\left(\frac{16}{81}\right)^{\sin^2\theta} + \left(\frac{16}{81}\right)^{\cos^2\theta} = \frac{26}{27}\,.$$

PART B
(5 questions × 10 marks = 50 marks)

1. Find the sum of the first 100 terms of the following double arithmetic series:
$$1 \times 4 + 5 \times 7 + 9 \times 10 + 13 \times 13 + 17 \times 16 + \cdots\,.$$

2. 

Points A, B, C, and P are located on a circle as shown. Prove that if triangle ABC is equilateral and AP intersects BC at D, then

$$\frac{1}{PD} = \frac{1}{PB} + \frac{1}{PC}.$$

3. Determine all triangles ABC which satisfy the condition

$$\tan(A - B) + \tan(B - C) + \tan(C - A) = 0.$$

4. Find all integers a such that $(n - a)(n - 10) + 1$ can be written as a product $(n + b)(n + c)$, where b and c are integers.

5. A transformation is given by

$$P(x, y) \to P'(y + 246, -x - 430).$$

Giving reasons, describe this transformation in simplest possible terms.

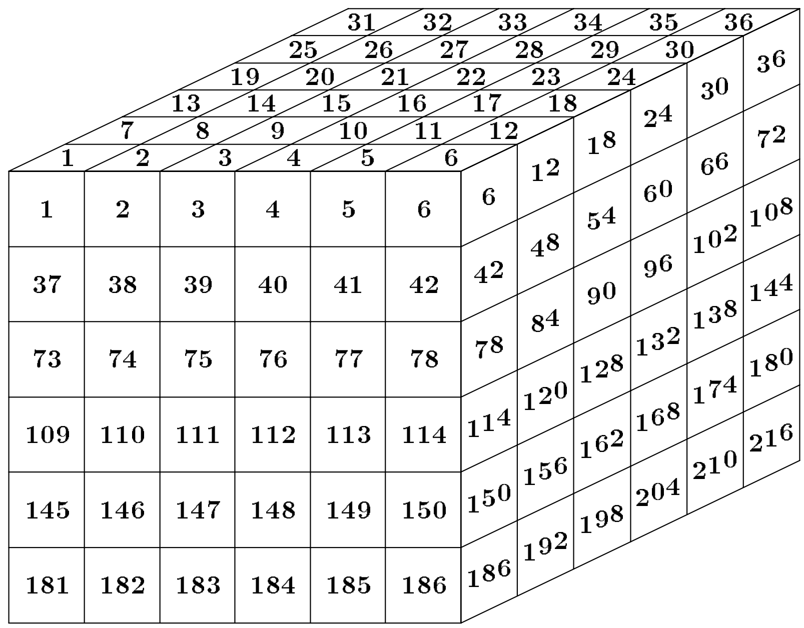
Swedish Mathematics Olympiad

1990 Qualifying Round

1. A boy has got two-thirds of the way over a railroad bridge, when he catches sight of a train coming towards him. He can just get off the bridge, and so escape the train, by running as fast as he can either towards the train or away from it. The train is approaching at a speed of 60 km/h. How fast can the boy run?

2. Which six-digit numbers of the form $abcabc$ are divisible by 33?

3. A large cube, $6 \times 6 \times 6$, is constructed of 216 unit cubes. These are numbered from 1 to 216 as shown in the figure on the next page. The first layer is numbered from 1 to 36, the first row in that layer from 1 to 6, the second from 7 to 12, and so on, always from left to right. The next layer is similarly numbered from 37 to 72, and so on. Choose 36 unit cubes so that no two of the chosen cubes come from the same row, column, or rank (parallel to one of the edges of the cube). Let S denote the sum of the numbers assigned to the 36 cubes. What are the possible values of S?



4. A is one of the angles in a given triangle. The corresponding side has length a, and the other two sides have lengths b and c. Show that

$$\sin A \leq \frac{a}{2\sqrt{bc}}.$$

5. Find all real solutions of the system of equations

$$\begin{aligned} x_1|x_1| &= x_2|x_2| + (x_1 - a)|x_1 - a| \\ x_2|x_2| &= x_3|x_3| + (x_2 - a)|x_2 - a| \\ &\cdots \\ x_n|x_n| &= x_1|x_1| + (x_n - a)|x_n - a| \end{aligned}$$

where a is a given positive integer.

6. Four houses are situated at the corners of a rectangle $ABCD$ with sides 3000 metres and 500 metres. The owners of the houses intend to sink a common well and run water mains from the well to each house. They have access to 1020 metres of pipe and a device that can join together two or more pieces of pipe. Where should the well be placed so that the length of pipe is sufficient?

1990 Final Round

1. The positive divisors of $n = 1900!$ are $d_1, d_2, \ldots, d_k$. Show that

$$\frac{d_1}{\sqrt{n}} + \frac{d_2}{\sqrt{n}} + \cdots + \frac{d_k}{\sqrt{n}} = \frac{\sqrt{n}}{d_1} + \frac{\sqrt{n}}{d_2} + \cdots + \frac{\sqrt{n}}{d_k}.$$

2. The points $A_1, A_2, \ldots, A_{2n}$ lie, in this order, on a straight line, and $|A_k A_{k+1}| = k$ for $k = 1, 2, \ldots, 2n-1$. The point P is situated on the line so that the sum $\sum_{k=1}^{2n} |PA_k|$ is as small as possible.

   Find this sum.

3. The numbers a and b are such that $\sin x + \sin a \geq b \cos x$ for all x. Find a and b.

4. A quadrilateral $ABCD$ is inscribed in a circle. The angle bisectors of A and B meet at a point E. Draw a line through E parallel to CD which meets AD in L and BC in M. Show that $|LA| + |MB| = |LM|$.

5. Find all (not necessarily strict) monotonic, positive functions f which are defined on the positive reals, and which satisfy

$$f(xy) \cdot f\left(\frac{f(y)}{x}\right) = 1$$

   for all $x, y > 0$.

6. Find all positive integers x and y such that $y \leq 500$ and

$$\frac{117}{158} > \frac{x}{y} > \frac{97}{131}.$$

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (⋆) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}$"×11" *or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 November 1999**. *They may also be sent by email to* **crux-editors@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX). *Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2388 [1998, 503] Correction. *Proposed by Daniel Kupper, Büllingen, Belgium.*

Suppose that $n \geq 1 \in \mathbb{N}$ is given and that, for each integer $k \in \{0, 1, \ldots, n-1\}$, the numbers a_k, b_k, $z_k \in \mathbb{C}$ are given, with the z_k^2 distinct. Suppose that the polynomials

$$A_n(z) = z^n + \sum_{k=0}^{n-1} a_k z^k \quad \text{and} \quad B_n(z) = z^n + \sum_{k=0}^{n-1} b_k z^k$$

satisfy $A_n(z_j) = B_n(z_j^2) = 0$ for all $j \in \{0, 1, \ldots, n-1\}$.

Find an expression for $b_0, b_1, \ldots, b_{n-1}$ in terms of $a_0, a_1, \ldots, a_{n-1}$.

2389 [1998, 503] Correction. *Proposed by Nikolaos Dergiades, Thessaloniki, Greece.*

Suppose that f is continuous on $\mathbb{R}^n$ and satisfies the condition that when any two of its variables are replaced by their arithmetic mean, the value of the function increases; for example:

$$f(a_1, a_2, a_3, \ldots, a_n) \leq f\left(\frac{a_1+a_3}{2}, a_2, \frac{a_1+a_3}{2}, a_4, \ldots, a_n\right).$$

Let $m = \dfrac{a_1 + a_2 + \ldots + a_n}{n}$. Prove that

$$f(a_1, a_2, a_3, \ldots, a_n) \leq f(m, m, m, \ldots, m).$$

2426. *Proposed by Mohammed Aassila, Strasbourg, France.*

(a) Show that there are two polynomials, $p(x)$ and $q(x)$, both having three integer roots and such that $p(x) - q(x)$ is a non-zero constant.

(b)* Do there exist two polynomials, $p(x)$ and $q(x)$, both having $n > 3$ integer roots and such that $p(x) - q(x)$ is a non-zero constant?

2427. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Suppose that G is the centroid of triangle ABC, and that

$$\angle GAB + \angle GBC + \angle GCA = 90^\circ .$$

Characterize triangle ABC.

2428. *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Given triangle ABC with $\angle BAC = 90^\circ$. The incircle of triangle ABC touches AB and AC at D and E respectively. Let M be the mid-point of BC, and let P and Q be the incentres of triangles ABM and ACM respectively. Prove that

1. $PD \parallel QE$;
2. $PD^2 + QE^2 = PQ^2$.

2429. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*
Suppose that D, E and F are points on the side AB (or its production) of triangle ABC. Suppose further that CD is a median, that CE is the bisector of $\angle ACB$, and that CF is its external bisector.

The circumcircle, Γ, of triangle EFC intersects CD again at P. Suppose that Γ_A and Γ_B are the circumcircles of triangles CPA and CPB respectively.

Show that Γ_A and Γ_B are tangent to AB at A and B respectively.

2430. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*
Points A and B lie outside circle Γ. Find a point C on Γ with the following property:

AC and BC intersect Γ again at D and E respectively, with $DE \parallel AB$.

2431. *Proposed by Jill Taylor, student, Mount Allison University, Sackville, New Brunswick.*
Let $n \in \mathbb{N}$. Prove that there exist triangles with integer area, integer side lengths, one side n and perimeter $4n$, where n is not necessarily prime.

For a given n, are such triangles uniquely determined?

[Compare problem **2331**.]

2432. *Proposed by K.R.S. Sastry, Bangalore, India.*

In $\triangle ABC$, we use the standard notation: O is the circumcentre, H is the orthocentre. Let M be the mid-point of BC, $OH = m$, $OM = n$ ($m, n \in \mathbb{N}$), and suppose that $OH \parallel BC$.

How many sides of $\triangle ABC$ can have integer lengths?

2433. *Proposed by K.R.S. Sastry, Bangalore, India.*

In $\triangle ABC$, let e denote the length of the segment of the Euler line between the orthocentre and the circumcentre.

Prove or disprove that $\triangle ABC$ is right angled if and only if e equals one half of the length of one of the sides of $\triangle ABC$.

2434. *Proposed by K.R.S. Sastry, Bangalore, India.*

In $\triangle ABC$, let $\angle ABC = 60^\circ$. Point P is on the line segment AC such that $\angle CBP = \angle BAC$. Point Q is on the line segment BP such that $BQ = BC$.

Prove that Q lies on the altitude through A of $\triangle ABC$ if and only if $\angle BAC = 40^\circ$.

2435. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Show that, for $x > 0$, the following functions are increasing:

$$f(x) = \frac{\left(1+\frac{1}{x}\right)^x}{(1+x)^{\frac{1}{x}}} \quad \text{and} \quad g(x) = \left(1+\frac{1}{x}\right)^x - (1+x)^{\frac{1}{x}}.$$

2436. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Find all real solutions of

$$2\cosh(xy) + 2^y - \left[(2\cosh(x))^y + 2\right] = 0\,.$$

2437. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

Let P be a point in the plane of triangle ABC. If the mid-points of the line segments AP, BP, CP all lie on the nine-point circle of triangle ABC, prove that P must be the orthocentre of triangle ABC.

2438. *Proposed by Peter Hurthig, Columbia College, Burnaby, BC.*

Show how to tile an equilateral triangle with congruent pentagons. Reflections are allowed. (Compare problem **1988**.)

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2321. [1998: 109] *Proposed by David Doster, Choate Rosemary Hall, Wallingford, Connecticut, USA.*

Suppose that $n \geq 2$. Prove that

$$\sum_{k=2}^{n} \left\lfloor \frac{n^2}{k} \right\rfloor = \sum_{k=n+1}^{n^2} \left\lfloor \frac{n^2}{k} \right\rfloor .$$

Here, as usual, $\lfloor x \rfloor$ means the greatest integer less than or equal to x.

Solution by Florian Herzig, student, Cambridge, UK.

We show more generally that

$$\sum_{k=2}^{n} \left\lfloor \frac{m}{k} \right\rfloor = \sum_{k=n+1}^{n^2} \left\lfloor \frac{m}{k} \right\rfloor \qquad (1)$$

for all $n^2 \leq m < (n+1)^2$ and $n \geq 2$. The proof is by induction on m. For $m = 4$ this is easily verified. For the induction step, first assume that $n^2 < m < (n+1)^2$ for some n and that (1) holds for $m-1$. Then the left-hand side increases (from its value for $m-1$) by the number of divisors d of m with $2 \leq d \leq n$. If $n^2 < m < (n+1)^2$, and d divides m, where $2 \leq d \leq n$, then $\frac{m}{d}$ also divides m, and

$$n < \frac{n^2+1}{n} \leq \frac{n^2+1}{d} \leq \frac{m}{d} \leq \frac{n^2+2n}{d} \leq \frac{n^2+2n}{2} \leq n^2 ;$$

that is,

$$n+1 \leq \frac{m}{d} \leq n^2 .$$

Conversely, if $n+1 \leq \frac{m}{d} \leq n^2$, where d divides m, then

$$1 < \frac{m}{n^2} \leq d \leq \frac{m}{n+1} < n + 1 ;$$

that is,

$$2 \leq d \leq n .$$

Consequently, there is a one-to-one correspondence between the divisors of m in the interval $[2, n]$ and those in $[n+1, n^2]$. Therefore both sides of (1) increase by the same amount as m increases by 1.

On the other hand, if $m = n^2 > 4$ for some n and (1) is true for $m-1$, then the left-hand side increases by the number N of divisors d of n^2 with $2 \leq d < n$ and by $\left\lfloor \frac{n^2}{n} \right\rfloor = n$; that is, by $N+n$. The right-hand side increases by the number of divisors d of n^2 with $n \le d \le (n-1)^2$, which is $N+1$, and by

$$\sum_{k=(n-1)^2+1}^{n} \left\lfloor \frac{n^2}{k} \right\rfloor - \left\lfloor \frac{n^2}{n} \right\rfloor = (2n - 1) - n = n - 1 .$$

Therefore both sides increase by the same value also in this case, and the result follows by induction.

Also solved by ZAVOSH AMIR-KHOSRAVI, student, North Toronto Collegiate Institute, Toronto; MICHEL BATAILLE, Rouen, France; MANSUR BOASE, student, Cambridge, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; PARAGIOU THEOKLITOS, Limassol, Cyprus; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer.

2322. [1998: 109] *Proposed by K.R.S. Sastry, Dodballapur, India.*

Suppose that the ellipse $\mathcal{E}$ has equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Suppose that Γ is any circle concentric with $\mathcal{E}$. Suppose that A is a point on $\mathcal{E}$ and B is a point of Γ such that AB is tangent to both $\mathcal{E}$ and Γ.

Find the maximum length of AB.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

Let the common tangent touch the ellipse at $A = (x_1, y_1)$ and the circle (of radius R) at $B = (x_2, y_2)$. Assume, without loss of generality, that $b < R < a$ and (since it is not perpendicular to an axis of the ellipse) that AB has the equation

$$y = px + q . \tag{1}$$

(x_1, y_1) satisfies (1) and also the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 . \tag{2}$$

Plugging (1) into (2) and setting the discriminant equal to zero, we find

$$x_1 = -\frac{pa^2}{q}, \qquad q^2 = b^2 + p^2a^2 . \tag{3}$$

Similarly, (x_2, y_2) satisfies (1) and also the equation

$$x^2 + y^2 = R^2 ,$$

so that

$$x_2 = -\frac{pR^2}{q}, \qquad q^2 = R^2(1 + p^2) . \tag{4}$$

From (3) and (4) we find that

$$p^2 = \frac{R^2 - b^2}{a^2 - R^2} .$$

Since $y_2 - y_1 = p(x_2 - x_1)$ and $x_2 - x_1 = \frac{p}{q}(a^2 - R^2)$, we have

$$\begin{aligned} AB^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (1 + p^2)(x_2 - x_1)^2 \\ &= \frac{(R^2 - b^2)(a^2 - R^2)}{R^2} \\ &= a^2 + b^2 - R^2 - \frac{a^2 b^2}{R^2} \\ &= (a - b)^2 - \left(R - \frac{ab}{R}\right)^2 \\ &\le (a - b)^2 , \end{aligned}$$

with equality if and only if $R = \sqrt{ab}$, in which case the maximum value of AB is $a - b$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; THEODORE CHRONIS, Athens, Greece; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; NICOLAS THÉRIAULT, étudiant, Université Laval, Québec; and the proposer.

2325*. [1998: 109] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that q is a prime and n is a positive integer. Suppose that $\{a_k\}$ $(0 \le k \le n)$ is given by

$$\sum_{k=0}^{n} a_k x^k = \frac{1}{q^n} \sum_{k=0}^{n} \binom{qn}{qk} (qx - 1)^k .$$

Prove that each a_k is an integer.

Solution by G.P. Henderson, Garden Hill, Campbellcroft, Ontario.

We are to prove that

$$F = \frac{1}{q^n} \sum_{k=0}^{n} \binom{qn}{qk} (qx - 1)^k$$

is an integer polynomial.

Let the (complex) q-th roots of the number $qx - 1$ be $y_1, y_2, \ldots, y_q$ so that $y_r^q = qx - 1$ for $r = 1, 2, \ldots, q$.

Lemma. If $P(y)$ is an integer polynomial, then

$$\frac{1}{q}\sum_{r=1}^{q} P(y_r)$$

is an integer polynomial in x.

Proof. Suppose

$$P(y) \;=\; \sum_{m=0}^{b} c_m y^m\,.$$

Then

$$\frac{1}{q}\sum_{r=1}^{q} P(y_r) \;=\; \frac{1}{q}\sum_{m=0}^{b} c_m \sum_{r=1}^{q} y_r^m\,.$$

If m is a multiple of q, say $m = qk$, the inner sum is

$$\sum_{r=1}^{q} y_r^{qk} \;=\; \sum_{r=1}^{q}(qx-1)^k \;=\; q(qx-1)^k\,,$$

and if m is not a multiple of q, the inner sum is zero. [*Editorial comment*: This follows because, with m and q relatively prime, $y_1^m, \ldots, y_q^m$ are the (distinct) q-th roots of $(qx-1)^m$ and thus sum to zero.] Therefore

$$\frac{1}{q}\sum_{r=1}^{q} P(y_r) \;=\; \sum_{k=0}^{\lfloor b/q\rfloor} c_{qk}(qx-1)^k\,, \tag{1}$$

which proves the lemma. □

Set $b = qn$ and $c_m = \binom{qn}{m}$. Then

$$P(y_r) \;=\; \sum_{m=0}^{qn}\binom{qn}{m} y_r^m \;=\; (1+y_r)^{qn}$$

for $r = 1, \ldots, q$. Thus, dividing (1) by q^n:

$$\frac{1}{q^{n+1}}\sum_{r=1}^{q}(1+y_r)^{qn} \;=\; \frac{1}{q^n}\sum_{k=0}^{n}\binom{qn}{qk}(qx-1)^k \;=\; F\,. \tag{2}$$

We have

$$(1+y_r)^q \;=\; 1+\sum_{m=1}^{q-1}\binom{q}{m}y_r^m + y_r^q \;=\; q[x+Q(y_r)]\,,$$

where $Q(y)$ is the integer polynomial

$$Q(y) \;=\; \sum_{m=1}^{q-1}\frac{1}{q}\binom{q}{m}y^m\,.$$

Using this in (2),

$$F \;=\; \frac{1}{q}\sum_{r=1}^{q}[x+Q(y_r)]^n \;=\; \sum_{s=0}^{n}\binom{n}{s}x^{n-s}\frac{1}{q}\sum_{r=1}^{q}[Q(y_r)]^s\,. \qquad (3)$$

Applying the lemma to the integer polynomials $[Q(y)]^s$, we see that F is an integer polynomial.

Note. If $q = 2$, we can obtain an explicit expression for F. The roots y_1 and y_2 are $\pm\sqrt{2x-1}$, and $Q(y) = y$. From (3),

$$\begin{aligned} F &= \frac{1}{2}\left[(x+\sqrt{2x-1})^n + (x-\sqrt{2x-1})^n\right] \\ &= x^n + \binom{n}{2}x^{n-2}(2x-1) + \binom{n}{4}x^{n-4}(2x-1)^2 \\ &\qquad + \cdots + \binom{n}{2r}x^{n-2r}(2x-1)^r\,, \end{aligned}$$

where $r = \lfloor n/2 \rfloor$.

No other solutions were received.

2326*. [1998: 175, 301] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Prove or disprove that if A, B and C are the angles of a triangle, then

$$\frac{2}{\pi} \;<\; \sum_{\text{cyclic}} \frac{\left(1-\sin\frac{A}{2}\right)\left(1+2\sin\frac{A}{2}\right)}{\pi - A} \;\leq\; \frac{9}{2\pi}\,.$$

Solution by Michael Lambrou, University of Crete, Crete, Greece.

Set $A' = (\pi - A)/2$ and similarly for B and C so that $A'+B'+C' = \pi$ and A', B', C' are the angles of an *acute*-angled triangle. Conversely, every acute-angled triangle arises in this way if we set $A = \pi - 2A'$, etc. So the substitution $A = \pi - 2A'$ transforms the inequality to be proved into

$$\frac{4}{\pi} < \sum \frac{\cos A' - \cos 2A'}{A'} \leq \frac{9}{\pi} \qquad (1)$$

for all acute-angled triangles. For notational convenience drop the primes and write A, B, C in place of A', B', C' once again. We may further assume $\pi/2 > A \geq B \geq C > 0$. For $0 < x \leq \pi/2$ set $f(x) = \dfrac{\cos x - \cos 2x}{x}$, and extend this, by continuity, to $x = 0$ [*Editorial note*: by defining $f(0) = 0$ — it is easy to show with calculus that $\lim_{x\to 0} f(x) = 0$].

For $x \in [0, \pi/2]$, we have $f''(x) = g(x)/x^3$, where

$$g(x) \;=\; (\cos x - \cos 2x)''x^2 - 2(\cos x - \cos 2x)'x + 2(\cos x - \cos 2x)\,,$$

so that

$$g'(x) \;=\; (\cos x - \cos 2x)'''x^2 \;=\; x^2 \sin x(1 - 16\cos x)\,.$$

Hence $g'(x) = 0$ if and only if $x = 0$ or $x = \arccos(1/16) \approx 86^\circ$, and g decreases in $[0, \arccos(1/16)]$ and increases in $[\arccos(1/16), \pi/2]$ and so has absolute minimum at $x = \arccos(1/16)$. But $g(0) = 0$, $g(\pi/2) = -\pi^2 + \pi + 2 < 0$ so $g(x) \le 0$ for all x in $[0, \pi/2]$. It follows that $f''(x) \le 0$ for all x in $[0, \pi/2]$ and so f is concave down. By Jensen's inequality we have

$$f(A) + f(B) + f(C) \;\le\; 3f\left(\frac{A+B+C}{3}\right) \;=\; 3f\left(\frac{\pi}{3}\right) \;=\; \frac{9}{\pi}\,,$$

showing the right hand side of (1).

The concavity of f together with $f(\pi/4) = 2\sqrt{2}/\pi > 2/\pi = f(\pi/2)$ shows that the least value of f in the restricted interval $[\pi/4, \pi/2]$ occurs at $x = \pi/2$. We also have $f(0) = 0 < f(\pi/2)$ so $f(x) \ge 0$ for all $x \in [0, \pi/2]$ with strict inequality $f(x) > 0$ if $x \ne 0$.

Finally, since $\pi/2 \ge A \ge B \ge C$, we have that $B \ge \pi/4$, so that $f(A)$, $f(B) \ge f(\pi/2) = 2/\pi$. Hence

$$f(A) + f(B) + f(C) \;>\; f(A) + f(B) \;\ge\; f\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = \frac{4}{\pi}\,,$$

which proves the left hand side of (1).

Editorial note. Both bounds in (1) are best possible: the lower bound $4/\pi$ is attained by the degenerate $\pi/2$, $\pi/2$, 0 triangle, and the upper bound $9/\pi$ by the equilateral triangle.

Also solved by HAYO AHLBURG, Benidorm, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; and HEINZ-JÜRGEN SEIFFERT, Berlin, Germany.

Konečný, Seiffert and the proposer rewrote the sum in the problem as

$$\sum \frac{\sin(A/2) + \cos A}{\pi - A}\,,$$

which may be a more attractive form, though not perhaps quite as attractive as Lambrou's equivalent form (which was also found by Ahlburg).

The problem arose from Janous's solution to his ***CRUX with MAYHEM*** *problem* **2190**.

2327. [1998: 175] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

The sequence $\{a_n\}$ is defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and

$$a_{n+1} = a_n - a_{n-1} + \frac{a_n^2}{a_{n-2}}, \qquad n \geq 3 .$$

Prove that each $a_n \in \mathbb{N}$, and that no a_n is divisible by 4.

Composite solutions by Edward J. Barbeau, University of Toronto, Toronto, Ontario and Florian Herzig, student, Cambridge, UK.

Note first that $\{a_n\}$ is increasing and so has no zero terms.
[Ed.: A simple induction suffices.]

Rearranging, we have $\dfrac{a_{n+1} + a_{n-1}}{a_n + a_{n-2}} = \dfrac{a_n}{a_{n-2}}$.

Setting $n = 3, 4, \ldots, m$, and taking the product, we get, for all $m \geq 2$,

$$\frac{a_{m+1} + a_{m-1}}{a_3 + a_1} = \frac{a_m a_{m-1}}{a_2 a_1} .$$

Since $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, we get $a_{m+1} + a_{m-1} = 2a_m a_{m-1}$, or $a_{m+1} = a_{m-1}(2a_m - 1)$; whence $a_n \in \mathbb{N}$ for all n, by induction. Since $2a_m - 1$ is odd, another induction shows that a_{m+1} is not divisible by 4, since a_{m-1} is not.

Also solved by ZAVOSH AMIR-KHOSRAVI, student, North-Toronto Collegiate Institute, Toronto, Ontario; MICHEL BATAILLE, Rouen, France; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; ANTHONY FONG, student, Eric Hamber Secondary School, Vancouver, BC; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; HADI SALMASIAN, student, Sharif University of Technology, Tehran, Iran; JOEL SCHLOSBERG, student, Bayside, NY, USA; NICOLAS THÉRIAULT, student, Université Laval, Montréal, Québec; TODD THOMPSON, student, University of Arizona, Tucson, AZ, USA; and the proposer. There was also a partially incorrect solution submitted.

The recurrence relation shown in the solution above was also obtained by nine other solvers. However, only Herzig, Lambrou and the proposer actually derived the relation, while the others all used induction. Strictly speaking, a complete proof should include a statement and proof for the fact that $a_n \neq 0$ for all n. However, only a few solvers pointed this out explicitly.

Janous considered more generally the sequence defined by

$$a_{n+1} = ka_n - ka_{n-1} + \frac{a_n^2}{a_{n-2}}, \qquad (n \geq 3),$$

where k, a_1, a_2, $a_3 \in \mathbb{N}$, and showed that if $\dfrac{ka_1 + a_3}{a_1 a_2} \in \mathbb{N}$ and $\dfrac{ka_1 + a_3}{a_1 a_2} > k$, then $a_n \in \mathbb{N}$ for all n.

2328*. [1998: 176] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

It is known from Wilson's Theorem that the sequence $\{y_n : n \geq 0\}$, with $y_n = \dfrac{n!+1}{n+1}$, contains infinitely many integers; namely, $y_n \in \mathbb{N}$ if and only if $n+1$ is prime.

(a) Determine all integer members of the sequences $\{y_n(a) : n \geq 0\}$, with $y_n = \dfrac{n!+a}{n+a}$, in the cases $a = 2, 3, 4$.

(b) Determine all integer members of the sequences $\{y_n(a) : n \geq 0\}$, with $y_n = \dfrac{n!+a}{n+a}$, in the cases $a \geq 5$.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

For all $a \in \mathbb{N}$, we have that $y_n(a) = \dfrac{n!+a}{n+a}$ is an integer when $n = 1$ and $n = 2$. If $n = a > 2$, then $y_n(a) = \dfrac{(a-1)!+1}{2}$ is not an integer.

Case 1: $n + a$ is not prime.

Suppose $n > a$. There are two possibilities:

(i) $n + a = pq$ where $\gcd(p, q) = 1$. Then $2n > n + a \geq 2q$ which implies $q \leq n-1$. Similarly $p \leq n-1$. This means $p|(n-1)!$ and $q|(n-1)!$, which further implies $pq|n!$. Thus $n!+a \equiv a \pmod{n+a}$, whence the number $\dfrac{n!+a}{n+a}$ is not an integer.

(ii) $n + a = p^k$, where $k > 1$ and p is prime. Arguing as above we get $p^{k-1}|(n-1)!$. If the number $\dfrac{n!+a}{n+a}$ is an integer, then $p^{k-1}|(n!+a)$. [*Ed*. note: we actually have $p^k|(n!+a)$, but that is not needed in the proof.] From $p^{k-1}|n!$ we conclude that $p^{k-1}|a$; from this and $p^{k-1}|(n+a)$ we get $p^{k-1}|n$ and finally

$$p^{k-1}|(n-1)! \text{ and } p^{k-1}|n \implies p^{k+k-2}|n! \implies n+a = p^k|n!$$

or $n! + a \equiv a \pmod{n+a}$. So the number $\dfrac{n!+a}{n+a}$ is not an integer.

Therefore, in case 1 if the number $\dfrac{n!+a}{n+a}$ is an integer we must have $n < a$.

Case 2: $n + a$ is prime.

From Wilson's Theorem we have $(n + a - 1)! + 1 \equiv 0 \pmod{n+a}$.

If $n! + a \equiv 0 \pmod{n+a}$, then

$$\begin{aligned} n!(n+1)\cdots(n+a-1) + a(n+1)\cdots(n+a-1) &\equiv 0 \pmod{n+a}, \\ -1 + a(n+1)\cdots(n+a-1) &\equiv 0 \pmod{n+a}, \\ -1 + a(-a+1)(-a+2)\cdots(-2)(-1) &\equiv 0 \pmod{n+a}, \\ -1 - (-1)^a a! &\equiv 0 \pmod{n+a}. \end{aligned}$$

In this case the number $\dfrac{n!+a}{n+a}$ is an integer if $n+a$ is a prime divisor of $s = -1 - (-1)^a a!$. So there are a finite number of terms of $y_n(a)$ such that $\dfrac{n!+a}{n+a}$ is an integer, since when $a \neq 1$ we have $s \neq 0$.

Examples:

Let $a = 2$. Then $s = -3$ with 3 the only prime divisor, so $n + 2 = 3$ gives $n = 1$. Thus the number $\dfrac{n!+2}{n+2}$ is an integer only when $n = 1, 2$.

Let $a = 3$. Then $s = 5$. So $n + 3 = 5$ gives $n = 2$ and the number $\dfrac{n!+3}{n+3}$ is an integer only when $n = 1, 2$.

Let $a = 4$. Then $s = -25$ with 5 the only prime divisor, so $n + 4 = 5$ gives $n = 1$. We also need to check $n < a$: if $n = 3$, then $\dfrac{n!+4}{n+4} = \dfrac{10}{7}$, which is not an integer. Thus the number $\dfrac{n!+4}{n+4}$ is an integer only when $n = 1, 2$.

Let $a = 5$. Then $s = 119$ with prime divisors 7 and 17, so $n + 5 = 7$ or 17 implies $n = 2$ or 12, respectively. We also need to check $n < a$: if $n = 3, 4$, then $\dfrac{n!+5}{n+5} = \dfrac{11}{8}, \dfrac{29}{9}$, respectively, neither of which is an integer. So the number $\dfrac{n!+5}{n+5}$ is an integer only when $n = 1, 2, 12$.

As a last example, if $a = 22$, then the number $\dfrac{n!+22}{n+22}$ is an integer only when $n = 1, 2, 12, 499, 93\,799\,610\,095\,769\,625$.

Also solved by MANSUR BOASE, student, Cambridge, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; RICHARD I. HESS, Rancho Palos Verdes, California, USA; and the proposer.

Hess was the only solver who specifically showed that $y_n(a)$ is never an integer when $n = 0$ and $a \geq 2$. He also gave a complete list of the values of n which yield integers for $y_n(a)$ for values of $6 \leq a \leq 17$ (the list below gives only the values of $n > 2$):

$a = 6$	:	4 , 97
$a = 7$	:	5 032
$a = 8$	:	6 , 53 , 653
$a = 9$	:	2 990
$a = 10$	:	329 881
$a = 11$	:	6 , 12 , 7 842
$a = 12$	:	10 , 2 834 317
$a = 13$	:	1 720 , 3 593 190
$a = 14$	:	9 , 12 , 3 790 360 473
$a = 15$	:	6 , 16 , 38 , 1 510 244
$a = 16$	:	4 , 45 , 121 , 123 , 1 059 495
$a = 17$	:	56 , 256 443 711 660

Janous observes that there are quite a few new questions coming from this problem, such as:

What is $\ell = \limsup\limits_{a\to\infty} i(a)$, *where* $i(a)$ *is the number of integer members of the set* $\left\{\dfrac{n!+a}{n+a} : n = 1, 2, 3 \ldots\right\}$?

If $n(k) = \#\{a \geq 2 : i(a) = k\}$, *what can be said about* $n(k)$? *Especially, in the event of* $\ell = \infty$, *is it true that, for all* $k \geq 2$, *we always have* $n(k) \geq 1$? *If not, for what values of* k *is* $n(k) = 0$?

2330. [1998: 176] *Proposed by Florian Herzig, student, Perchtoldsdorf, Austria.*

Prove that

$$e = 3 - \frac{1!}{1\cdot 3} + \frac{2!}{3\cdot 11} - \frac{3!}{11\cdot 53} + \frac{4!}{53\cdot 309} - \frac{5!}{309\cdot 2119} + \cdots,$$

where

$$\begin{aligned} 11 &= 3\cdot 3 + 2\cdot 1, \\ 53 &= 4\cdot 11 + 3\cdot 3, \\ 309 &= 5\cdot 53 + 4\cdot 11, \\ 2119 &= 6\cdot 309 + 5\cdot 53, \\ &\vdots \end{aligned}$$

Solution by Con Amore Problem Group, Royal Danish School of Educational Studies, Copenhagen, Denmark.

We have to prove that

$$e = 3 - \frac{1!}{a_1a_2} + \frac{2!}{a_2a_3} - \frac{3!}{a_3a_4} + \cdots + (-1)^n\cdot\frac{n!}{a_na_{n+1}} + \cdots, \qquad (1)$$

where a_1, a_2, ..., are such that $a_1 = 1$, $a_2 = 3$, and for $n \geq 3$:

$$a_n = na_{n-1} + (n-1)a_{n-2}. \qquad (2)$$

Along with $a_1, a_2, \ldots$, we consider $d_1, d_2, \ldots$, defined by

$$d_n = n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{(-1)^n}{n!}\right) \tag{3}$$

for $n = 1, 2, \ldots$. These are well-known as the derangement (or displacement) numbers, so called because d_n is the number of permutations of $(1, 2, \ldots, n)$ where all the elements are displaced, but this need not concern us here. We just note for later use that

$$d_3 = 2 \text{ and } d_4 = 9 \tag{4}$$

and

$$\begin{aligned} \frac{d_{n+1}}{(n+1)!} - \frac{d_n}{n!} &= \frac{(-1)^{n+1}}{(n+1)!}, \\ \text{or} \quad (n+1)d_n - d_{n+1} &= (-1)^n \end{aligned} \tag{5}$$

for $n = 1, 2, \ldots$, whence $(n+2)d_{n+1} - d_{n+2} = d_{n+1} - (n+1)d_n$, or

$$d_{n+2} = (n+1)d_{n+1} + (n+1)d_n \tag{6}$$

for $n = 1, 2, \ldots$. Now define for $n = 1, 2, \ldots$:

$$\alpha_n = \frac{d_{n+2}}{n+1}. \tag{7}$$

Then by (4) and (6), $\alpha_1 = d_3/2 = 1$, and $\alpha_2 = d_4/3 = 3$, and for $n \geq 3$ we have

$$(n+1)\alpha_n = (n+1)n\alpha_{n-1} + (n+1)(n-1)\alpha_{n-2},$$

or $\alpha_n = n\alpha_{n-1} + (n-1)\alpha_{n-2}$. Comparison with (2) shows that for $n = 1, 2, \ldots$, we have $\alpha_n = a_n$. Thus by (7):

$$a_n = \frac{d_{n+2}}{n+1} \tag{8}$$

for $n = 1, 2, \ldots$. Now (8) and (5) imply

$$\begin{aligned} (-1)^n \cdot \frac{n!}{a_n a_{n+1}} &= (-1)^{n+2} \cdot \frac{n!(n+1)(n+2)}{d_{n+2}d_{n+3}} \\ &= \big((n+3)d_{n+2} - d_{n+3}\big) \cdot \frac{(n+2)!}{d_{n+2}d_{n+3}}, \end{aligned}$$

and so, for $n = 1, 2, \ldots$:

$$(-1)^n \cdot \frac{n!}{a_n a_{n+1}} = \frac{(n+3)!}{d_{n+3}} - \frac{(n+2)!}{d_{n+2}}. \tag{9}$$

Using (9) and (3), and noting that $3 = 3!/d_3$, we get

$$\begin{aligned}
& 3 - \frac{1!}{a_1 a_2} + \frac{2!}{a_2 a_3} - \frac{3!}{a_3 a_4} + \cdots + (-1)^n \cdot \frac{n!}{a_n a_{n+1}} \\
= \; & \frac{3!}{d_3} + \left(\frac{4!}{d_4} - \frac{3!}{d_3}\right) + \left(\frac{5!}{d_5} - \frac{4!}{d_4}\right) + \cdots + \left(\frac{(n+3)!}{d_{n+3}} - \frac{(n+2)!}{d_{n+2}}\right) \\
= \; & \frac{(n+3)!}{d_{n+3}} \; = \; \left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{(-1)^{n+3}}{(n+3)!}\right)^{-1} \\
\to \; & (e^{-1})^{-1} \; = \; e \qquad \text{as } n \to \infty \,,
\end{aligned}$$

which proves (1).

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.

Herzig used continued fractions to solve the problem.

2331. [1998: 176] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

Let p be an odd prime. Show that there is at most one non-degenerate integer triangle with perimeter $4p$ and integer area. Characterize those primes for which such triangles exist.

Composite solution by Jill S. Taylor, student, Mount Allison University, Sackville, New Brunswick; and the proposer.

Consider such a triangle of sides a, b, c, and semi-perimeter s, with area $\sqrt{s(s-a)(s-b)(s-c)}$. Then $1 \leq a,\, b,\, c \leq 2p - 1$.

Since $s = 2p$ and the area of the triangle is an integer, one of $(s-a)$, $(s-b)$, $(s-c)$ must be divisible by p. Without loss of generality, assume that $p|(s-a)$. Then $s - a = p$ since $1 \leq s - a \leq 2p - 1$.

Hence $(s-b) + (s-c) = a = p$ and $(s-b)(s-c)$ must be twice a square. Clearly, $(s-b)$ and $(s-c)$ are relatively prime. [*Ed.*: If $d \in \mathbb{N}$ is such that $d|(s-b)$ and $d|(s-c)$, then $d|p$, and thus $d = 1$ or $d = p$. However, if $d = p$, then $p|(2p-b)$, $(2p-c)$ would imply that $b = c = p$, and so, $2s = a + b + c = 3p$, which is a contradiction.]

It follows that $s - b = x^2$ and $x - c = 2y^2$ for relatively prime integers x and y. Hence $p = x^2 + 2y^2$.

Conversely, if $p = x^2 + 2y^2$, then p, $p + x^2$ and $2p - x^2$ would be the sides of a triangle with the described properties, since

$$\begin{aligned}
2s \; &= \; p + (p + x^2) + (2p - x^2) \; = \; 4p \qquad \text{and} \\
s(s-p)(s-(p+x^2))(s-(2p-x^2)) \; &= \; 2p^2(p - x^2)x^2 \\
&= \; 2p^2(2x^2y^2) = (2pxy)^2 \,.
\end{aligned}$$

By well-known results in elementary number theory (see [3]), the representation $p = x^2 + 2y^2$ is possible and unique if and only if $p \equiv 1, 3 \pmod 8$. Hence such a triangle exists (and is unique) if and only if $p \equiv 1, 3 \pmod 8$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; JOEL SCHLOSBERG, student, Bayside, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; KENNETH M. WILKE, Topeka, Kansas, USA.

The first four triangles with the described proprties are $(3, 4, 5; 6)$, $(11, 13, 20; 66)$, $(17, 25, 26; 204)$, *and* $(19, 20, 37; 114)$, *(where the components denote the side lengths and the areas of the triangles, respectively). These were given by Hess, Leversha and the proposer. Hess and Leversha actually listed the first six and eleven such triangles, respectively.*

The result cited in the solution above can be found in many books on number theory. Besides [3], *Konečný quoted* [1] *and two others. Both Seiffert and Wilke quoted* [2].

References:

[1] Ethan D. Bolker, *Elementary Number Theory*, W.A. Benjamin Inc., New York, 1970, pp. 113–116.

[2] T. Nagell, *Introduction to Number Theory*, 2nd Ed., Chelsea, 1964, pp. 188–191.

[3] W. Sierpiński, *A Selection of Problems in the Theory of Numbers*, Pergamon Press Ltd., Oxford, England, 1964, p. 72.

2332. [1998: 177] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Suppose x and y are integers. Solve the equation

$$x^2y^2 - 7x^2y + 12x^2 - 21xy - 4y^2 + 63x + 70y - 174 = 0 .$$

Solution by Michel Bataille, Rouen, France.

The equation reduces to the equivalent form

$$(x-2)(y-3)((x+2)(y-4) - 21) = 0 .$$

The first two factors provide two families of solutions: $(2, a)$ and $(b, 3)$, where a and b are integers. Any other solution is such that $(x+2)(y-4) = 21$; that is, $(x+2, y-4)$ is one of the pairs $(1, 21)$, $(21, 1)$, $(3, 7)$, $(7, 3)$, $(-1, -21)$, $(-21, -1)$, $(-3, -7)$, and $(-7, -3)$. Examining each case separately, we obtain seven new solutions:

$$(-1, 25)\,,\ (19, 5)\,,\ (1, 11)\,,\ (5, 7)\,,\ (-3, -17)\,,\ (-5, -3)\,,\ \text{and}\ (-9, 1)\,.$$

Also solved by HAYO AHLBURG, Benidorm, Spain; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ZAVOSH AMIR-KHOSRAVI, student, North Toronto Collegiate Institute, Toronto; SAM BAETHGE, Nordheim, Texas, USA; EDWARD J. BARBEAU, University of Toronto, Toronto, Ontario; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; THEODORE CHRONIS, Athens, Greece; GORAN CONAR, student, Gymnasium Varaždin,

Varaždin, Croatia; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; ANTHONY FONG, student, Eric Hamber Secondary School, Vancouver; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines and GIOVANNI MAZZARELLO, Ferrovie dello Stato, Florence, Italy; DOUGLASS L. GRANT, University College of Cape Breton, Sydney, Nova Scotia; YEO KENG HEE, Hwa Chong Junior College, Singapore; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; JESSIE LEI, student, Vincent Massey Secondary School, Windsor, Ontario; GERRY LEVERSHA, St. Paul's School, London, England; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; CHRISTO SARGOTIS, Thessaloniki, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA; DIGBY SMITH, Mount Royal College, Calgary, Alberta; PANOS E. TSAOUSSOGLOU, Athens, Greece; ERIC UMEGÅRD, Västerås, Sweden; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer. There was one incomplete solution submitted.

Most of the submitted solutions are similar to the one given above.

2333. [1998: 177] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

You are given that D and E are points on the sides AC and AB respectively of $\triangle ABC$. Also, DE is not parallel to CB. Suppose F and G are points of BC and ED respectively such that

$$\overline{BF} : \overline{FC} \;=\; \overline{EG} : \overline{GD} \;=\; \overline{BE} : \overline{CD}\,.$$

Show that GF is parallel to the angle bisector of $\angle BAC$.

I. *Solution by Michael Lambrou, University of Crete, Crete, Greece.*

With respect to position vectors originating from A, write $\overrightarrow{AB} = \mathrm{b}$, $\overrightarrow{AC} = \mathrm{c}$, so that for some p, $q \in (0,1)$ (see remark) we have $\mathrm{e} := \overrightarrow{AE} = p\mathrm{b}$, $\mathrm{d} := \overrightarrow{AD} = q\mathrm{c}$. If the given equal ratios are denoted by $\lambda > 0$ (see remark), we have

$$\mathrm{f} := \overrightarrow{AF} \;=\; \frac{\lambda\mathrm{c} + \mathrm{b}}{\lambda + 1} \quad \text{and} \quad \mathrm{g} := \overrightarrow{AG} \;=\; \frac{\lambda\mathrm{d} + \mathrm{e}}{\lambda + 1} \;=\; \frac{\lambda q\mathrm{c} + p\mathrm{b}}{\lambda + 1}\,.$$

Moreover, the condition $\overline{BE} : \overline{CD} \;=\; \lambda$ becomes $\left|\overrightarrow{BE}\right| = \lambda\left|\overrightarrow{CD}\right|$, and so $(1-p)|\mathrm{b}| = \lambda(1-q)|\mathrm{c}|$. Thus,

$$\begin{aligned}\overrightarrow{GF} \;&=\; \mathrm{f} - \mathrm{g} \;=\; \frac{\lambda(1-q)}{\lambda+1}\mathrm{c} + \frac{1-p}{\lambda+1}\mathrm{b} \\ &=\; \frac{\lambda(1-q)|\mathrm{c}|}{\lambda+1}\frac{\mathrm{c}}{|\mathrm{c}|} + \frac{(1-p)|\mathrm{b}|}{\lambda+1}\frac{\mathrm{b}}{|\mathrm{b}|} \;=\; \frac{(1-p)|\mathrm{b}|}{\lambda+1}\left(\frac{\mathrm{c}}{|\mathrm{c}|} + \frac{\mathrm{b}}{|\mathrm{b}|}\right)\,,\end{aligned}$$

which is parallel to $\left(\frac{\mathrm{c}}{|\mathrm{c}|} + \frac{\mathrm{b}}{|\mathrm{b}|}\right)$. But this last is a sum of unit vectors along AB, AC and so is along the bisector of A, completing the proof.

Remarks. 1. The condition "DE not parallel to CB" has not been used. In the parallel case it turns out that GF coincides with the bisector.

[*Editor's additional remark*: Most solvers made a similar remark, with different interpretations according to whether or not one's definition of *parallel lines* permits the lines to coincide.]

2. We have taken here D, E, F in the interior of the corresponding sides of $\triangle ABC$. Thus we have $0 < p$, $q < 1$ and $\lambda > 0$. However, this is not necessary and the proof can be adapted to the more general case. For example, if $\lambda > 0$ then p, q must be both greater than 1 or both less than 1.

II. *Solution by Nikolaos Dergiades, Thessaloniki, Greece.*

Let K, L be chosen so that the quadrangles $EBKG$ and $GLCD$ are parallelograms, and therefore KG is equal and parallel to BE while LG is equal and parallel to CD. Consequently, $\angle BAC = \angle KGL$, and since $BK||LC$ and

$$\frac{BK}{LC} = \frac{EG}{GD} = \frac{BF}{FC}$$

[the last equality by assumption], we deduce that $\triangle BKF$ is similar to $\triangle CLF$, so that KL passes through F. Therefore,

$$\frac{KF}{FL} = \frac{BF}{FC} = \frac{BE}{CD} = \frac{KG}{LG}$$

[the middle equality by assumption], which implies that in $\triangle KGL$, we have that GF is the bisector of $\angle KGL$ and (since $KG||BA$ and $LG||CA$) parallel to the bisector of $\angle BAC$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; FLORIAN HERZIG, student, Cambridge, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

2334. [1998: 177] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that ABC is a triangle with incentre I, and that BI, CI meet AC, AB at D, E respectively. Suppose that P is the intersection of AI with DE. Suppose that $PD = PI$.

Find angle ACB.

Solution by Gerry Leversha, St. Paul's School, London, England.

We are given that $PD = PI$; hence $\angle AID = \angle EDI$. But

$$\angle AID = \angle ABI + \angle BAI = \frac{A+B}{2}$$

Therefore $$\angle EDI = \frac{A+B}{2}.$$

We need to calculate some angles.

$$\begin{aligned}
\angle DEI &= 180^\circ - \angle EDI - \angle EID \\
&= 180^\circ - \frac{A+B}{2} - \left(180^\circ - \frac{B+C}{2}\right) \;=\; \frac{C-A}{2}\,, \\
\angle DEA &= 180^\circ - \angle EAD - \angle ADE \\
&= 180^\circ - A - (180^\circ - \angle EDI - \angle BDC) \\
&= -A + \frac{A+B}{2} + \left(180^\circ - C - \frac{B}{2}\right) \\
&= 180^\circ - C - \frac{A}{2}\,.
\end{aligned}$$

Now let $ED = x$ and use the Sine Rule on triangles AED and CED:

$$\frac{x}{AD} = \frac{\sin A}{\sin\left(C + \frac{A}{2}\right)} \qquad \text{and} \quad \frac{x}{DC} = \frac{\sin\frac{C}{2}}{\sin\left(\frac{C-A}{2}\right)}\,.$$

Now we divide and recall that $\dfrac{DC}{AD} = \dfrac{a}{c}$ by the Angle Bisector Theorem.

Therefore $$\frac{\sin A}{\sin C} = \frac{a}{c} = \frac{\sin A \sin\left(\frac{C-A}{2}\right)}{\sin\frac{C}{2}\sin\left(C+\frac{A}{2}\right)}.$$

Thus,

$$\begin{aligned}
\sin\frac{C}{2}\sin\left(C+\frac{A}{2}\right) &= \sin C \sin\left(\frac{C-A}{2}\right) \\
&= 2\sin\frac{C}{2}\cos\frac{C}{2}\sin\left(\frac{C-A}{2}\right)\,,
\end{aligned}$$

and $$\sin\left(C+\frac{A}{2}\right) = 2\cos\frac{C}{2}\sin\left(\frac{C-A}{2}\right) = \sin\left(C-\frac{A}{2}\right) - \sin\frac{A}{2}.$$

Hence $$\sin\frac{A}{2} = \sin\left(C-\frac{A}{2}\right) - \sin\left(C+\frac{A}{2}\right) = -2\cos C\sin\frac{A}{2},$$

and $$\cos C = -\frac{1}{2}\,, \qquad \text{so that } C = 120^\circ.$$

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NIKOLAOS DERGIADES, Thessaloniki, Greece; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer. There was one incorrect solution.

2335. [1998: 177] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Triangle ABC has circumcircle Γ. A circle Γ' is internally tangent to Γ at P, and touches sides AB, AC at D, E respectively. Let X, Y be the feet of the perpendiculars from P to BC, DE respectively.

Prove that $PX = PY \sin\frac{A}{2}$.

Solution by Florian Herzig, student, Cambridge, UK.

Rename the circles Γ and Γ' (respectively) as Γ_1 and Γ_2 to avoid later confusion. Invert the configuration in any circle with centre P, and denote the image of a point or circle X by X'. Then Γ_1' and Γ_2' are parallel lines, and the latter touches the circumcircles of $PC'E'A'$ and $PA'D'B'$ (because the corresponding lines touch Γ_2). Therefore $B'C' = 2D'E'$ and also

$$\angle E'PD' \;=\; \angle E'PA' + \angle A'PD' \;=\; \frac{1}{2}(\angle C'PA' + \angle A'PB') \;=\; \frac{1}{2}\angle C'PB'\,.$$

Moreover, $\angle C'PB' = \angle BPC = 180^o - A$. Since X is the foot of the perpendicular to BC from P (the centre of inversion), the image of BC is the circle $B'C'P$ having PX' as diameter.

[*Editor's comment*. Herzig provided a simple justification of the claim, but proofs are readily found in any text with a section on inversions, such as Coxeter and Greitzer's *Geometry Revisited*.]

Similarly, PY' is a diameter of circle $E'PD'$. Hence PX' is twice the circumradius of circle $B'PC'$, and PY' of $E'PD'$. By the Sine Law, therefore,

$$PX' \;=\; \frac{B'C'}{\sin A}\,, \quad \text{and} \quad PY' \;=\; \frac{D'E'}{\sin\angle E'PD'} \;=\; \frac{D'E'}{\sin(90^o - \frac{A}{2})}\,.$$

Finally,

$$\frac{PX}{PY} \;=\; \frac{PY'}{PX'} \;=\; \frac{D'E'\cdot\sin A}{B'C'\cdot\cos\frac{A}{2}} \;=\; \sin\frac{A}{2}\,,$$

as we wanted to show.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

The configuration investigated here has appeared before in **CRUX** *— in Leon Bankoff's "Mixtilinear Adventure"* [1983: 2-7] *and in S. Shirali's "On a Generalized Ptolemy Theorem"* [1996: 49-53]. *See also Nathan Altshiller Court, College Geometry, p. 239 theorem 537. We thank Bellot and Seimiya for these references. Bellot also found the configuration in a 1991 Bulgarian journal and in the Cambridge Mathematical Tripos of 1929. It is remarkable how he is so often able to find relevant references. This editor wonders if the method can be applied to finding my glasses (which occasionally get misplaced.)*

2336. [1998: 177] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

The bisector of angle A of a triangle ABC meets BC at D. Let Γ and Γ' be the circumcircles of triangles ABD and ACD respectively, and let P, Q be the intersections of AD with the common tangents to Γ, Γ' respectively.

Prove that $PQ^2 = AB \cdot AC$.

Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.

We first will prove that, with the notation of the problem, calling t the length of the segment of common tangent to both circles (AD the common chord),

$$PQ^2 \;=\; AD^2 + t^2\,. \tag{1}$$

It is easy to see that P is the midpoint of the common tangent segment (for example, calculating the power of P with respect to both circles in two forms: if L, N are the points of tangency, then $PL^2 = PA \cdot PD = PN^2 \Longrightarrow PL = PN$.)

Then calling $x = AP = QD$, we have

$$PQ \;=\; AD + 2x \Longrightarrow PQ^2 \;=\; AD^2 + 4x(AD + x)\,,$$

and as

$$\left(\frac{t}{2}\right)^2 \;=\; x(AD + x)\,,$$

the formula (1) follows.

As AD is the internal bisector of angle A, we have the well known formulae

$$AD \;=\; \frac{2bc}{b+c}\cos\frac{A}{2}\,, \tag{2}$$

$$AD^2 \;=\; cb\left[1 - \frac{a^2}{(b+c)^2}\right]\,. \tag{3}$$

Therefore, we must prove that, if AD is the bisector of the angle A,

$$AD^2 + t^2 \;=\; cb \qquad \text{or, using (3),} \quad t \;=\; \frac{a\sqrt{bc}}{b+c}\,. \tag{4}$$

To this end, we first find the radius R_1 of the circle ABD with aid of the Sine Law:

$$\frac{BD}{\sin\frac{A}{2}} \;=\; 2R_1 \Longleftrightarrow R_1 \;=\; \frac{ab}{2(b+c)\sin\frac{A}{2}}\,, \tag{5}$$

and analogously the radius of the circle ADC:

$$R_2 = \frac{ab}{2(b+c)\sin\frac{A}{2}}. \tag{6}$$

If O_1, O_2 are the centres of these circles, we have

$$t^2 = \overline{O_1O_2}^2 - (R_1 - R_2)^2, \tag{7}$$

and, calling $M = O_1O_2 \cap AD$, we obtain:

$$\overline{O_1M}^2 = R_1^2 - \frac{AD^2}{4}; \quad \overline{O_2M}^2 = R_2^2 - \frac{AD^2}{4},$$

from which, using (2), (5) and the Sine Law, we find

$$\overline{O_1M}^2 = \frac{c^2(a^2 - b^2\sin^2 A)}{4(b+c)^2\sin^2\frac{A}{2}} = \frac{c^2a^2\cos^2 B}{4(b+c)^2\sin^2\frac{A}{2}};$$

that is, $O_1M = \dfrac{ca\cos B}{2(b+c)\sin\frac{A}{2}}$, and analogously, $O_2M = \dfrac{ba\cos C}{2(b+c)\sin\frac{A}{2}}$, from which results

$$O_1O_2 = \frac{a^2}{2(b+c)\sin\frac{A}{2}}.$$

Going back to (7), we obtain

$$t^2 = \frac{a^2[a^2 - (c-b)^2]}{4(b+c)^2\sin^2\frac{A}{2}} = \frac{a^2(a+c-b)(a-c+b)}{4(b+c)^2\sin^2\frac{A}{2}} = \frac{a^2bc}{(b+c)^2},$$

which is precisely (4), and the problem is solved.

Also solved by NIKOLAOS DERGIADES, Thessaloniki, Greece; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer. There was one incorrect solution.