# SOLUTIONS

*No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.*

**2223**. [1997: 111] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

We are given a bag with $n$ identical bolts and $n$ identical nuts, which are to be used to secure the $n$ holes of a gadget.

The $2n$ pieces are drawn from the bag at random one by one. Throughout the draw, bolts and nuts are screwed together in the holes, but if the number of bolts exceeds the number of available nuts, the bolt is put into a hole until one obtains a nut, whereas if the number of nuts exceeds the number of bolts, the nuts are piled up, one on top of the other, until one obtains a bolt.

Let $L$ denote the discrete random variable which measures the height of the pile of nuts.

Find $E[L] + E[L^2]$.

*Solution by Gerry Leversha, St. Paul's School, London, England.*

[*Editor's comment: Leversha noticed the typographical error in the original statement of the problem and recognized correctly that what was required was $E[L] + E[L^2]$. He found the equivalent: $E[L^2 + L]$.*]

Denote the sequences of bolts and nuts by 0's and 1's respectively; thus, for $n = 2$ there are six possible sequences: $1100$, $1010$, $1001$, $0110$, $0101$, and $0011$, and, in general, there are $\binom{2n}{n}$ different sequences. Call the maximum height of the pile of nuts represented by such a sequence the **value** of the sequence. In the sequences above the heights are respectively $2, 1, 1, 1, 0, 0$.

We approach this problem by deriving a probability generating function $G_n(x)$ for the case of $n$ nuts and $n$ bolts. Define $g_n(x) = \binom{2n}{n} G_n(x)$; then $g_n(x) = \sum_{r=0}^{n} a_r^n x^r$, where $a_r^n$ is the number of sequences of length $2n$ which have value $r$. The first few such polynomials are as follows:

$$\begin{aligned}
g_0(x) &= 1\\
g_1(x) &= x + 1\\
g_2(x) &= x^2 + 3x + 2\\
g_3(x) &= x^3 + 5x^2 + 9x + 5\\
g_4(x) &= x^4 + 7x^3 + 20x^2 + 28x + 14
\end{aligned}$$

Notice in each case that the sum of the coefficients is $g_n(1) = \binom{2n}{n}$, as required. In fact, the polynomials can be calculated from the following inductive definition:

$$xg_{n+1}(x) = (x+1)^2 g_n(x) - (x+1)g_n(0) \qquad (*)$$

starting with $g_0(x) = 1$. It is easy to check that $g_n(x)$ is indeed a polynomial of degree $n$.

$(*)$ follows by virtue of the following relationships between the coefficients:

$$\begin{aligned}
a_{n+1}^{n+1} &= a_n^n \\
a_n^{n+1} &= 2a_n^n + a_{n-1}^n \\
a_r^{n+1} &= a_{r+1}^n + 2a_r^n + a_{r-1}^n \qquad (1 \le r \le n-1) \\
a_0^{n+1} &= a_1^n + a_0^n
\end{aligned}$$

The first of these is immediate, since it merely states that the leading coefficient is 1, and there is obviously only one way to obtain a value of $n+1$ in an $n \pm 1$ sequence, namely by having all the 1's at the front.

The second statement follows because a value of $n$ can be obtained either by placing 10 in front of the unique sequence of value $n$, or by placing 01 in front of the sequence of value $n$, or by placing 1 in a sequence of value $n-1$ in front of that part of the sequence which creates the value.

The third statement is the most difficult to see. It states that a sequence of value $r$, where $r$ is at least 1 and at most $n-1$, can be made either by placing 01 or 10 in front of an existing $r$ sequence, or by devaluing an existing $r+1$ sequence by placing an extra 0 in front of the part which matters, and an extra 1 at the end, or by adding 1 to an existing $r-1$ sequence.

The final statement says that a 0 sequence can be created either by putting 01 in front of an existing 0 sequence or by devaluing an existing 1 sequence by placing an extra 0 in front of the critical part.

These claims are best checked by looking at, say, the sequences for $n = 3$ and seeing how the sequences for $n = 4$ are constructed.

We can now proceed to a calculation of $E[L+L^2]$; this is done by finding the value of $G_n''(1) + 2G_n'(1) = \dfrac{g_n''(1) + 2g_n'(1)}{g_n(1)}$. In fact, we can now make the following claim: $E[L^2 + L] = n$.

This is proved using induction on $n$. It is trivially true for $n = 0$, so we assume that it is true for $n = k$; that is, we assume that

$$g_k''(1) + 2g_k'(1) = kg_k(1).$$

By differentiating twice the defining equation for $g_{k+1}(x)$, we have

$$\begin{aligned}
xg_{k+1}'(x) + g_{k+1}(x) &= (x+1)^2 g_k'(x) + 2(x+1)g_k(x) - g_k(0), \\
xg_{k+1}''(x) + 2g_{k+1}'(x) &= (x+1)^2 g_k''(x) + 4(x+1)g_k'(x) + 2g_k(x).
\end{aligned}$$

Hence, putting $x = 1$, we have

$$\begin{aligned} g''_{k+1}(1) + 2g'_{k+1}(1) &= 4g''_k(1) + 8g'_k(1) + 2g_k(1) \\ &= (4k+2)g_k(1) \quad \text{(by the inductive hypothesis)} \end{aligned}$$

Hence, for $n = k+1$, $\quad E[L^2 + L] = \dfrac{(4k+2)g_k(1)}{g_{k+1}(1)}$.

However, we know that $g_k(1) = \dbinom{2k}{k}$, and so

$$E[L^2 + L] = \frac{(4k+2)(2k)!(k+1)!(k+1)!}{k!k!(2k+2)!},$$

and it is straightforward to check that this reduces to $k+1$. This finishes the induction step and establishes the claim.

*Also solved by the proposer (by a completely different method).*

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**2227**. [1997: 166] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Evaluate

$$\prod_p \left[ \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2p)^{2k}} \right].$$

where the product is extended over all prime numbers.

*Composite solution by David Doster, Choate Rosemary Hall, Wallingford, Connecticut, USA and Kee-Wai Lau, Hong Kong.*

It is well known that for $|x| < \frac{1}{4}$, we have

$$\sum_{k=0}^{\infty} \binom{2k}{k} x^k = (1-4x)^{-1/2}, \text{ and hence, } \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2p)^{2k}} = \left(1 - \frac{1}{p^2}\right)^{-1/2}.$$

It is also known that $\zeta(s) = \displaystyle\prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$ for $s > 1$, where $\zeta$ is the Riemann Zeta function, and that $\zeta(2) = \frac{\pi^2}{6}$. (See, for example, Hardy and Wright, *An Introduction to the Theory of Numbers*, 5$^{\text{th}}$ edition, p. 246.) Therefore,

$$\prod_p \left( \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2p)^{2k}} \right) = \prod_p \left(1 - \frac{1}{p^2}\right)^{-1/2} = (\zeta(2))^{1/2} = \frac{\pi}{\sqrt{6}}.$$

*Also solved by MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CON AMORE*

*PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; RICHARD I. HESS, Rancho Palos Verdes, California, USA; ROBERT B. ISRAEL, University of British Columbia, Vancouver, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.*

*Most of the submitted solutions are similar to, or virtually the same as, the one given above. By using the same argument, one can easily show that the value of the slightly more general product* $P(\lambda) = \prod_p \left(\frac{\binom{2k}{k}}{(4p^\lambda)^k}\right)$ *is* $(\zeta(\lambda))^{1/2}$. *This was pointed out by Janous and the proposer.*

**2228**. [1997: 167] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Let $A$ be the set of all real numbers from the interval $(0,1)$ whose decimal representation consists only of $1$'s and $7$'s; that is, let

$$A = \left\{\sum_{k=1}^{\infty} \frac{a_k}{10^k} : a_k \in \{1,7\}\right\}.$$

Let $B$ be the set of all reals that cannot be expressed as finite sums of members of $A$. Find $\sup B$.

*Solution by the proposer.*

We prove that $\sup B = 1$.

Fix $a \in [1,7]$ and consider $b = (a-1)/6$. Since $b \in [0,1]$, $b$ can be easily expressed as a sum of nine reals from $[0,1]$ each consisting only of 0's and 1's in decimal representation. Now multiplying each by $6$ and next adding $.1111\ldots$ to the result, we obtain (since $1 = 9 \times .1111\ldots$) an expression for $a$ as a sum of nine reals, each consisting only of $1$'s and $7$'s.

Now take any $a > 7$ and subtract from $a$ as many $.1111\ldots$'s as needed to get into the interval $[1,7]$. This procedure shows that any $a \geq 1$ is a finite sum of members of $A$; that is, $\sup B \leq 1$.

Now let $x \in (8/9, 1)$ be a finite sum of members of $A$ (so that $x \notin B$). Fixing our attention, assume that $x$ is a sum of $k$ members of $A$; since $x \in (8/9,1)$ and each member of $A$ is at least $.1111\ldots$, we know that $2 \leq k \leq 8$. Consider

$$y = \frac{x - k(.1111\ldots)}{6}.$$

Then $y$ is a sum of $k$ real numbers, each consisting only of $1$'s and $0$'s. This implies, since $k \leq 8$, that $y$ does not contain a digit $9$ in its decimal representation. Thus we have shown that a number $x \in (8/9,1)$ must be in $B$ if the following property holds:

(*): *all seven numbers* $\frac{1}{6}(x - k(.1111\dots))$, $k = 2, 3, \dots, 8$, *contain a* 9 *in their decimal representation.*

Denote the interval

$$\left(\frac{8-k}{54}, \frac{9-k}{54}\right)$$

by $J_k$ ($k = 2, 3, \dots, 8$). The set $C_k$ of all numbers from $J_k$ having a unique decimal representation and containing at least one 9 is easily seen to be open and dense in $J_k$. [*Editorial note*. For the benefit of readers not familiar with these terms, here are a couple of explanations. "$C_k$ is open" means that for any $c \in C_k$, there is some interval $(c - \epsilon, c + \epsilon)$ $(\epsilon > 0)$ which is completely contained in $C_k$; that is, all numbers close enough to $c$ also have a 9 in their decimal representation. "$C_k$ is dense in $J_k$" means that for any number $y \in J_k$, and for any $\epsilon > 0$, the interval $(c - \epsilon, c + \epsilon)$ must contain some member of $C_k$; that is, there are numbers with at least one digit 9 as close as you like to $y$, even if $y$ itself does not have a 9 in it.]

For each $k = 2, 3, \dots, 8$, the map

$$\varphi_k(x) = \frac{x - k(.1111\dots)}{6} = \frac{9x - k}{54}$$

is increasing and continuous and takes $(8/9, 1)$ onto $J_k$. Therefore the sets $\varphi_k^{-1}(C_k)$, $k = 2, 3, \dots, 8$ are open and dense in $(8/9, 1)$, and thus so is the intersection

$$\varphi_2^{-1}(C_2) \cap \varphi_3^{-1}(C_3) \cap \dots \cap \varphi_8^{-1}(C_8).$$

[*Editorial note*. In other words, the intersection of finitely many open and dense sets is open and dense. Proof left to the reader! Or look at problem 16, Chapter 2 of Rudin's *Principles of Mathematical Analysis*.] But all elements $x$ in this intersection have the property (*), and thus lie in $B$, which means that $B$ is dense in $(8/9, 1)$. This completes the proof that $\sup B = 1$.

*Remark*. In the last part of the proof we have used Baire's Theorem, in fact its easier version for the intersection of finitely many open and dense sets instead of countably many. Also, the argument used in the solution shows that there are plenty of elements of $B$ in $(8/9, 1)$! So I believe that a reader not familiar with topology might find an explicit example of a sequence of elements of $B$ which get arbitrarily close to 1.

*Also solved by GERALD ALLEN, CHARLES DIMINNIE, TREY SMITH and ROGER ZARNOWSKI (jointly), Angelo State University, San Angelo, Texas, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; and DAVID STONE and VREJ ZARIKIAN, Georgia Southern University, Statesboro, Georgia, USA. Two other readers sent in incomplete solutions.*

*The other solutions to this problem actually contain the explicit examples that the proposer asks for in his remark above. Allen, Diminnie, Smith and Zarnowski give* $b_n = .99\dots944999$, *where there are* $(3n + 1)$ 9*'s before the two* 4*'s. Stone and Zarikian use* $b_n = .99\dots9546$, *where there are* $(3n - 2)$ 9*'s before the* 546.

*Lambrou uses somewhat more complicated $b_n$'s. In all cases the solvers then must do some calculation to show that the $b_n$'s all lie in $B$.*

**2231**. [1997: 167] *Proposed by Herbert Gülicher, Westfalische Wilhelms-Universität, Münster, Germany.*

In quadrilateral $P_1P_2P_3P_4$, suppose that the diagonals intersect at the point $M \neq P_i$ $(i = 1,2,3,4)$. Let $\angle MP_1P_4 = \alpha_1$, $\angle MP_3P_4 = \alpha_2$, $\angle MP_1P_2 = \beta_1$ and $\angle MP_3P_2 = \beta_2$.

Prove that

$$\lambda_{13} := \frac{|P_1M|}{|MP_3|} = \frac{\cot\alpha_1 \pm \cot\beta_1}{\cot\alpha_2 \pm \cot\beta_2},$$

where the $+(-)$ sign holds if the line segment $P_1P_3$ is located inside (outside) the quadrilateral.

*Preliminary comment.* No solver made use of directed angles and segments, which would have reduced the problem to a single case. The proposer's angles are assumed to be positive, and nobody explicitly went through all four resulting possibilities: $M$ can lie between $P_1$ and $P_3$ or not, and between $P_2$ and $P_4$ or not. Since $P_1P_2P_3P_4$ need not have an interior, a more careful statement of the proposer's two cases would distinguish whether or not $P_2$ and $P_4$ lie on opposite sides of the line $P_1P_3$. We shall feature the opposite-side case, and leave the other to the reader.

*Solution by Michael Lambrou, University of Crete, Crete, Greece* (*modified by the editor to explicitly distinguish the two subcases*).

Assume $M$ to be any point of the line $P_1P_3$ except $P_1, P_3$, and let $P_2P_4$ be any other line through $M$ with $P_2, P_4$ on opposite sides of $M$. Let $\omega = \angle P_1MP_4$. Then $\sin\angle P_1P_4M = \sin(\pi-(\alpha_1+\omega)) = \sin(\alpha_1+\omega)$, while $\sin\angle P_3P_4M = \sin(\omega\pm\alpha_2)$ where '$+$' corresponds to the subcase where $M$ is outside the segment $P_1P_3$, while '$-$' corresponds to where $M$ is between $P_1$ and $P_3$. From the sine rule applied to triangles $MP_1P_4$ and $MP_3P_4$ we have

$$P_1M\frac{\sin\alpha_1}{\sin(\alpha_1+\omega)} = MP_4 = MP_3\frac{\sin\alpha_2}{\sin(\omega\pm\alpha_2)}.$$

Hence,

$$\begin{aligned}\frac{P_1M}{MP_3} &= \frac{\sin\alpha_2\sin(\alpha_1+\omega)}{\sin(\omega\pm\alpha_2)\sin\alpha_1} \\ &= \frac{\sin\alpha_2\sin\alpha_1\cos\omega + \sin\alpha_2\cos\alpha_1\sin\omega}{\sin\omega\cos\alpha_2\sin\alpha_1 \pm \cos\omega\sin\alpha_2\sin\alpha_1} \\ &= \frac{\cot\omega + \cot\alpha_1}{\cot\alpha_2 \pm \cot\omega} \qquad (1)\end{aligned}$$

(where we divided both the numerator and the denominator by the product of three sines to get the last equality).

Similarly from triangles $MP_1P_2$ and $MP_3P_2$ we have

$$\begin{aligned}\frac{P_1M}{MP_3} &= \frac{\sin\beta_2\sin(\omega-\beta_1)}{\sin\beta_1\sin(\omega\mp\beta_2)}\\ &= \frac{\cot\beta_1+\cot\omega}{\cot\beta_2\mp\cot\omega}.\end{aligned} \tag{2}$$

The result follows by combining (1) and (2) using the following property of proportions: if $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$ then $\frac{a}{b}=\frac{(c+e)}{(d+f)}$.

*Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; ISTVÁN REIMAN, Budapest, Hungary; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.*

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**2232**. [1997: 168] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Find all solutions of the inequality:

$$n^2+n-5<\left\lfloor\frac{n}{3}\right\rfloor+\left\lfloor\frac{n+1}{3}\right\rfloor+\left\lfloor\frac{n+2}{3}\right\rfloor<n^2+2n-2,\ (n\in\mathbb{N}).$$

(Note: If $x$ is a real number, then $\lfloor x\rfloor$ is the largest integer not exceeding $x$.)

*Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

The only solution is $n=2$. Let

$$M=\left\lfloor\frac{n}{3}\right\rfloor+\left\lfloor\frac{n+1}{3}\right\rfloor+\left\lfloor\frac{n+2}{3}\right\rfloor.$$

Then

$$M\le\frac{n}{3}+\frac{n+1}{3}+\frac{n+2}{3}=n+1.$$

Hence $n^2+n-5<n+1$, or $n^2<6$, and so $n=1$ or $2$. But when $n=1$, $M=1=n^2+2n-2$, and so $n=1$ is not a solution. By inspection, $n=2$ is a solution.

*Also solved by CHARLES ASHBACHER, Cedar Rapids, Iowa, USA; MICHEL BATAILLE, Rouen, France; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, Oregon, USA; MICHAEL*

*LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; SEAM MCILROY, student, Vancouver, BC; ANNE MARTIN, Westmont College, Santa Barbara, California, USA; CAN ANH MINH, Berkeley, California, USA; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; CHRISTOS SARAGOTIS, student, Aristotle University, Thessaloniki, Greece; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; REZA SHAHIDI, student, University of Waterloo, Waterloo, Ontario, Canada; D.J. SMEENK, Zaltbommel, the Netherlands; DIGBY SMITH, Mount Royal College, Calgary, Alberta; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; and the proposer. There were also three incomplete and one incorrect solution submitted.*

*Some solvers initially prove the equality*

$$\left\lfloor\frac{n}{3}\right\rfloor+\left\lfloor\frac{n+1}{3}\right\rfloor+\left\lfloor\frac{n+2}{3}\right\rfloor=n,$$

*or some more general form of it. The following general form is well-known.*

$$\sum_{i=0}^{n-1}\left\lfloor x+\frac{i}{n}\right\rfloor=\lfloor nx\rfloor \text{ for } n\in N, n\neq 0, \text{ and } x\in R$$

*(See for example, D.O. Shklarsky, N.N. Chentzov, and I.M. Yaglom, The USSR Olympiad Problem Book, W.H. Freeman and Company, 1962, p. 24, problem 101, (3).)*

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**2233**. [1997: 168] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $x$, $y$, $z$ be non-negative real numbers such that $x+y+z=1$, and let $p$ be a positive real number.

(a) If $0<p\leq 1$, prove that

$$x^p+y^p+z^p \;\geq\; C_p\left((xy)^p+(yz)^p+(zx)^p\right),$$

where

$$C_p \;=\; \begin{cases} 3^p & \text{if} \quad p\leq\frac{\log 2}{\log 3-\log 2}, \\ 2^{p+1} & \text{if} \quad p\geq\frac{\log 2}{\log 3-\log 2}. \end{cases}$$

(b)$^\star$ Prove the same inequality for $p>1$.

Show that the constant $C_p$ is best possible in all cases.

*Solution by G.P. Henderson, Campbellcroft, Ontario.*

(a) We are to prove that $F(x,y,z)\geq 0$ where

$$F=\sum x^p-C\sum x^py^p$$

(the sums here and below are cyclic over $x,y,z$). Here

$$C=\min(3^p,2^{p+1})=3^p$$

for $0 < p \leq 1$. For any real $u, v, w$,

$$uv + vw + wu \leq \frac{1}{3}(u+v+w)^2.$$

[*Editorial note*. This follows from the identity

$$(u+v+w)^2 = 3(uv+vw+wu) + \tfrac{1}{2}\left[(u-v)^2 + (v-w)^2 + (w-u)^2\right],$$

as other solvers point out.] Therefore

$$F \geq \sum x^p - \frac{C}{3}\left(\sum x^p\right)^2 = \left(\sum x^p\right)\left(1 - 3^{p-1}\sum x^p\right) \geq 0,$$

the last inequality following since

$$\max \sum x^p = 3^{1-p}.$$

[For example, since $p \leq 1$,

$$\left(\frac{x^p + y^p + z^p}{3}\right)^{1/p} \leq \frac{x+y+z}{3} = \frac{1}{3}$$

by the power mean inequality. — *Ed.*] This is the best possible $C$ because $F(1/3, 1/3, 1/3) = 0$.

(b) With no loss of generality we assume $x \geq y \geq z$, which implies

$$z \leq \frac{1}{3} \qquad \text{and} \qquad x \geq \frac{1-z}{2}.$$

Give $z$ a fixed value in $[0, 1/3]$ and set $y = 1 - x - z$ [so that $dy/dx = -1$]. Then $F$ is a function of $x$ only and

$$\begin{aligned}\frac{dF}{dx} &= px^{p-1} - py^{p-1} - Cpx^{p-1}y^p + Cx^p py^{p-1} + Cpy^{p-1}z^p - Cpz^p x^{p-1} \\ &= p(x^{p-1} - y^{p-1})(1 - Cz^p) + pCx^{p-1}y^{p-1}(x-y).\end{aligned}$$

Since $C \leq 3^p$ and $z^p \leq 3^{-p}$ (and $p > 1$), we see that $dF/dx \geq 0$ and we only need to prove $F \geq 0$ when $x$ has its minimum value, $(1-z)/2$. That is, we are to prove $F(x, x, z) \geq 0$ where $x = (1-z)/2 = y$ and $0 \leq z \leq 1/3$; that is,

$$2x^p + z^p - C(x^{2p} + 2x^p z^p) \geq 0.$$

This is equivalent to showing that

$$\min_{0 \leq z \leq 1/3} \frac{2x^p + z^p}{x^{2p} + 2x^p z^p} \geq C.$$

Calling the function to be minimized $G(z)$, we find [using $dx/dz = -1/2$]

$$\begin{aligned}\operatorname{sgn}\left(\frac{dG}{dz}\right) &= \operatorname{sgn}\Big((x^{2p}+2x^pz^p)(-px^{p-1}+pz^{p-1}) \\ &\qquad -(2x^p+z^p)(-px^{2p-1}-px^{p-1}z^p+2px^pz^{p-1})\Big) \\ &= \operatorname{sgn}\left(px^{p-1}z^{2p}+px^{2p-1}z^p-3px^{2p}z^{p-1}+px^{3p-1}\right) \\ &= \operatorname{sgn}\left[\left(\frac{z}{x}\right)^{2p}+\left(\frac{z}{x}\right)^{p}-3\left(\frac{z}{x}\right)^{p-1}+1\right].\end{aligned}$$

Set

$$t = \frac{z}{x} = \frac{2z}{1-z}\,;$$

that is,

$$z = \frac{t}{t+2}\,, \qquad 0 \le t \le 1.$$

As $t$ increases from $0$ to $1$, $z$ increases from $0$ to $1/3$. Then $\operatorname{sgn}(dG/dz) = \operatorname{sgn} H(t)$ where

$$H(t) = t^{2p}+t^p-3t^{p-1}+1.$$

We have

$$\frac{dH}{dt} = t^{p-2}(2pt^{p+1}+pt-3p+3).$$

The expression in brackets is a continuous, increasing function. It is negative at $t=0$ and positive at $t=1$. Therefore it has a unique root, $t_0$, in $[0,1]$. It follows that $H$ decreases to a minimum at $t_0$ then increases. Since $H(0)=1$ and $H(1)=0$, $H(t_0)<0$ and we see that $H$ has a unique root $t_1$ in $(0,t_0)$. Therefore

$$\frac{dG}{dz} \ge 0 \quad \text{for} \quad 0 \le t \le t_1 \qquad \text{and} \qquad \frac{dG}{dz} \le 0 \quad \text{for} \quad t_1 \le t \le 1.$$

Hence, putting $z_1 = t_1/(t_1+2)$, $G$ increases in $0 \le z \le z_1$, is a maximum at $z_1$ and decreases in $z_1 \le z \le 1/3$. Therefore

$$\min G = \min[G(0), G(1/3)] = \min(2^{p+1}, 3^p) = C.$$

Since the minimum is attained in both cases, $C$ is best possible.

*Both parts also solved by RICHARD I. HESS, Rancho Palos Verdes, California, USA; and MICHAEL LAMBROU, University of Crete, Crete, Greece. Part* (a) *only solved by HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. One other reader sent in a comment.*

*Seiffert and the proposer had the same proof for part* (a) *as Henderson (and incidentally, the proof seems to hold for negative $p$ as well). For part* (b) *both Hess and Lambrou used multivariable calculus.*

**2234**. [1997: 168] *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Given triangle $ABC$, its centroid $G$ and its incentre $I$, construct, using only an unmarked straightedge, its orthocentre $H$.

*Solution by the proposer edited to fit the rewording of the problem from his original submission.*

We will first establish a lemma:

**Lemma**. Given a line segment and its midpoint, and any other point $O$ in the plane, a line passing through $O$ and parallel to the given line segment may be constructed using only an unmarked straightedge.

Proof: Let $AD$ be the given line segment and let $K$ be its midpoint. Let $P$ be any point on $AO$ extended. Connect $P$ with $D$ and with $K$. Draw segment $OD$. Let $M$ be the intersection of $OD$ and $PK$, and let $N$ be the intersection of $AM$ with $PD$. We will show that $ON$ is parallel to $AD$. Suppose instead that $ON_1$ is parallel to $AD$ with $N_1$ on $PD$. Let $M_1$ be the intersection of $AN_1$ and $OD$, let $K_1$ be the intersection of $PM_1$ and $AD$, and let $L_1$ be the intersection of $ON_1$ with $PK_1$. Then $\triangle AM_1K_1 \sim \triangle N_1M_1L_1$ and $\triangle OM_1L_1 \sim \triangle DM_1K_1$. Therefore,

$$\frac{AK_1}{L_1N_1} = \frac{K_1M_1}{M_1L_1} = \frac{K_1D}{OL_1}. \tag{1}$$

Also $\triangle APK_1 \sim \triangle OPL_1$ and $\triangle DK_1P \sim \triangle N_1L_1P$, which implies

$$\frac{AK_1}{OL_1} = \frac{K_1P}{L_1P} = \frac{K_1D}{L_1N_1}. \tag{2}$$

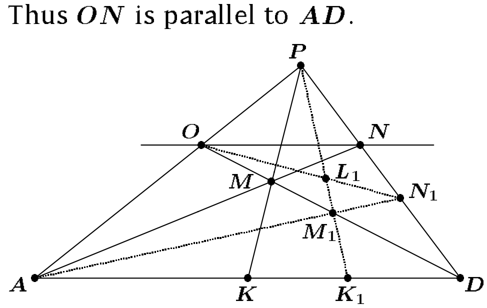
From (1) and (2) it follows that

$$AK_1 \cdot OL_1 = L_1N_1 \cdot K_1D, \qquad AK_1 \cdot L_1N_1 = OL_1 \cdot K_1D.$$

Consequently,

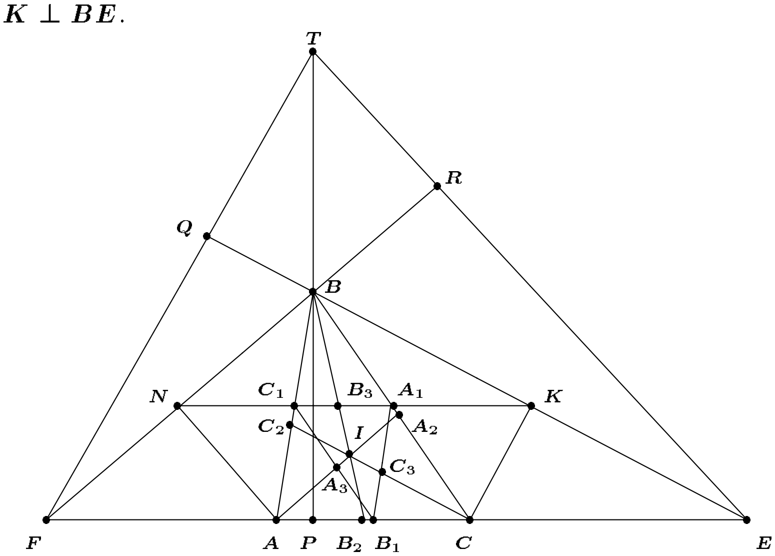
$$(AK_1)^2 \cdot OL_1 \cdot L_1N_1 = (K_1D)^2 \cdot OL_1 \cdot L_1N_1\,,$$

from which we have $(AK_1)^2 = (K_1D)^2$, and $AK_1 = K_1D$. Thus point $K_1$ coincides with point $K$, $PK_1$ coincides with $PK$, point $M_1$ with $M$ and point $N_1$ with $N$. Thus $ON$ is parallel to $AD$.



Using the centroid $G$ draw the medians $AA_1$, $BB_1$, $CC_1$, where $A_1$, $B_1$, $C_1$ are the midpoints of $BC$, $CA$, $AB$, respectively. Similarly, using the incentre $I$ draw the angle bisectors $AA_2$, $BB_2$, $CC_2$, where $A_2$, $B_2$, $C_2$ lie on $BC$, $CA$, $AB$, respectively. Draw triangle $A_1B_1C_1$ and denote $C_1A_1 \cap BB_2$ by $B_3$, $A_1B_1 \cap CC_2$ by $C_3$, $B_1C_1 \cap AA_2$ by $A_3$. Thus $A_3$, $B_3$, $C_3$ are the midpoints of $AA_2$, $BB_2$, $CC_2$, respectively.

By applying the lemma twice we may draw a straight line $BE$ through $B$ parallel to angle bisector $CC_2$ and draw a straight line $BF$ parallel to angle bisector $AA_2$ (with points $E$ and $F$ located on the line $CA$). We see that triangles $FAB$ and $BCE$ are isosceles ($\angle FBA = \angle BFA$, $\angle EBC = \angle BEC$). Draw a straight line $A_1K$ parallel to $CA$ with $K$ on $BE$. Thus $K$ is the midpoint of $BE$. Similarly get the midpoint $N$ of $FB$. Thus $AN \perp FB$ and $CK \perp BE$.




Now we may draw lines parallel to $FB$ and $BE$ through the points $C_1$ and $A_1$ respectively, and get the midpoints of $AN$ and $CK$. Then applying the lemma we may draw $FT$ parallel to $CK$ and $ET$ parallel to $AN$. Denote $ET \cap FB$ by $R$ and $BE \cap FT$ by $Q$. Thus $FR \perp ET$, $EQ \perp FT$ and point $B$ is the point of intersection of the altitudes of triangle $FTE$. Draw line $TB$, denote $TB \cap CA$ by $P$ and get $BP \perp CA$; that is, $BP$ is an altitude of triangle $ABC$. We can construct a second altitude similarly. Their intersection is $H$, the orthocentre.

*Also solved or answered by JORDI DOU, Barcelona, Spain and TOSHIO SEIMIYA, Kawasaki, Japan. There was one incorrect solution.*

**2235**. [1997: 168] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands, not Walther Janous, Ursulinengymnasium, Innsbruck, Austria as was printed.*

Triangle $ABC$ has angle $\angle CAB = 90°$. Let $\Gamma_1(O, R)$ be the circumcircle and $\Gamma_2(T, r)$ be the incircle. The tangent to $\Gamma_1$ at $A$ and the polar line of $A$ with respect to $\Gamma_2$ intersect at $S$. The distances from $S$ to $AC$ and $AB$ are denoted by $d_1$ and $d_2$ respectively.

Show that

(a) $ST \| BC$,

(b) $|d_1 - d_2| = r$.

[For the benefit of readers who are not familiar with the term "polar line", we give the following definition as in, for example, *Modern Geometries*, 4th Edition, by James R. Smart, Brooks/Cole, 1994:

*The line through an inverse point and perpendicular to the line joining the original point to the centre of the circle of inversion is called the polar of the original point, whereas the point itself is called the pole of the line.*]

*Solution by István Reiman, Budapest, Hungary.*

Assume without loss of generality that $AB > AC$. [Should $AB = AC$ the polar line and tangent would be parallel so that $S$ would be at infinity, while $AB < AC$ would involve only minor changes in notation.] Let $P$ be the point where $\Gamma_2$ touches $AB$, and $Q$ be where it touches $AC$. Thus $AQTP$ is a square whose sides have length $r$. Next, let $U$ and $V$ be the feet of the perpendiculars from $S$ to $AB$ and to $AC$ respectively, so that $AUSV$ is a rectangle.

The polar of $A$ is the line $PQ$. Since $PQT$ is an isosceles right triangle, $PQ$ makes a $45^o$ angle with $AC$, which implies that $QSV$ is also an isosceles right triangle. Consequently $SV = QV = d_1$, and $AV - SV = d_2 - d_1 = AV - QV = AQ = r$ (which proves (b)).

Now, $\angle QAS = \angle CBA$, since both angles are subtended by the chord $AC$ of the circle $\Gamma_1$. Moreover, $\angle QAS = \angle QTS$ because these angles are symmetric about the line $PQ$. Since the corresponding side vectors $\overrightarrow{TQ}$ and $\overrightarrow{BA}$ have the same direction, it follows that so do the vectors $\overrightarrow{TS}$ and $\overrightarrow{BC}$, and we conclude that the lines $ST$ and $BC$ are parallel as desired.

*Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; ADRIAN BIRKA, student, Lakeshore Catholic High School, Niagara Falls, Ontario; MANSUR BOASE, student, St. Paul's School, London, England;; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; JIMMY CHUI, student, Earl Haig Secondary School, North York, Ontario; FILIP CRNOGORAC, student, Western Canada High School, Calgary, Alberta; JORDI*

*DOU, Barcelona, Spain; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece (2 solutions); JAMES LEE, student, Eric Hamber Secondary School, Vancouver, BC; GERRY LEVERSHA, St. Paul's School, London, England; DAVID NICHOLSON, student, Fenelon Falls Secondary School, Fenelon Falls, Ontario; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; CHRISTOPHER SO, student, Franch Liberman Catholic High School, Scarborough, Ontario; KAREN YEATS, student, St. Patrick's High School, Halifax, Nova Scotia; and the proposer.*

**2236**. [1997: 169] *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Let $ABC$ be an arbitrary triangle and let $P$ be an arbitrary point in the interior of the circumcircle of $\triangle ABC$. Let $K$, $L$, $M$, denote the feet of the perpendiculars from $P$ to the lines $AB$, $BC$, $CA$, respectively.

Prove that $[KLM] \leq \dfrac{[ABC]}{4}$.

Note: $[XYZ]$ denotes the area of $\triangle XYZ$.

*Almost identical solutions were submitted by Niels Bejlegaard, Stavanger, Norway; Mansur Boase, student, St. Paul's School, London, England; István Reiman, Budapest, Hungary; Toshio Seimiya, Kawasaki, Japan; and Panos E. Tsaoussoglou, Athens, Greece.*

For $\triangle KLM$, with $O$ as circumcentre, we have

$$[KLM] \;=\; \frac{R^2 - OP^2}{4R^2}[ABC] \;\leq\; \frac{[ABC]}{4}\,.$$

[Various different references were given.]

*Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.*

**2237**. [1997: 169] *Proposed by Meletis D. Vasiliou, Elefsis, Greece.*

$ABCD$ is a square with incircle $\Gamma$. Let $\ell$ be a tangent to $\Gamma$. Let $A'$, $B'$, $C'$, $D'$ be points on $\ell$ such that $AA'$, $BB'$, $CC'$, $DD'$ are all perpendicular to $\ell$.

Prove that $AA' \cdot CC' = BB' \cdot DD'$.

I. *Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.*

We give a solution with coordinates. Clearly, without loss of generality, we can take $A(-1,1)$, $B(1,1)$, $C(1,-1)$, $D(-1,-1)$, and the equation of the incircle is $x^2+y^2=1$. Then, if $P(\cos t, \sin t)$ is any point of the incircle, the equation of the line $\ell$, tangent to the incircle, is

$$(\cos t)x + (\sin t)y = 1.$$

(The cases in which $\ell$ is parallel to either of the axes are trivial.) Then, calculating

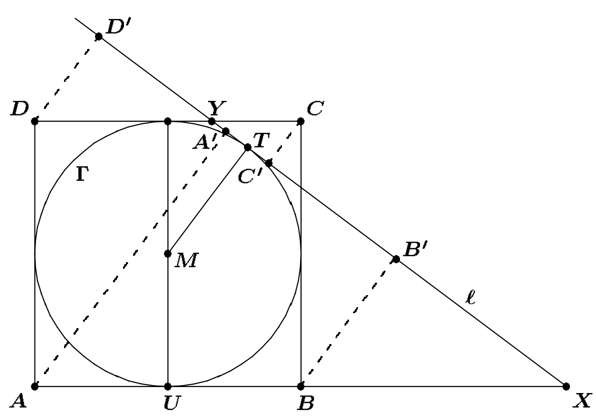
$$\begin{aligned} AA' &= d(A,\ell) = |-\cos t + \sin t - 1| \\ BB' &= d(B,\ell) = |\cos t + \sin t - 1| \\ CC' &= d(C,\ell) = |\cos t - \sin t - 1| \\ DD' &= d(D,\ell) = |-\cos t - \sin t - 1| \end{aligned}$$

we easily obtain

$$AA' \cdot CC' = |\sin 2t| = |-\sin 2t| = BB' \cdot DD',$$

and we are done.

II. *Solution by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.*




First we note that the equation holds obviously if the point of tangency of $\ell$ and $\Gamma$ is the midpoint of a side of the square. Otherwise, by rotational symmetry about the centre $M$ of $ABCD$, we may assume that, without loss of generality, $\ell$ touches the arc of $\Gamma$ "next to $C$", say at $T$. Let $X$ and $Y$ be the points of intersection of $l$ with $AB$ produced and $CD$, respectively, and $U$ and $V$ the midpoints of $AB$ and $CD$, respectively. Note that quadrangles $MUXT$ and $YVMT$ are similar, since their respective angles correspond, and $XU = XT$ and $MV = MT$. Hence it follows that

$$\begin{aligned} \frac{AA'}{BB'} = \frac{AX}{BX} &= \frac{AU+UX}{UX-UB} = \frac{MU+UX}{UX-MU} \\ &= \frac{YV+VM}{VM-YV} = \frac{YV+VD}{CV-YV} = \frac{YD}{CY} = \frac{DD'}{CC'}. \end{aligned}$$

III. *Solution by Heinz-Jürgen Seiffert, Berlin, Germany.*

There exist vectors $\mathbf{E}$ and $\mathbf{F}$ with $\mathbf{E}\cdot\mathbf{F} = 0$ and $\|\mathbf{F}\| = 1$, such that

$$\ell = \{\mathbf{E} + t\mathbf{F} \mid t \in \mathbb{R}\}.$$

We have $\mathbf{A} = x\mathbf{E} + y\mathbf{F}$ and $\mathbf{B} = u\mathbf{E} + v\mathbf{F}$ for some reals $x, y, u, v$. If $\mathbf{A}' = \mathbf{E} + t_A\mathbf{F}$, where $t_A \in \mathbb{R}$, then from $(\mathbf{A} - \mathbf{A}')\cdot\mathbf{F} = 0$, it follows that $t_A = y$, so that $\mathbf{A}' = \mathbf{E} + y\mathbf{F}$. Similarly, we find $\mathbf{B}' = \mathbf{E} + v\mathbf{F}$, and since $\mathbf{C} = -\mathbf{A}$ and $\mathbf{D} = -\mathbf{B}$, we have $\mathbf{C}' = \mathbf{E} - y\mathbf{F}$ and $\mathbf{D}' = \mathbf{E} - v\mathbf{F}$. From $2r^2 = \|\mathbf{A}\|^2 = \|\mathbf{B}\|^2$, where $r$ denotes the radius of $\Gamma$, and $\mathbf{A}\cdot\mathbf{B} = 0$, we obtain

$$2r^2 = x^2r^2 + y^2 = u^2r^2 + v^2 \qquad \text{and} \qquad xur^2 + yv = 0.$$

The first equation gives $v^2 = r^2(2-u^2)$, so that, using the second equation, we have

$$\begin{aligned} x^2u^2r^4 = y^2v^2 &= y^2r^2(2-u^2) \\ \text{or} \qquad u^2(x^2r^2+y^2) &= 2y^2. \end{aligned}$$

Using the first equation again, we get $y^2 = u^2r^2$ and then $2 = x^2 + u^2$, which by $AA' = \|\mathbf{A} - \mathbf{A}'\| = |x-1|r$, $BB' = \|\mathbf{B} - \mathbf{B}'\| = |u-1|r$, $CC' = \|\mathbf{C} - \mathbf{C}'\| = |x+1|r$, and $DD' = \|\mathbf{D} - \mathbf{D}'\| = |u+1|r$, implies the desired equation.

*Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; SAM BAETHGE, Nordheim, Texas, USA; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; DENISE CHEUNG, student, Albert Campbell Collegiate Institute, Scarborough, Ontario; JIMMY CHUI, student, Earl Haig Secondary School, North York, Ontario; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; JORDI DOU, Barcelona, Spain; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, Oregon, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece (4 methods); KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; ISTVÁN REIMAN, Budapest, Hungary; K.R.S. SASTRY, Dodballapur, India; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.*

*Most of the solutions were similar to either* I *or* II *above. Several solvers made generalizations in different directions.*

*Janous considers a regular* $2n$*-gon* $P_{2n} := A_1A_2\cdots A_{2n}$ *with incircle* $\Gamma$ *and* $\ell$*, a tangent line to* $\Gamma$*; then for the orthogonal projections* $A_1', A_2', \ldots, A_{2n}'$ *of the vertices of* $P_{2n}$ *to* $\ell$ *we have*

$$\prod_{\substack{k=1\\k \text{ even}}}^{2n} A_kA_k' = \prod_{\substack{k=1\\k \text{ odd}}}^{2n} A_kA_k' .$$

*Konečný observes that the condition "*$AA'$*,* $BB'$*,* $CC'$*,* $DD'$ *are all* perpendicular *to* $\ell$*" is not necessary; it is sufficient to state that they make the same angle with* $\ell$*; that is, they are parallel to one another.*

*Sastry shows that if we start with* $ABCD$ *being a rhombus with incircle* $\Gamma$*, and with* $A'$*,* $B'$*,* $C'$*,* $D'$ *defined as before, we can prove that the equation holds for all lines* $\ell$ *tangent to* $\Gamma$ *if and only if* $ABCD$ *is a square.*

*Finally, Seimiya comments that if we start with* $ABCD$ *any quadrilateral having an incircle* $\Gamma$ *with centre* $O$*, and with* $\ell$ *being a line tangent to* $\Gamma$*, then when* $A'$*,* $B'$*,* $C'$*,* $D'$ *are the feet of the perpendiculars from* $A$*,* $B$*,* $C$*,* $D$*, respectively, to* $\ell$*, we have*

$$\frac{AA'\cdot CC'}{BB'\cdot DD'} = \frac{AO\cdot CO}{BO\cdot DO}, \quad \text{a constant.}$$

**2238**. [1997: 242] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

A four-digit number $\overline{abcd}$ is said to be *faulty* if it has the following property:

> The product of the two last digits $c$ and $d$ equals the two-digit number $\overline{ab}$, while the product of the digits $c-1$ and $d-1$ equals the two digit number $\overline{ba}$.

Determine all faulty numbers!

*Solution by David R. Stone, Georgia Southern University, Statesboro, Georgia, USA.*

Implicit in the defining properties is that $a \neq 0$, $c \geq 1$, $d \geq 1$, and $b \neq 0$. Translating the given conditions, if $\overline{abcd}$ is to be faulty we must have

$$c\cdot d = \overline{ab} = 10a + b, \tag{1}$$

and

$$(c-1)\cdot(d-1) = \overline{ba} = 10b + a.$$

These imply

$$10b + a = cd - d - c + 1 = 10a + b - d - c + 1,$$

so

$$9(a-b)=c+d-1.$$

Thus $1 \leq 9(a-b) \leq 17$, forcing $9(a-b)=9$, or $a=b+1$. Hence $c+d=10$. Substituting into (1), we get $c(10-c)=10(b+1)+b$, or

$$c^2-10c+(11b+10)=0.$$

By the quadratic formula, $c=5 \pm \sqrt{15-11b}$, which forces $15-11b=4$, or $b=1$. Thus $a=2$ and $c=7$ or $c=3$, which forces $d=3$ or $d=7$, respectively. That is, the only two faulty numbers are

**2137** and **2173.**

(Checking, $3 \times 7 = 21$ and $(3-1) \times (7-1) = 12$.)

*Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHARLES ASHBACHER, Cedar Rapids, Iowa, USA; SAM BAETHGE, Nordheim, Texas, USA; FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts, USA; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; CON AMORE PROBLEM GROUP, The Royal Danish School of Educational Studies, Copenhagen, Denmark; MAYUMI DUBREE, student, Angelo State University, San Angelo, Texas, USA; KEITH EKBLAW, Walla Walla, Washington, USA; JEFFREY K. FLOYD, Newnan, Georgia, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, Oregon, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; J.A. MCCALLUM, Medicine Hat, Alberta; GRADY MYDLAK, student, University College of the Cariboo, Kamloops, BC; CHRISTOS SARAGIOTIS, student, Aristotle University, Thessaloniki, Greece; JOEL SCHLOSBERG, student, Robert Louis Stevenson School, New York, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; REZA SHAHIDI, student, University of Waterloo, Waterloo, Ontario; SKIDMORE COLLEGE PROBLEM GROUP, Saratoga Springs, New York, USA; D.J. SMEENK, Zaltbommel, the Netherlands; DIGBY SMITH, Mount Royal College, Calgary, Alberta; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.*

*Although nobody raised the issue, readers must have wondered why the proposer chose the term "faulty" for numbers satisfying this condition. He reveals his reason at the end of his solution with the following remark: "**Crux** problems 2137 and 2173 both have been corrected in **Crux**, 2137 even twice! They are not faulty anymore!" (See [1996: 317] and [1997: 48] for 2137, and [1997: 169] for 2173.)*

**2239**. [1997: 242] *Proposed by Kenneth Kam Chiu Ko, Mississauga, Ontario.*

Suppose that $1 \le r \le n$ and consider all subsets of $r$ elements of the set $\{1, 2, 3, \ldots, n\}$. The elements of these subsets are arranged in ascending order of magnitude. For $i$ from $1$ to $r$, let $t_i$ denote the $i^{\text{th}}$ smallest element in the subset. Let $T(n, r, i)$ denote the arithmetic mean of the elements $t_i$.

Prove that $T(n, r, i) = i\,\dfrac{n+1}{r+1}$.

*Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let $X(n) = \{1, 2, \ldots, n\}$. Since there are $\binom{n}{r}$ subsets of $X(n)$ with $r$ elements,

$$T(n, r, i) = \frac{\sum t_i}{\binom{n}{r}},$$

where the summation is over all $r$-subsets. Thus we are to show that

$$\sum t_i = i\frac{n+1}{r+1}\binom{n}{r} = i\binom{n+1}{r+1}. \tag{1}$$

For each $k \in X(n)$, let $u_k$ denote the number of $r$-subsets

$$(a_1, a_2, \ldots, a_{i-1}, k, a_{i+1}, \ldots, a_r)$$

of $X(n)$ with $k$ being the $i^{\text{th}}$ smallest element. Clearly, we must have $k \ge i$. There are $\binom{k-1}{i-1}$ ways of choosing the $i-1$ elements before $k$, and $\binom{n-k}{r-i}$ ways of choosing the $r - i$ elements after $k$. Thus

$$u_k = \binom{k-1}{i-1}\binom{n-k}{r-i}$$

from which we get

$$\sum t_i = \sum_{k \ge i} k u_k = \sum_{k \ge i} k\binom{k-1}{i-1}\binom{n-k}{r-i}. \tag{2}$$

From (1) and (2) we see that it remains to prove that

$$\sum_{k \ge i} \frac{k}{i}\binom{k-1}{i-1}\binom{n-k}{r-i} = \binom{n+1}{r+1},$$

or, equivalently,

$$\sum_{k \ge i} \binom{k}{i}\binom{n-k}{r-i} = \binom{n+1}{r+1}. \tag{3}$$

The equation (3) is well-known (for example, formula (11) on page 207 of *Applied Combinatorics*, 2nd edition by Alan Tucker). For completeness we give a combinatorial proof of this identity. Note first that $\binom{n+1}{r+1}$ is the number of $(r+1)$-subsets of $X(n+1)$. Again, we partition the family of all these subsets according to their $(i+1)^{\text{th}}$ smallest element, where $i$ is fixed, $1 \le i \le r$. Specifically, for each $k \in X(n+1)$, we count the number, $v_k$, of $(r+1)$-subsets with $k+1$ being the $(i+1)^{\text{th}}$ smallest element. Clearly, $k \ge i$, and, by the same argument as before, with $n, r, i, k$ replaced by $n+1, r+1, i+1$, and $k+1$, respectively, we have

$$v_k = \binom{k}{i}\binom{n-k}{r-i}.$$

Summing over $k$, we get

$$\binom{n+1}{r+1} = \sum_{k \ge i} v_k = \sum_{k \ge i} \binom{k}{i}\binom{n-k}{r-i}$$

which establishes (3) and completes the proof.

*Also solved by NIELS BEJLEGAARD, Stavanger, Norway; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; ADRIAN BIRKA, student, Lakeshore Catholic High School, Port Colbourne, Ontario; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; JIMMY CHUI, student, Earl Haig Secondary School, North York, Ontario; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; ALAN LING, student, University of Toronto, Toronto, Ontario; CHRISTOPHER SO, student, Francis Liberman Catholic High School, Scarborough, Ontario; and the proposer. There was also one incomplete solution submitted.*

*Francisco Bellot Rosado noted that this problem is a generalization of the second problem from the IMO 1981 (Washington), where the question was to find the arithmetic mean of the smallest elements of the $r$-subsets. He also pointed out that a solution to the general question is given in a Romanian Olympiad book: Cuculescu, I.,* Olimpiadele Internationale de Matematica ale elevilor, *Ed. Tehnica, Bucarest 1984, p. 315.*