

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as a **Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of *Crux Mathematicorum with Mathematical Mayhem*.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Haven, CT 06520-8283 USA. The electronic address is still

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Shreds and Slices

From the Archives

An excerpt from “Wilson’s Theorem”, by Oliver Johnson, Volume 5, Issue 2, *Mathematical Mayhem*:

One of the more beautiful results in number theory is Wilson’s Theorem. [So named because Sir John Wilson didn’t discover it, and Leibniz who did discover it didn’t prove it—Lagrange did. This is of course an example of the lawyers doing the best out of everything; Wilson was a judge, and it is worth considering how many of Fermat’s theorems weren’t just won in lawsuits rather than proved by himself... So sue me Pierre.]

Primitive Roots and Quadratic Residues, Part 1

For the following we will need some definitions, none of which should be too abstruse or unfamiliar. For a positive integer n , the polynomial $x^n - 1$ has n distinct roots in the field of complex numbers \mathbb{C} . These roots are called the n^{th} roots of unity, and denoted by μ_n . An element ζ in μ_n is called *primitive* if $\zeta^k \neq 1$ for $1 \leq k \leq n - 1$; in other words, n is the smallest positive exponent k we must raise ζ to the power of to obtain 1. It can be shown that there are exactly $\phi(n)$ primitive n^{th} roots of unity. If n is a prime p , then for ζ to be primitive, it suffices that $\zeta \neq 1$; in other words, any p^{th} root of unity not equal to 1 is primitive. This turns out to be the case we are interested in.

Now, recall quadratic residues modulo n : these are the set of non-zero squares in modulo n . For example, the non-zero squares modulo 10 are 1, 4, 9, $16 \equiv 6$, $25 \equiv 5$, $36 \equiv 6$, and so on, so $\{1, 4, 5, 6, 9\}$ is the set of quadratic residues modulo 10. Let A_n denote the quadratic residues modulo n , and B_n the remaining numbers, so $B_{10} = \{2, 3, 7, 8\}$. If n is a prime p , then both A_p and B_p contain exactly $(p-1)/2$ elements, and this is where things finally get interesting.

Let p be a prime, let ζ be a primitive p^{th} root of unity, and let

$$x = \sum_{a \in A_p} \zeta^a, \quad y = \sum_{b \in B_p} \zeta^b.$$

Then both $x + y$ and xy are always integers. Actually $x + y = -1$, and this is not hard to see. Since $\zeta^p = 1$, or

$$\zeta^p - 1 = (\zeta - 1)(\zeta^{p-1} + \zeta^{p-2} + \dots + \zeta + 1) = 0$$

and $\zeta \neq 1$, the latter factor must be 0. Also, A_p and B_p form a partition of $\{1, 2, \dots, p-1\}$. Hence,

$$x + y = \zeta + \zeta^2 + \dots + \zeta^{p-1} = -1.$$

But why xy is an integer is a little deeper. Let us consider an example. For $p = 7$, $x = \zeta + \zeta^2 + \zeta^4$ and $y = \zeta^3 + \zeta^5 + \zeta^6$. Then

$$\begin{aligned} xy &= (\zeta + \zeta^2 + \zeta^4)(\zeta^3 + \zeta^5 + \zeta^6) \\ &= \zeta^4 + \zeta^6 + \zeta^7 + \zeta^5 + \zeta^7 + \zeta^8 + \zeta^7 + \zeta^9 + \zeta^{10} \\ &= 3 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^6 = 2. \end{aligned}$$

In particular, this shows that to calculate xy , we do not need to resort to any messy cis notation, and that it is quite accessible by just pencil and paper. Playing around with other primes reveals the following values:

p	3	5	7	11	13
xy	1	-1	2	3	-3

The absolute value of xy always seems to be $(p \pm 1)/4$ (the sign is the one which makes the resulting value an integer). Why is this? And what determines the sign of xy ? We encourage readers to play around with this, and to send in any interesting results. We will divulge the reasons behind this phenomenon, as well as some of the deeper theory it will lead to, in the next issue.

Hint: If indeed $x + y$ and xy are integers, then x and y are the roots of a quadratic equation with integer coefficients. What is this quadratic? What does this quadratic say about x and y ?



Mayhem Problems

The Mayhem Problems editors are:

Richard Hoshino *Mayhem High School Problems Editor,*
Cyrus Hsia *Mayhem Advanced Problems Editor,*
David Savitt *Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only solutions — the next issue will feature only problems.

We warmly welcome proposals for problems and solutions. We request that solutions from the previous issue be submitted by 1 February 1999, for publication in issue 4 of 1999. Also, starting with this issue, we would like to re-open the problems to all **CRUX with MAYHEM** readers, not just students, so now all solutions will be considered for publication.

High School Solutions

Editor: Richard Hoshino, 17 Norman Ross Drive, Markham, Ontario, Canada. L3S 3E8 <rhoshino@undergrad.math.uwaterloo.ca>

H221. Let $P = 19^5 + 660^5 + 1316^5$. It is known that 25 is one of the forty-eight positive divisors of P . Determine the largest divisor of P that is less than 10,000.

Solution.

If k divides $a + b$, then k divides $a^5 + b^5$ since the latter is a multiple of $a + b$. If k also divides c , then k divides $a^5 + b^5 + c^5$, since k divides c^5 . Notice that $19 + 660 + 1316 = 1995 = 3 \cdot 5 \cdot 7 \cdot 19$. Since 3 divides 660 and 3 also divides $19 + 1316 = 1335$, by our result, 3 divides $660^5 + 19^5 + 1316^5 = P$. Similarly, since 5 divides 660, 7 divides 1316 and 19 divides 19, we can show that P is also divisible by 5, 7 and 19. Therefore, P can be expressed in the form $3 \cdot 5^2 \cdot 7 \cdot 19 \cdot R$ for some positive integer R , since we are given that 25 is a divisor of P .

If R is prime, we know that P will have $2 \cdot 3 \cdot 2 \cdot 2 \cdot 2 = 48$ divisors. (Recall that if p_1, p_2, \dots, p_n are distinct primes, then $p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$ has $(a_1 + 1)(a_2 + 1) \cdots (a_n + 1)$ positive divisors.)

Thus, R must be prime; otherwise P will have more than 48 divisors, and so $P = 3 \cdot 5^2 \cdot 7 \cdot 19 \cdot R = 9975 \cdot R$, for some large prime R . Since $1316^5 > 1000^5 = 10^{15}$, R will certainly be larger than 10,000. (Using **Maple**, one can show that R is 408,255,160,853.) Hence, it follows that the largest divisor of P less than 10,000 is 9975.

H222. McGregor becomes very bored one day and decides to write down a three digit number ABC , and the six permutations of its digits. To his surprise, he finds that ABC is divisible by 2, ACB is divisible by 3, BAC is divisible by 4, BCA is divisible by 5, CAB is divisible by 6 and CBA is a divisor of 1995. Determine ABC .

Solution by Evan Borenstein, student, Woodward Academy, College Park, Georgia.

Since BCA is divisible by 5, A must be 0 or 5. But we are given that ABC is a three-digit number, so $A = 5$. Since ABC is divisible by 2 and CAB is divisible by 6, both B and C must be even. Since BAC is divisible by 4, $10A + C = 50 + C$ must be a multiple of 4. So C can be 2 or 6. If $C = 2$, B must be 2 or 8, since ACB is divisible by 3 and B is even. If $C = 6$, B must be 4. Hence, there are only three possibilities for ABC : 522, 582 and 546. And of these three, only the second one will make CBA a divisor of 1995, since $1995 = 285 \cdot 7$. Thus we conclude that ABC is 582.

Also solved by Joel Schlosberg, student, Robert Louis Stevenson School, New York, NY, USA.

H223. There are n black marbles and two red marbles in a jar. One by one, marbles are drawn at random out of the jar. Jeanette wins as soon as two black marbles are drawn, and Fraserette wins as soon as two red marbles are drawn. The game continues until one of the two wins. Let $J(n)$ and $F(n)$ be the two probabilities that Jeanette and Fraserette win, respectively.

1. Determine the value of $F(1) + F(2) + \cdots + F(3992)$.
2. As n approaches infinity, what does $J(2) \times J(3) \times J(4) \times \cdots \times J(n)$ approach?

Solution.

1. If Fraserette wins, the balls must be drawn in one of the following three ways: red, red; red, black, red; or black, red, red. This must be the case, as otherwise two black balls will be drawn and Jeanette will win. Hence, the probability that Fraserette wins is the sum of the probabilities of each of three cases above.

Thus,

$$\begin{aligned} F(n) &= \frac{2}{n+2} \cdot \frac{1}{n+1} + \frac{2}{n+2} \cdot \frac{n}{n+1} \cdot \frac{1}{n} + \frac{n}{n+2} \cdot \frac{2}{n+1} \cdot \frac{1}{n} \\ &= \frac{2}{(n+1)(n+2)} + \frac{2}{(n+1)(n+2)} + \frac{2}{(n+1)(n+2)} \\ &= \frac{6}{(n+1)(n+2)}. \end{aligned}$$

Note that $F(n) = \frac{6}{n+1} - \frac{6}{n+2}$, so we can use a telescoping series to

calculate the desired sum. We have

$$\begin{aligned}
 F(1) + F(2) + \cdots + F(3992) &= \left(\frac{6}{2} - \frac{6}{3}\right) + \left(\frac{6}{3} - \frac{6}{4}\right) + \left(\frac{6}{4} - \frac{6}{5}\right) \\
 &\quad + \cdots + \left(\frac{6}{3993} - \frac{6}{3994}\right) \\
 &= \frac{6}{2} - \frac{6}{3994} \\
 &= 3 - \frac{3}{1997} \\
 &= \frac{5988}{1997}.
 \end{aligned}$$

$$2. \text{ Now } J(n) = 1 - F(n) = 1 - \frac{6}{(n+1)(n+2)} = \frac{n^2+3n-4}{(n+1)(n+2)} = \frac{(n-1)(n+4)}{(n+1)(n+2)}.$$

Thus,

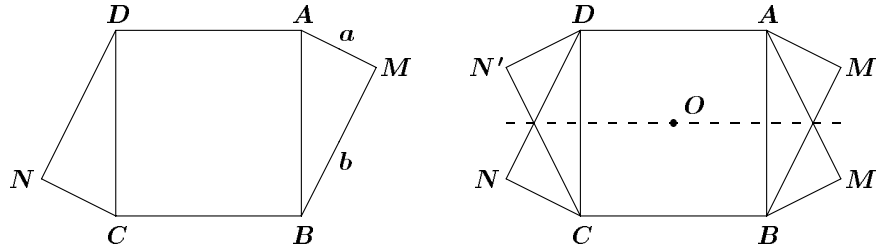
$$\begin{aligned}
 &J(2) \times J(3) \times J(4) \times \cdots \times J(n) \\
 &= \frac{1 \cdot 6}{3 \cdot 4} \cdot \frac{2 \cdot 7}{4 \cdot 5} \cdot \frac{3 \cdot 8}{5 \cdot 6} \cdot \frac{4 \cdot 9}{6 \cdot 7} \cdots \frac{(n-1) \cdot (n+4)}{(n+1) \cdot (n+2)} \\
 &= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-2) \cdot (n-1)}{3 \cdot 4 \cdot 5 \cdot 6 \cdots n \cdot (n+1)} \\
 &\quad \cdot \frac{6 \cdot 7 \cdot 8 \cdot 9 \cdots (n+3) \cdot (n+4)}{4 \cdot 5 \cdot 6 \cdot 7 \cdots (n+1) \cdot (n+2)} \\
 &= \frac{1 \cdot 2}{n \cdot (n+1)} \cdot \frac{(n+3) \cdot (n+4)}{4 \cdot 5} \\
 &= \frac{1}{10} \cdot \frac{(n+3)(n+4)}{n(n+1)} \\
 &= \frac{1}{10} \cdot \frac{n+3}{n} \cdot \frac{n+4}{n+1}.
 \end{aligned}$$

Thus, as n approaches infinity, $\frac{n+3}{n} = 1 + \frac{3}{n}$ and $\frac{n+4}{n+1} = 1 + \frac{3}{n+1}$ both approach 1, and so $J(2) \times J(3) \times J(4) \times \cdots \times J(n)$ approaches $\frac{1}{10}$.

H224. Consider square $ABCD$ with side length 1. Select a point M exterior to the square so that $\angle AMB$ is 90° . Let $a = AM$ and $b = BM$. Now, determine the point N exterior to the square so that $CN = a$ and $DN = b$. Find, as a function of a and b , the length of the line segment MN .

I. Solution by Adrian Chan, student, Upper Canada College, Toronto, Ontario.

Let O be the center of the square. Consider a horizontal reflection through the line parallel to DA and passing through O . Let the image of M and N about this line be M' and N' , respectively. There exists a point where MN intersects $M'N'$. Let this point be K . Since K lies on both lines, this point must lie on this horizontal line of reflection.



Now, the diagram is also symmetrical about the line parallel to DC passing through O . Then K must lie on this line as well. This leaves us with K coinciding with O , since the two lines intersect at O .

Now, because of the symmetry, $NO = OM$. That is, $OM = \frac{MN}{2}$. Now consider quadrilateral $OAMB$. Since $\angle AOB$ is right and so is $\angle AMB$, then the quadrilateral is cyclic.

By Ptolemy's Theorem on this quadrilateral, we have $OM \cdot AB = AM \cdot OB + MB \cdot OA = a \cdot \frac{1}{\sqrt{2}} + b \cdot \frac{1}{\sqrt{2}}$, since $OA = OB = \frac{1}{\sqrt{2}}$. Since $OB = 1$, we have $OM = \frac{\sqrt{2}}{2}(a + b)$, and so $MN = \sqrt{2}(a + b)$, since $MN = 2OM$. Thus the length of line segment MN is $\sqrt{2}(a + b)$.

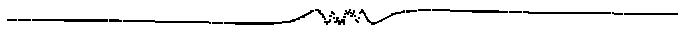
II. Solution by Joel Schlosberg, student, Robert Louis Stevenson School, New York, NY, USA.

Let $\angle BAM = x$. Then $a = \cos x$ and $b = \sin x$. Let O be the centre of $ABCD$ and R be the midpoint of AB . Then $\angle BAM = \angle RAM = \angle RMA = x$, and so $\angle ORM = \angle ARO + \angle ARM = \frac{\pi}{2} + (\pi - 2x) = \frac{3\pi}{2} - 2x$. Now $OR = RA = RM = \frac{1}{2}$. By the cosine law on triangle ROM , we have:

$$\begin{aligned} OM^2 &= OR^2 + RM^2 - 2OR \cdot OM \cos \angle ORM \\ &= \frac{1}{4} + \frac{1}{4} - 2 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \cos \left(\frac{3\pi}{2} - 2x \right) \\ &= \frac{1}{2} - \frac{1}{2} \cdot (-\sin 2x) \\ &= \frac{1}{2} - \frac{1}{2} \cdot 2 \sin x \cos x = \frac{1}{2} + ab. \end{aligned}$$

Thus using the fact that $a^2 + b^2 = 1$, we have $4OM^2 = 2(1 + 2ab) = 2(a^2 + b^2 + 2ab) = 2(a + b)^2$. Hence, $2OM = \sqrt{2}(a + b)$, by taking the square root of both sides (Note: a , b and OM are all positive).

Since $2OM = MN$, we have $MN = \sqrt{2}(a + b)$.



Advanced Solutions

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A197. Calculate

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin(2N+1)\theta}{\sin \theta} d\theta,$$

where N is a non-negative integer.

Solution by D.J. Smeenk, Zaltbommel, the Netherlands, with minor modifications.

Let $f(N)$ denote $\frac{\sin(2N+1)\theta}{\sin \theta}$. Then

$$f(N+1) - f(N) = \frac{\sin(2N+3)\theta - \sin(2N+1)\theta}{\sin \theta}.$$

Now

$$\begin{aligned} & \sin(2N+3)\theta - \sin(2N+1)\theta \\ &= \sin(2N+1)\theta \cos 2\theta + \cos(2N+1)\theta \sin 2\theta - \sin(2N+1)\theta \\ &= \sin(2N+1)\theta(\cos 2\theta - 1) + \cos(2N+1)\theta \sin 2\theta \\ &= -2\sin(2N+1)\theta \sin^2 \theta + 2\cos(2N+1)\theta \sin \theta \cos \theta \\ &= 2\sin \theta(\cos(2N+1)\theta \cos \theta - \sin(2N+1)\theta \sin \theta) \\ &= 2\sin \theta \cos(2N+2)\theta. \end{aligned}$$

These two equations imply $f(N+1) - f(N) = 2\cos(2N+2)\theta$. So,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(N+1) d\theta - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(N) d\theta = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2N+2)\theta d\theta = 0.$$

This means

$$f(N) = f(0) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin \theta}{\sin \theta} d\theta = \pi.$$

A198. Given positive real numbers a , b , and c such that $a+b+c=1$, show that $a^a b^b c^c + a^b b^c c^a + a^c b^a c^b \leq 1$.

Solution.

Using the weighted AM-GM inequality three times, we have the following:

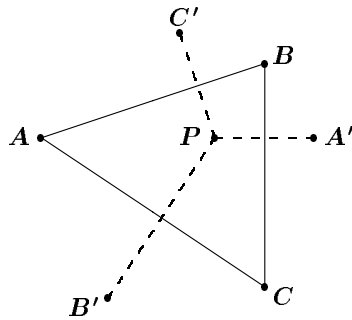
$$\begin{aligned} \frac{c \cdot a + a \cdot b + b \cdot c}{c + a + b} &\geq (a^c b^a c^b)^{\frac{1}{a+b+c}}, \\ \frac{b \cdot a + c \cdot b + a \cdot c}{b + c + a} &\geq (a^b b^c c^a)^{\frac{1}{a+b+c}}, \\ \frac{a \cdot a + b \cdot b + c \cdot c}{a + b + c} &\geq (a^a b^b c^c)^{\frac{1}{a+b+c}}. \end{aligned}$$

Adding these inequalities together gives

$$\begin{aligned} 1 = a + b + c &= \frac{(a + b + c)^2}{a + b + c} \\ &= \frac{a^2 + b^2 + c^2 + 2ab + 2ac + 2bc}{a + b + c} \\ &\geq a^a b^b c^c + a^b b^c c^a + a^c b^a c^b. \end{aligned}$$

A199. Let P be a point inside triangle ABC . Let A' , B' , and C' be the reflections of P through the sides BC , AC , and AB respectively. For what points P are the six points A , B , C , A' , B' , and C' concyclic?

Solution.



We show that AP must be perpendicular to BC . Similar arguments will show the same for BP and CP . Thus, P must be the orthocentre of triangle ABC .

To see that AP is perpendicular to BC , consider angles $C'A'A$ and $C'A'P$ as shown in the diagram. Then $\angle C'A'A = \angle C'BA = \angle PBA = \angle PBF = \angle FDP = \angle C'A'P$ from the quadrilaterals $AC'BA'$ and $BFPD$ being concyclic and the last equality from similar triangles PDF and $PC'A'$. Thus, P lies on AA' so AP is perpendicular to BC .

A200. Given positive integers n and k , for $0 \leq i \leq k - 1$, let

$$S_{n,k,i} = \sum_{j \equiv i \pmod{k}} \binom{n}{j}.$$

Do there exist positive integers n , $k > 2$, such that $S_{n,k,0}$, $S_{n,k,1}$, \dots , $S_{n,k,k-1}$ are all equal?

Solution.

The answer is NO. To see this, consider the k^{th} roots of unity. In particular, since $k > 2$, there is a k such that $\omega^k = 1$, $\omega \neq -1$. Now consider

the expansion of $(1 + \omega)^n$:

$$\begin{aligned}
 (1 + \omega)^n &= \sum_{j=0}^n \binom{n}{j} \omega^j \\
 &= \sum_{j \equiv 0 \pmod{k}} \binom{n}{j} + \omega \sum_{j \equiv 1 \pmod{k}} \binom{n}{j} \\
 &\quad + \cdots + \omega^{k-1} \sum_{j \equiv k-1 \pmod{k}} \binom{n}{j} \\
 &= S_{n,k,0} + S_{n,k,1} \omega + \cdots + S_{n,k,k-1} \omega^{k-1}.
 \end{aligned}$$

Now if all the $S_{n,k,i}$ are equal, say to A , then we have $(1 + \omega)^n = A(1 + \omega + \cdots + \omega^{k-1}) = A \cdot 0 = 0$. Thus $\omega = -1$, contradiction.

Challenge Board Solutions

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C74. Prove that the k -dimensional volume of a parallelepiped in \mathbb{R}^n spanned by vectors $\vec{v}_1, \dots, \vec{v}_k$ is the square root of the determinant of the $k \times k$ matrix $\{\vec{v}_i \cdot \vec{v}_j\}_{i,j}$.

Solution.

First, note that by restricting to any k -dimensional subspace of the n -space which contains $\vec{v}_1, \dots, \vec{v}_k$, we may assume without loss of generality that $k = n$. Let M be the $n \times n$ matrix whose i^{th} column is \vec{v}_i , and let P be the parallelepiped spanned by the \vec{v}_i . Under the coordinate transformation ϕ sending \vec{x} to $M\vec{x}$, the i^{th} elementary basis vector $\vec{e}_i = (0, \dots, 1, \dots, 0)$ is sent to \vec{v}_i , and so ϕ transforms the unit cube $[0, 1]^n$ onto P . We find then, that

$$\text{Volume}(P) = \int_P 1 \, dV = \int_{[0,1]^n} |\det \phi'| \, dV.$$

Since $\phi' = M$, it follows that $\text{Volume}(P) = |\det M| = (\det M^T M)^{1/2}$, and since the i, j^{th} -entry of $M^T M$ is indeed $\vec{v}_i \cdot \vec{v}_j$, we are done.

Remark. The above proof that $\text{Volume}(P) = |\det M|$ is actually a bit bogus, since use is usually made of that result when deriving the change-of-variables formula for integration. So, for those of you who are dissatisfied: let $V(\vec{v}_1, \dots, \vec{v}_n)$ be the oriented volume of the parallelepiped spanned by $\vec{v}_1, \dots, \vec{v}_n$ in that order. One can then verify that:

$$(1) \quad V(\vec{e}_1, \dots, \vec{e}_n) = 1,$$

$$(2) V(\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_j, \dots, \vec{v}_n) = -V(\vec{v}_1, \dots, \vec{v}_j, \dots, \vec{v}_i, \dots, \vec{v}_n), \text{ and}$$

$$(3) V(\vec{v}_1, \dots, \vec{v}_i + c\vec{v}'_i, \dots, \vec{v}_n) = V(\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_n) \\ + cV(\vec{v}_1, \dots, \vec{v}'_i, \dots, \vec{v}_n),$$

where the last assertion follows because $V(\vec{v}_1, \dots, \vec{w}, \dots, \vec{v}_n)$ is directly proportional to the component of \vec{w} which lies perpendicular to the hyperplane spanned by $\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n$. However, we know from linear algebra that the determinant is the unique function satisfying conditions (1)-(3), so we conclude that $V = \det$, the oriented volume of P is $\det M$, and indeed $\text{Volume}(P) = |\det M|$.

In clearing out the **Mayhem** archives, we also dug up this solution to an old problem.

S21. *Proposed by Colin Springer.*

There are n houses situated around a certain lake, in a circle. Each is painted one of k colours, chosen at random. Find the probability that no two neighbouring houses are of the same colour.

Solution by Philip Oppenheimer, South Norwalk, CT.

For convenience, we use the following notation: Let $P(n, k)$ denote the desired probability. Let $h(a)$ denote the colour of the a^{th} house, starting from a fixed house, going around the circle.

Without loss of generality, we may fix $h(1) = 1$. Under this condition, let $p(n, a, b)$ denote the probability that $h(a) = b$, if indeed no two adjacent houses are painted the same colour.

With $h(1) = 1$, there are $k - 1$ ways to colour the second house (out of k colours), the third house, etc., until the $(n - 1)^{\text{th}}$ house. If $h(n - 1) = 1$, then there are $k - 1$ ways to colour the n^{th} house. However, if $h(n - 1) \neq 1$, then there are $k - 2$ ways to colour the n^{th} house. Hence,

$$P(n, k) = \left(\frac{k-1}{k}\right)^{n-2} \left(\frac{k-1}{k} p(n, n-1, 1) + \frac{k-2}{k} (1 - p(n, n-1, 1))\right).$$

But for all a ,

$$p(n, a, 1) = \frac{1}{k-1} (1 - p(n, a-1, 1)) \\ = \frac{1}{k-1} - \frac{1}{k-1} p(n, a-1, 1)$$

so that

$$\begin{aligned}
 p(n, a, 1) - \frac{1}{k} &= \frac{1}{k(k-1)} - \frac{1}{k-1} p(n, a-1, 1) \\
 &= \frac{-1}{k-1} \left(p(n, a-1, 1) - \frac{1}{k} \right) \\
 &= \left(\frac{-1}{k-1} \right)^2 \left(p(n, a-2, 1) - \frac{1}{k} \right) \\
 &= \vdots \\
 &= \left(\frac{-1}{k-1} \right)^{a-1} \left(p(n, 1, 1) - \frac{1}{k} \right) \\
 &= \frac{(-1)^{a-1}}{(k-1)^{a-1}} \cdot \frac{k-1}{k} \\
 &= \frac{(-1)^{a-1}}{k(k-1)^{a-2}}.
 \end{aligned}$$

Therefore,

$$p(n, n-1, 1) = \frac{(-1)^n}{k(k-1)^{n-3}},$$

and

$$\begin{aligned}
 P(n, k) &= \left(\frac{k-1}{k} \right)^{n-2} \left[\frac{k-1}{k} \left(\frac{1}{k} + \frac{(-1)^n}{k(k-1)^{n-3}} \right) \right. \\
 &\quad \left. + \frac{k-2}{k} \left(1 - \frac{1}{k} - \frac{(-1)^n}{k(k-1)^{n-3}} \right) \right] \\
 &= \left(\frac{k-1}{k} \right)^{n-2} \left(\frac{k-1}{k^2} + \frac{(-1)^n(k-1)}{k^2(k-1)^{n-3}} \right. \\
 &\quad \left. + \frac{(k-1)(k-2)}{k^2} - \frac{(-1)^n(k-2)}{k^2(k-1)^{n-3}} \right) \\
 &= \left(\frac{k-1}{k} \right)^{n-2} \left(\frac{(k-1)^2}{k^2} + \frac{(-1)^n}{k^2(k-1)^{n-3}} \right) \\
 &= \left(\frac{k-1}{k} \right)^n + \frac{(-1)^n(k-1)}{k^n}.
 \end{aligned}$$

Swedish Mathematics Olympiad

1986 Qualifying Round

1. Show that

$$(1986!)^{1/1986} < (1987!)^{1/1987}.$$

2. Show that for $t > 0$,

$$t^2 + \frac{1}{t^2} - 3 \left(t + \frac{1}{t} \right) + 4 \geq 0.$$

3. A circle C_1 with radius 1 is internally tangent to a circle C_2 with radius 2. Let ℓ be a line through the centres of the circles C_1 and C_2 . A circle C_3 is tangent to C_1 , C_2 , and ℓ . Find the radius of C_3 .
4. In how many ways can 11 apples and 9 pears be shared among 4 children, so that every child gets 5 fruit? (The apples are identical, as are the pears.)
5. P is a polynomial of degree greater than 2 with integer coefficients and such that $P(2) = 13$ and $P(10) = 5$. It is known that P has a root which is an integer. Find it.
6. The numbers $1, 2, \dots, n$ are placed in some order at different points on the circumference of a circle. Form the product of each pair of neighbouring numbers. How should the numbers be placed in order for the sum of these products to be as large as possible?

1986 Final Round

1. Show that the polynomial

$$x^6 - x^5 + x^4 - x^3 + x^2 - x + \frac{3}{4}$$

has no real roots.

2. $ABCD$ is a quadrilateral, and O is the intersection of the diagonals AC and BD . The triangles AOB and COD have areas S_1 and S_2 respectively, and the area of $ABCD$ is S . Show that

$$\sqrt{S_1} + \sqrt{S_2} \leq \sqrt{S}.$$

Show also that equality holds if and only if the lines AB and CD are parallel.

3. Let N be a positive integer, $N \geq 3$. Form all pairs (a, b) of consecutive integers such that $1 \leq a < b \leq N$ and consider the quotient $q = \frac{b}{a}$ for every such pair. Remove all pairs with $q = 2$. Show that of the remaining pairs, there are as many with $q < 2$ as there are with $q > 2$.
4. Show that the only positive solution of

$$\begin{aligned}x + y^2 + z^3 &= 3 \\y + z^2 + x^3 &= 3 \\z + x^2 + y^3 &= 3\end{aligned}$$

is $x = y = z = 1$.

5. In the arrangement of pn real numbers below, the difference between the greatest and least numbers in every row is at most d , where $d > 0$:

$$\begin{array}{cccc}a_{11} & a_{12} & \cdots & a_{1n} \\a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\a_{p1} & a_{p2} & \cdots & a_{pn}\end{array}$$

In each column, the numbers are now ordered by size, so that the greatest appears in the first row, the next greatest in the second row, and so on. Show that the difference between the greatest and least numbers in each of the rows is still at most d .

6. A finite number of intervals on the real line together cover the interval $[0, 1]$. Show that one can choose a number of these intervals such that no two have any points in common and whose total length is at least $1/2$.

J.I.R. McKnight Problems Contest 1982

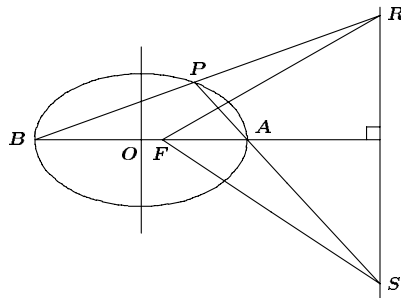
1. (a) Given the equal positive rationals $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, prove that

$$\left(\frac{ma^4 + nc^4 + pe^4}{mb^4 + nd^4 + pf^4} \right)^{1/4}$$

is equal to each of the given rationals.

(b) Given that $a^4 + b^4 = 7^c$ and that a and b are the roots of $x^2 - 5x + 3$, find c .

2. Consider AB , the major axis of an ellipse centred at the origin with focus F as shown. Let P be any point on the ellipse. Draw the lines BP and AP and extend them so that they cross the directrix of F at R and S respectively. Prove that $\angle RFS$ is a right angle.



3. Solve the system of equations:

$$xy + yz + zx = -4$$

$$y + z - yz = 4$$

$$x - y - z = 3$$

4. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, prove that $x^2 + y^2 + z^2 + 2xyz = 1$.
5. A shopkeeper orders 19 large and 3 small packets of marbles, all alike. When they arrive at the shop it is discovered that all the packets have come open with the marbles loose in the container. If the total number of marbles is 224, can you help the shopkeeper put up the packets with the proper number of marbles in each?
6. A radar tracking station is located at ground level vertically below the path of an approaching aircraft flying at 900 km/h at a constant height of 10000 m. Find the rate in degree/s at which the radar beam to the aircraft is turning at the instant when the aircraft is at a horizontal distance of 3 km from the station.

