THE ACADEMY CORNER

No. 19

Bruce Shawyer

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Memorial University Undergraduate Mathematics Competition

September 25, 1997

Solutions by Solomon W. Golomb, USC, Los Angeles, CA, USA, who writes: "It's reassuring to know that I can still do freshman/high school mathematics, after all these years".

1. Determine whether or not the following system has any real solutions. If so, state how many real solutions exist.

$$x + \frac{1}{x} = y, \qquad y + \frac{1}{y} = z, \qquad z + \frac{1}{z} = x. \tag{1}$$

Solution. If (1) is to have real solutions, then $xyz \neq 0$. Hence

$$x^2 + 1 = xy, \qquad y^2 + 1 = yz, \qquad z^2 + 1 = zx.$$

Multiplying gives $\left(x^2+1\right)\left(y^2+1\right)\left(z^2+1\right) \;=\; x^2y^2z^2$.

But, for real x, y, z, we have $x^2 + 1 > x^2$, $y^2 + 1 > y^2$, $z^2 + 1 > z^2$, so there are no real solutions.

2. The surface area of a closed cylinder is twice the volume. Determine the radius and height of the cylinder given that the radius and height are both integers.

Solution. "The surface area of a closed cylinder is twice the volume" is absurd. It is dimensionally incorrect. One has units of area, the other has units of volume.

What was no doubt intended was $2\pi rh + 2\pi r^2 = 2\pi r^2 h$, $h + r = rh$, $\frac{1}{r} + \frac{1}{h} = 1$, with no integer solution except $r = h = 2$. But what are the *units* in which $h + r = rh$? (No physical object corresponds to this solution, independent of the arbitrary choice of units.)

3. Prove that

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2} < 2.$$

Solution (a).

$$\begin{aligned} 1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2} &< \sum_{k=1}^{\infty}\frac{1}{k^2} \\ &= \frac{\pi^2}{6} < \frac{10}{6} \\ &= \frac{5}{3} < 2. \end{aligned}$$

Solution (b). (For those who do not know the value of $\zeta(2)$.)

$$\begin{aligned} 1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2} &< 1+\int_1^{n+1}\frac{dt}{t^2} \\ &< 1+\int_1^{\infty}\frac{dt}{t^2} = 1-\frac{1}{t}\Big|_1^{\infty} \\ &= 1+1 = 2. \end{aligned}$$

4. Describe the set of points (x, y) in the plane for which

$$\sin(x+y)=\sin x+\sin y.$$

Solution. Since $\sin(x+y) = \sin x\cos y + \sin y\cos x$, it follows that $\sin(x+y) = \sin x + \sin y$ has no solutions in the first quadrant, in which $\cos x < 1$, $\cos y < 1$, so that $\sin(x+y) < \sin x + \sin y$.

There are no solutions in the third quadrant, in which

$$\sin(x+y)=\sin x\cos y+\sin y\cos x > 0 > \sin x+\sin y.$$

The line $y = -x$, which bisects the second and fourth quadrants, is clearly a solution line: $\sin(x+(-x)) = 0$ and $\sin x + \sin(-x) = 0$.

To show that there are no other solutions, observe:

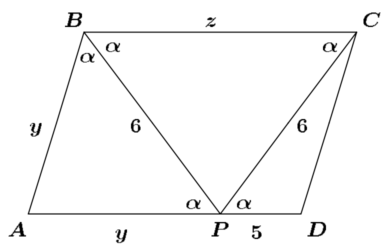
any solution (x, y) in the second or fourth quadrant corresponds to a first-quadrant solution of $\sin(x-y) = \sin x - \sin y$.

If $x \neq y$, suppose (without loss of generality) that $x > y$, so that $\sin x > \sin y$ while $\cos y > \cos x$. But then

$$\sin(x-y)=\sin x\cos y-\sin y\cos x > \sin x-\sin y.$$

5. In a parallelogram $ABCD$, the bisector of angle ABC intersects AD at P. If $PD = 5$, $BP = 6$ and $CP = 6$, find AB.

Solution.



We label the parallelogram as shown.

All angles α are equal, because BCP is an isosceles triangle, BP bisects $\angle ABC$, and alternate interior angles are equal.

Thus BPA is also an isosceles triangle, similar to triangle BCP, and $z = 5 + y$ because $BC = AD$.

From similar triangles, we see that $\frac{6}{z} = \frac{y}{6}$, $36 = yz = y(5 + y)$, and $y^2 + 5y - 36 = 0$. By the quadratic formula, the positive root is $y = 4$, which is the length of side AB. (Triangle CDP is a 4–5–6 triangle!)

6. Show that, where $k + n \leq m$,

$$\sum_{i=0}^{n} \binom{n}{i} \binom{m}{k+i} = \binom{m+n}{n+k}.$$

Solution. With $k+n \leq m$, we represent $\binom{m+n}{n+k}$ by taking j elements from n, and the remaining $(n + k) - j$ elements from m, for every j, $0 \leq j \leq n$. Thus

$$\begin{aligned} \binom{m+n}{n+k} &= \sum_{j=0}^{n} \binom{n}{j} \binom{m}{n+k-j} \\ &= \sum_{i=n}^{0} \binom{n}{n-i} \binom{m}{k+i} \\ &= \sum_{i=0}^{n} \binom{n}{i} \binom{m}{k+i}, \end{aligned}$$

where we substituted $i = n - j$, and used $\binom{n}{a} = \binom{n}{n-a}$.

Solutions were also received from D. KIPP JOHNSON, Beaverton, Oregon, USA and D.J. SMEENK, Zaltbommel, the Netherlands.

THE OLYMPIAD CORNER

No. 190

R.E. Woodrow

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Since the "summer break" is coming up, we give three Olympiad contests from three different parts of the world. My thanks go to Bill Sands for collecting the contest materials for me when he was helping to coordinate marking of the IMO held in Toronto in 1995.

We first give the problems of the Grade XI and Grade XII versions of the Lithuanian Mathematical Olympiad.

44^{th} LITHUANIAN MATHEMATICAL OLYMPIAD (1995)

GRADE XI

1. You are given a set of 10 positive integers. Summing nine of them in ten possible ways we only get nine different sums: 86, 87, 88, 89, 90, 91, 93, 94, 95. Find those numbers.

2. What is the least possible number of positive integers such that the sum of their squares equals 1995?

3.

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Replace the asterisks in the "equilateral triangle" by the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 so that, starting from the second line, each number is equal to the absolute value of the difference of the nearest two numbers in the line above.

Is it always possible to inscribe the numbers $1, 2, \ldots, n$, in the way required, into the equilateral triangle with the sides having n asterisks?

4. A function $f : \mathbb{N} \to \mathbb{N}$ is such that

$$f(f(m) + f(n)) = m + n$$

for all $m, n \in \mathbb{N}$ ($\mathbb{N} = \{1, 2, \ldots\}$ denotes the set of all positive integers). Find all such functions.

5. In a trapezium $ABCD$, the bases are $AB = a$, $CD = b$, and the diagonals meet at the point O. Find the ratio of the areas of the triangle ABO and trapezium.

GRADE XII

1. Consider all pairs (x, y) of real numbers satisfying the inequalities

$$-1 \le x + y \le 1, \quad -1 \le xy + x + y \le 1.$$

Let M denote the largest possible value of x.

(a) Prove that $M \le 3$.
(b) Prove that $M \le 2$.
(c) Find M.

2. A positive integer n is called an *ambitious* number if it possesses the following property: writing it down (in decimal representation) on the right of any positive integer gives a number that is divisible by n. Find:

(a) the first 10 ambitious numbers;
(b) all the ambitious numbers.

3. The area of a trapezium equals 2; the sum of its diagonals equals 4. Prove that the diagonals are mutually orthogonal.

4. 100 numbers are written around a circle. Their sum equals 100. The sum of any 6 neighbouring numbers does not exceed 6. The first number is 6. Find the remaining numbers.

5. Show that, at any time, moving both the hour-hand and the minute-hand of the clock symmetrically with respect to the vertical $(6 - 12)$ axis results in a possible position of the clock-hands. How many straight lines containing the centre of the clock-face possess the same property?

Next we give the problems of the Korean Mathematical Olympiad.

8^{th} KOREAN MATHEMATICAL OLYMPIAD
First Round

Morning Session — 2.5 hours

1. Consider finitely many points in a plane such that, if we choose any three points A, B, C among them, the area of $\triangle ABC$ is always less than 1. Show that all of these finitely many points lie within the interior or on the boundary of a triangle with area less than 4.

2. For a given positive integer m, find all pairs (n, x, y) of positive integers such that m, n are relatively prime and $(x^2 + y^2)^m = (xy)^n$, where n, x, y can be represented by functions of m.

3. Let A, B, C be three points lying on a circle, and let P, Q, R be the midpoints of arcs BC, CA, AB, respectively. AP, BQ, CR intersect BC, CA, AB at L, M, N, respectively. Show that

$$\frac{AL}{PL} + \frac{BM}{QM} + \frac{CN}{RN} \geq 9.$$

For which triangle ABC does equality hold?

4. A partition of a positive integer n is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$. Each λ_i is called a summand. For example, $(4, 3, 1)$ is a partition of 8 whose summands are distinct. Show that, for a positive integer m with $n > \frac{1}{2}m(m+1)$, the number of all partitions of n into distinct m summands is equal to the number of all partitions of $n - \frac{1}{2}m(m+1)$ into r summands $(r \leq m)$.

Afternoon Session — 2.5 hours

5. If we select at random three points on a given circle, find the probability that these three points lie on a semicircle.

6. Show that any positive integer $n > 1$ can be expressed by a finite sum of numbers satisfying the following conditions:

(i) they do not have factors except 2 or 3;
(ii) any two of them are neither a factor nor a multiple of each other.

That is,

$$n = \sum_{i=1}^{N} 2^{\alpha_i} 3^{\beta_i},$$

where α_i, β_i $(i = 1, 2, \ldots, N)$ are nonnegative integers and, whenever $i \neq j$, the condition $(\alpha_i - \alpha_j)(\beta_i - \beta_j) < 0$ is satisfied.

7. Find all real valued functions f defined on real numbers except 0 such that

$$\frac{1}{x} f(-x) + f\left(\frac{1}{x}\right) = x, \quad x \neq 0.$$

8. Two circles O_1, O_2 of radii r_1, r_2 $(r_1 < r_2)$, respectively, intersect at two points A and B. P is any point on circle O_1. Lines PA, PB and circle O_2 intersect at Q and R, respectively.

(i) Express $y = QR$ in terms of r_1, r_2, and $\theta = \angle APB$.

(ii) Show that $y = 2r_2$ is a necessary and sufficient condition that circle O_1 be orthogonal to circle O_2.

Final Round

First Day — 4.5 hours

1. For any positive integer m, show that there exist integers a, b satisfying

$$|a| \leq m, \quad |b| \leq m, \quad 0 < a + b\sqrt{2} \leq \frac{1+\sqrt{2}}{m+2}\,.$$

2. Let A be the set of all non-negative integers. Find all functions $f : A \to A$ satisfying the following two conditions:

(i) for any $m, n \in A$,

$$2f(m^2 + n^2) = \{f(m)\}^2 + \{f(n)\}^2;$$

(ii) for any $m, n \in A$ with $m \geq n$,

$$f(m^2) \geq f(n^2).$$

3. Let $\triangle ABC$ be an equilateral triangle of side length 1, let D be a point on BC, and let r_1, r_2 be inradii of triangles ABD, ADC, respectively. Express $r_1 r_2$ in terms of $p = BD$, and find the maximum of $r_1 r_2$.

Second Day — 4.5 hours

4. Let O and R be the circumcentre and the circumradius of $\triangle ABC$, respectively, and let P be any point on the plane of ABC. Let perpendiculars PA_1, PB_1, PC_1, be dropped to the three sides BC, CA, AB. Express

$$\frac{[A_1B_1C_1]}{[ABC]}$$

in terms of R and $d = OP$, where $[ABC]$ denotes the area of $\triangle ABC$.

5. Let p be a prime number such that

(i) p is the greatest common divisor of a and b;

(ii) p^2 is a divisor of a. Prove that the polynomial $x^{n+2} + ax^{n+1} + bx^n + a + b$ cannot be decomposed into the product of two polynomials with integral coefficients, whose degrees are greater than one.

6. Let m, n be positive integers with $1 \leq n \leq m - 1$. A box is locked with several padlocks, all of which must be opened to open the box, and all of which have different keys. A total of m people each have keys to some of the locks. No n people of them can open the box but any $n + 1$ people can open the box. Find the smallest number l of locks and in that case find the number of keys that each person has.

Now, problems selected from the 1995 Israel Mathematical Olympiads.

Selected Problems From ISRAEL MATHEMATICAL OLYMPIADS 1995

1. n positive integers $d_1, d_2, \ldots, d_n$ divide 1995. Prove that there exist d_i and d_j among them, such that the denominator of the reduced fraction $\frac{d_i}{d_j}$ is at least n.

2. Two players play a game on an infinite board that consists of 1×1 squares. Player I chooses a square and marks it with O. Then, Player II chooses another square and marks it with X. They play until one of the players marks a whole row or a whole column of 5 consecutive squares, and this player wins the game. If no player can achieve this, the result of the game is a tie. Show that Player II can prevent Player I from winning.

3. Two thieves stole an open chain with $2k$ white beads and $2m$ black beads. They want to share the loot equally, by cutting the chain to pieces in such a way that each one gets k white beads and m black beads. What is the minimal number of cuts that is always sufficient?

4. α and β are two given circles that intersect each other at two points. Find the geometric locus of the centres of all circles that are orthogonal to both α and β.

5. Four points are given in space, in a general position (that is, they are not contained in a single plane). A plane π is called "an equalizing plane" if all four points have the same distance from π. Find the number of equalizing planes.

6. n is a given positive integer. A_n is the set of all points in the plane, whose x and y coordinate are positive integers between 0 and n. A point (i, j) is called "internal" if $0 < i, j < n$. A real function f, defined on A_n, is called a "good function" if it has the following property: for every internal point x, the value of $f(x)$ is the mean of its values on the four neighbouring points (the neighbouring points of x are the four points whose distance from x equals 1).

If f and g are two given good functions and $f(a) = g(a)$ for every point a in A_n which is not internal (that is, a boundary point), prove that $f \equiv g$.

7. Solve the system

$$\begin{aligned} x + \log(x + \sqrt{x^2+1}) &= y \\ y + \log(y + \sqrt{y^2+1}) &= z \\ z + \log(z + \sqrt{z^2+1}) &= x \end{aligned}$$

8. Prove the inequality

$$\frac{1}{kn}+\frac{1}{kn+1}+\frac{1}{kn+2}+\cdots+\frac{1}{(k+1)n-1}\geq n\left(\sqrt[n]{\frac{k+1}{k}}-1\right)$$

for any positive integers k, n.

9. PQ is a diameter of a half circle H. The circle O is tangent to H from the inside and touches diameter PQ at the point C. A is a point on H and B is a point on PQ such that AB is orthogonal to PQ and is also tangent to the circle O. Prove that AC bisects the angle PAB.

10. α is a given real number. Find all functions $f:(0,\infty)\mapsto(0,\infty)$ such that the equality

$$\alpha x^2 f\left(\frac{1}{x}\right)+f(x)=\frac{x}{x+1}$$

holds for all real $x>0$.

Next we turn to solutions to problems posed in the February 1997 number of the Corner. Some new solutions arrived after we went to press last issue, and at least one other batch of solutions was incorrectly filed and just turned up! Pavlos Maragoudakis, Pireas, Greece sent in solutions to problems 1, 3 and 4 of the Final Grade, Third Round, and also to Problem 1 of the 1^{st} Selection Round. The misplaced solutions were from D.J. Smeenk, who gave solutions to Problem 3 of the Final Grade, 3^{rd} Round and to Problem 3 of each of the second and third Selection Rounds. Because they are interesting and different, we give his two solutions to Problem 3 of the Final Grade. (Look at all the 3's in the above!)

3. [1997: 78, 1998: 13–14] *Latvian* 44 *Mathematical Olympiad*.

It is given that $a>0$, $b>0$, $c>0$, $a+b+c=abc$. Prove that at least one of the numbers a, b, c exceeds $17/10$.

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.

In any triangle ABC the following identity holds

$$\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma.$$

Hence, in this problem a, b, c can be considered to be the tangents of angles of an acute angled triangle. At least one of the angles is at least $\frac{\pi}{3}$, and $\tan\left(\frac{\pi}{3}\right)=\sqrt{3}>\frac{17}{10}$.

Second solution.

We may suppose $a \geq b \geq c$,

$$abc \;=\; a+b+c \;\geq\; 3c.$$

So $ab \geq 3$, $a \geq b$ and $a \geq \sqrt{3} > \frac{17}{10}$.

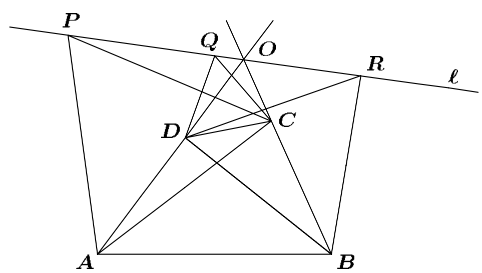
Now we turn to readers' solutions of problems from the March 1997 Corner and the 3rd Mathematical Olympiad of the Republic of China (Taiwan) [1997: 66].

3rd MATHEMATICAL OLYMPIAD OF THE REPUBLIC OF CHINA (Taiwan)

First Day — April 14, 1994

1. Let $ABCD$ be a quadrilateral with $\overline{AD} = \overline{BC}$ and let $\angle A + \angle B = 120^\circ$. Three equilateral triangles $\triangle ACP$, $\triangle DCQ$ and $\triangle DBR$ are drawn on $\overline{AC}$, $\overline{DC}$ and $\overline{DB}$ away from $\overline{AB}$. Prove that the three new vertices P, Q and R are collinear.

Solution by Toshio Seimiya, Kawasaki, Japan.



Let O be the intersection of AD and BC. Since $\angle A + \angle B = 120^\circ$, we get $\angle AOB = 60^\circ$. Let l be the exterior bisector of angle AOB. Since $\angle APC = 60^\circ = \angle AOC$, we have that O, P, A, C are concyclic. Hence $\angle POA = \angle PCA = 60^\circ$. The exterior angle of $\angle AOB$ is 120°, showing that PO bisects the exterior angle of $\angle AOB$. Thus P lies on l. Similarly Q and R lie on l. Hence, P, Q and R are collinear.

Comment. As is shown in the proof, the condition $AD = BC$ is not necessary.

2. Let a, b, c be positive real numbers, α be a real number. Suppose that

$$f(\alpha) = abc(a^\alpha + b^\alpha + c^\alpha)$$

$$g(\alpha) = a^{\alpha+2}(b+c-a) + b^{\alpha+2}(a-b+c) + c^{\alpha+2}(a+b-c)$$

Determine the magnitude between $f(\alpha)$ and $g(\alpha)$.

Solution by Panos E. Tsaoussoglou, Athens, Greece.

$$\begin{aligned} & bca^{\alpha+1} + acb^{\alpha+1} + abc^{\alpha+1} - a^{\alpha+2}(b+c-a) \\ & \qquad -b^{\alpha+2}(a-b+c) - c^{\alpha+2}(a+b-c) \\ = & \quad a^{\alpha+1}(bc - a(b+c) + a^2) + b^{\alpha+1}(ac - b(a+c) + b^2) \\ & \qquad +c^{\alpha+1}(ab - c(a+b) + c^2) \\ = & \quad a^{\alpha+1}(a-b)(a-c) + b^{\alpha+1}(b-a)(b-c) + c^{\alpha+1}(c-a)(c-b) \\ \geq & \quad 0 \end{aligned}$$

which is an inequality of Schur.

Next we move to solutions of Selected Problems from the Israel Mathematical Olympiads, 1994 [1997: 131].

SELECTED PROBLEMS FROM THE ISRAEL MATHEMATICAL OLYMPIADS, 1994

1. p and q are positive integers. f is a function defined for positive numbers and attains only positive values, such that $f(xf(y)) = x^p y^q$. Prove that $q = p^2$.

Solutions by Pavlos Maragoudakis, Pireas, Greece; and Michael Selby, University of Windsor, Windsor, Ontario. We give the solution of Maragoudakis.

For $x = \dfrac{1}{f(y)}$, we get $f(y) = \dfrac{y^{q/p}}{(f(1))^{1/p}}$.

For $y = 1$, we get $f(1) = 1$, so $f(y) = y^{q/p}$. Hence $f(x \cdot y^{q/p}) = x^p \cdot y^q$. For $y = z^{p/q}$ we get $f(x \cdot z) = x^p z^p$ or $f(x) = x^p$.

Thus $\frac{q}{p} = p$, whence $q = p^2$.

2. The sides of a polygon with 1994 sides are $a_i = \sqrt{4 + i^2}$, $i = 1, 2, \ldots, 1994$. Prove that its vertices are not all on integer mesh points.

Solution by Michael Selby, University of Windsor, Windsor, Ontario.

One assumes integer mesh points are lattice points.

Assume that P_i, the i^{th} vertex, has coordinates (x_i, y_i) where x_i, y_i are integers. Let $\overrightarrow{d_i} = (x_{i+1} - x_i, y_{i+1} - y_i) = (\alpha_i, \beta_i)$ be a vector representation of the i^{th} side. (The indices are read cyclically.) Then $|\overrightarrow{d_i}|^2 = a_i^2 = 4 + i^2$. Also $\overrightarrow{d_{1994}} = (x_1 - x_{1994}, y_1 - y_{1994})$.

We know that

$$\sum_{i=1}^{1994} (\overrightarrow{d_i})^2 = \sum_{i=1}^{1994} (4 + i^2), \qquad \text{or}$$

$$\sum_{i=1}^{1994}(\alpha_i^2+\beta_i^2) = 4\cdot 1994+\frac{(1994)(1995)(3989)}{6}$$
$$= 4\cdot 1994+(997)(665)(3989),$$

which is an odd integer.

However, we know $\sum_{i=1}^{1994}\alpha_i = \sum_{i=1}^{1994}\beta_i = 0$, since $\sum_{i=1}^{1994}\overrightarrow{d_i} = \overrightarrow{0}$.

Therefore $\left(\sum_{i=1}^{1994}(\alpha_i+\beta_i)\right)^2 = 0$.

Thus

$$\sum_{i=1}^{1994}(\alpha_i^2+\beta_i^2)+2\left(\sum_{ij}\alpha_i\beta_j+\sum_{i<j}\alpha_i\alpha_j+\sum_{i<j}\beta_i\beta_j\right) = 0$$

and

$$2\left(\sum_{i,j}\alpha_i\beta_j+\sum_{i<j}\alpha_i\alpha_j+\sum_{i<j}\beta_i\beta_j\right) = -\left(\sum_i(\alpha_i^2+\beta_i^2)\right),$$

giving an odd integer on the right, and an even one on the left, a contradiction.

Therefore, not all the vertices can be lattice points.

5. Find all real coefficients polynomials $p(x)$ satisfying

$$(x-1)^2p(x) = (x-3)^2p(x+2)$$

for all x.

Solutions by F.J. Flanigan, San Jose State University, San Jose, California, USA; and Michael Selby, University of Windsor, Windsor, Ontario. We give Flanigan's solution.

We consider polynomials $p(x)$ with coefficients in a field $\mathbb{F}$ of *arbitrary* characteristic and find as follows:

(i) If char($\mathbb{F}$) $= 0$, (in particular, if $\mathbb{F} = \mathbb{R}$), then $p(x) = a(x-3)^2$, where a is any scalar (possibly 0) in $\mathbb{F}$;

(ii) If char($\mathbb{F}$) $= 2$, then *every* $p(x)$ satisfies the equation (clear);

(iii) If char($\mathbb{F}$) $=$ an odd prime, l, then there are infinitely many solutions, including all $p(x) = a(x-3)^2(x^{l^\nu}-x+c)$ with $a, c \in \mathbb{F}$, and $\nu = 0, 1, 2, \ldots$. (Note that $p(x)$ has the form $a(x-3)^2$ if $\nu = 0$.)

To prove this, observe that if char($\mathbb{F}$) $\neq 2$, then $x-1$ and $x-3$ are coprime, whence $p(x) = (x-3)^2q(x)$ in $\mathbb{F}[x]$.

Thus our equation becomes

$$(x-1)^2(x-3)^2q(x) = (x-3)^2(x-1)^2q(x+2) \qquad (*)$$

whence $q(x) = q(x+2)$, as polynomials; that is, elements of $\mathbb{F}[x]$.

Now if char($\mathbb{F}$) $= 0$, then $(*)$ has only constant solutions.

(The most elementary proof of this: without loss of generality, $q(x) = x^n + ax^{n-1} + \cdots$. Then $q(x+2) - q(x) = 2nx^{n-1} + \cdots$, and this is non-zero if $n \geq 1$. Another proof: $(*)$ implies that $q(x)$ is periodic, which forces equations $q(x) = c$ to have infinitely many roots x, a contradiction).

This establishes the assertion (i).

Re: assertion (iii). Let char($\mathbb{F}$) $= l$ and $q(x) = x^{l^\nu} - x + c$.

Then for $x = 0, 1, \ldots, l-1$, (that is for each element of the prime field), we have $q(x) = c$ and so $q(x) = q(x+1) = q(x+2) = \ldots$, yielding polynomials of degree greater than or equal to l which satisfy $(*)$. This establishes the assertion (iii).

We next turn to solutions to Problems From the Bi-National Israel–Hungary Competition, 1994 [1997: 132].

PROBLEMS FROM THE BI-NATIONAL ISRAEL–HUNGARY COMPETITION, 1994

1. $a_1, \ldots, a_k, a_{k+1}, \ldots, a_n$ are positive numbers ($k < n$). Suppose that the values of $a_{k+1}, \ldots, a_n$ are fixed. How should one choose the values of $a_1, \ldots, a_n$ in order to minimize $\sum_{i,j,i\neq j} \frac{a_i}{a_j}$?

Solutions by F.J. Flanigan, San Jose State University, San Jose, California, USA; and Michael Selby, University of Windsor, Windsor, Ontario. We give Flanigan's solution.

To minimize the given rational function, choose

$$a_i = \left(\frac{a_{k+1} + \cdots + a_n}{\frac{1}{a_{k+1}} + \cdots + \frac{1}{a_n}}\right)^{1/2} = (\mathbb{A} \cdot \mathbb{H})^{1/2}, \quad i = 1, 2, \ldots, k$$

where $\mathbb{A}$ is the arithmetic and $\mathbb{H}$ the harmonic mean of $a_{k+1}, \ldots, a_n$.

To prove this, we will be forgiven if we change notation: let $x_i = a_i$, $i = 1, 2, \ldots, k$ and $b_r = a_{k+r}$, $r = 1, \ldots, m$ with $k + m = n$, and denote the given rational function $F(x_1, \ldots, x_k)$. Then we have $F(x_1, \ldots, x_k) = X + Y + B$, where

$$\begin{aligned} X &= \sum_{1\leq i<j\leq k} \left(\frac{x_i}{x_j} + \frac{x_j}{x_i}\right), \\ Y &= \sum_{1\leq i\leq k} \sum_{1\leq r\leq m} \left(\frac{x_i}{b_r} + \frac{b_r}{x_i}\right), \\ B &= \sum_{1\leq r<s\leq m} \left(\frac{b_r}{b_s} + \frac{b_s}{b_r}\right). \end{aligned}$$

Note that B is fixed and Y can be improved to

$$\begin{aligned} Y &= \sum_{1\le i\le k}\left(\left(\sum_{1\le r\le m}\frac{1}{b_r}\right)x_i+\left(\sum_{1\le r\le m}b_i\right)\frac{1}{x_i}\right)\\ &= \sum_i\left(\frac{m}{\mathbb{H}}x_i+\frac{m\mathbb{A}}{x_i}\right)\end{aligned}$$

where $\mathbb{A}$ is the arithmetic mean and $\mathbb{H}$ is the harmonic mean of the b_r.

Now we recall that the simple function $\alpha x+\frac{\beta}{x}$ (with α, β, x all positive) assumes its minimum when $\alpha x = \frac{\beta}{x}$; that is $x = \sqrt{\beta/\alpha}$. Thus each of the terms in Y (and so Y itself) assumes its minimum when we choose, for $i = 1, 2, \ldots, k$,

$$x_i = \sqrt{\frac{m\mathbb{A}}{(m/\mathbb{H})}} = \sqrt{\mathbb{A}\mathbb{H}},$$

as asserted.

But there is more. It is also known that each term in X, (and so X itself) assumes its minimum when $x_i = x_j$, with $1 \le i < j \le k$. Thus choosing all $x_i = \sqrt{\mathbb{A}\mathbb{H}}$ minimizes *both* X *and* Y and, since B is fixed, minimizes $F(x_1, \ldots, x_k)$ as claimed.

Comments:

(1) It is unusual to minimize the sum of two terms in the same variables by minimizing each term simultaneously.

(2) When $m = 2$, then $\sqrt{\mathbb{A}\mathbb{H}} = \mathbb{G}$, the geometric mean of b_1, b_2.

(3) $F_{\min} = k(k-1) + 2k(n-k)^2\sqrt{\mathbb{A}/\mathbb{H}} + B$.

3. m, n are 2 different natural numbers. Show that there exists a real number x, such that $\frac{1}{3} \le \{xn\} \le \frac{2}{3}$ and $\frac{1}{3} \le \{xm\} \le \frac{2}{3}$, where $\{a\}$ is the fractional part of a.

Solution by F.J. Flanigan, San Jose State University, San Jose, California, USA.

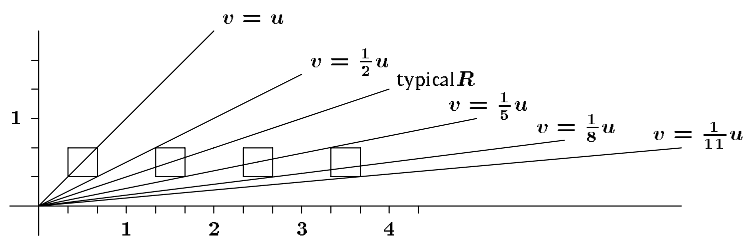
We work in the first quadrant of the standard uv–plane, studying the ray

$$R = \{(u, v) = (xm, xn) : x > 0\}.$$

The key is to observe that the problem is equivalent to showing that the ray R contacts at least one of the "small" $\frac{1}{3}$ by $\frac{1}{3}$ squares in the centres of the large standard 1 by 1 lattice squares (considered as a 3×3 checkerboard). (For if (xm, xn) lies in one of these small squares then $(\{xm\}, \{xn\})$ lies in the small square $\{(u, v) : \frac{1}{3} \le u, v \le \frac{2}{3}\}$ closest to the origin, as desired.)

To establish this contact, we assume, without loss of generality, that $0 < n < m$, so that R is given by $v = \frac{n}{m}u$, $u > 0$.

Consider the sequence of rays $v = \frac{1}{2}u$, $v = \frac{1}{5}u$, $v = \frac{1}{8}u$, $v = \frac{1}{11}u, \ldots$, with $u > 0$. These rays are determined by the lower right corners of the 1^{st}, 2^{nd}, 3^{rd}, $4^{\text{th}}, \ldots$, small central square.



It is now apparent that our ray R lies between, (or on) the ray $v = u$ and $v = \frac{1}{2}u$, **or** $v = \frac{1}{2}u$ and $v = \frac{1}{5}u$, **or** $\ldots$, and hence R will contact the first **or** the second **or** $\ldots$, small square, as required.

Comment: We can now estimate the least x for given m, n in the sequence $1, \frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots$ of slopes and use this to determine which small square is the first to be contacted by the ray R. From this one can estimate the coordinates of (xm, xn) in various ways.

4. An "n-m society" is a group of n girls and m boys. Show that there exist numbers n_0 and m_0 such that every n_0-m_0 society contains a subgroup of 5 boys and 5 girls in which all of the boys know all of the girls or none of the boys knows none of the girls.

Solution by Michael Lebedinsky, student, Henry Wise Wood School, Calgary, Alberta.

We will show that we can take $n_0 = 9$. For $n_0 \geq 9$, observe that for each girl there must be at least 5 boys whom she knows, or 5 boys whom she does not know. We associate to each girl an ordered pair, the first element of which is a subset of 5 of the boys all of whom she knows or all of whom she does not know, and the second element of which is 0 or 1 according as she knows the boys or not. There are $\binom{9}{5} \times 2 = 252$ such pairs. Invoking the Pigeonhole Principle, if $m_0 \geq 4 \times 252 + 1 = 1009$, at least 5 girls must be assigned the same ordered pair, producing 5 girls and 5 boys for which each girl knows each boy, or no girl knows any of the boys.

That completes this Corner for this issue. Enjoy solving problems over the next weeks — and send me your nice solutions as well as your Olympiad materials.

BOOK REVIEWS

Edited by ANDY LIU

Mathematical Challenge, by **Tony Gardiner**,
published by Cambridge University Press, 1996,
ISBN# 0-521-55875-1, softcover, 138+ pages.

More Mathematical Challenges, by **Tony Gardiner**,
published by Cambridge University Press, 1997,
ISBN# 0-521-58568-6, softcover, 140+ pages.

*Reviewed by **Ted Lewis**, University of Alberta.*

These two books are companion volumes which cover the recent phenomenal development in mathematics competitions in the United Kingdom. Although the country has a long and distinguished history in this endeavour, featuring the famed Cambridge Tripos, it was not until the late 1980's when a popularization movement, under the leadership of the author Tony Gardiner, made it a truly national event. This grassroots approach has nurtured the young talents into a force to be reckoned with consistently in the International Mathematical Olympiad.

The first book contains the problems for the UK Schools Mathematical Challenge Papers from 1988 to 1993. The target audience are children aged 12 to 14. Each paper consists of 25 multiple-choice questions, of which the first 15 are relatively straightforward. The paper is to be attempted in one hour, and students are not expected to finish it. Random guessing is discouraged, and calculators are forbidden. Answers, statistics and brief comments are given, but the reader has to work out the solutions.

The second book contains the problems for the UK Junior Mathematical Olympiad from 1989 to 1995. The target audience is children aged 11 to 15. Except in 1989, each paper consists of 10 problems in Section A and 6 problems in Section B. The 1989 paper consists of 13 problems, and may be regarded as a long Section B. The following paragraph is quoted from page 2 of the book.

> *Section A problems are direct, "closed" problems, each requiring a specific calculation and having a single numerical answer. Section B problems are longer and more "open". Thus, while the final mathematical solution is often quite short, and should involve a clear claim, followed by a direct deductive calculation or proof, there will generally be a preliminary phase of exploration and conjecture, in which one tries to sort out how to tackle the problem.*

There is a section titled "Comments and hints" and another titled "Outline solutions". Thus a student who is frustrated by a particular problem has several recourses for assistance. The outline solutions are precisely that, with gaps for the reader to fill in, but revealing enough details to guide a diligent student to the full solution.

We now present some sample problems.

1988/14.
Weighing the baby at the clinic was a problem. The baby would not keep still and

caused the scales to wobble. So I held the baby and stood on the scales while the nurse read off 79 kg. Then the nurse held the baby while I read off 69 kg. Finally I held the nurse while the baby read off 137 kg. What is the combined weight of all three (in kg)?

(a) 142 (b) 147 (c) 206 (d) 215 (e) 284

1993/18.
Sam the super snail is climbing a vertical gravestone 1 metre high. She climbed at a steady speed of 30 cm per hour, but each time the church clock strikes, the shock causes her to slip down 1 cm. The clock only strikes the hours, so at 1 o'clock she would slip back 1 cm, at 2 o'clock she would slip back 2 cm and so on. If she starts to climb just after the clock strikes 3 pm, when will she reach the top?

(a) 3:50 pm (b) 6:20 pm (c) 6:50 pm (d) 7:04 pm (e) 7:20 pm

1994/A6.
A normal duck has two legs. A lame duck has one leg. A sitting duck has no legs. Ninety nine ducks have a total of 100 legs. Given that there are half as many sitting ducks as normal ducks and lame ducks put together, find the number of lame ducks.

1995/B6.
I write out the numbers 1, 2, 3, 4 in a circle. Starting at 1, I cross out every second integer till just one number remains: 2 goes first, then 4, leaving 1 and 3; 3 goes next, leaving 1 — so "1" is the last number left. Suppose I start with 1, 2, 3, ..., n in a circle. For which values of n will the number "1" be the last number left?

Comments and hints for 1995/B6.
The question "For which values of $n \dots$?" is more complicated than it looks. It is not enough to answer "$n = 4$" (even though $n = 4$ works), since we clearly want to know *all* possible values of n that work. Hence a solution must not only show that certain values of n work, but must also somehow prove that *no other values of n could possibly work*.

Outline solution of 1995/B6.
The easy step is to realize that, if n is odd (and ≥ 3) then 1 will be crossed out at the start of the second circuit. Hence, for 1 to survive, either $n = 1$ or n must be e⋆e⋆ (say $n = 2m$). Suppose $n = 2m$. Then at the start of the second circuit, there remain exactly m numbers 1, 3, 5, ..., $2m - 1$. For 1 to survive next time it is essential that either $m = 1$ (so $n = 2$) or m must be e⋆e⋆ (say $m = 2p$). Continuing in this way we see that 1 will be the last uncrossed number if and only if n is a ⋆o⋆e⋆ of ⋆⋆o. [Ed. here ⋆ indicates some number.]

As a bonus, the first book provides 420 additional questions, and the second 40 Section A problems and 20 Section B problems. For one reason or another, these did not make it into the competition papers, but are nevertheless excellent questions.

A lot of thought has gone into the planning of these two books. They are to be *done actively* rather than read passively. The problems are the important features, not the solutions. Particularly valuable are the author's comments on how things are done and *why*. This dynamic package is a must for anyone interested in mathematics competitions for youngsters.

A Note on Special Numerals in Arbitrary Bases

Glenn Appleby, Peter Hilton and Jean Pedersen

1 Introduction

In [1], at the end of a section devoted to the Pigeonhole Principle, the author posed the following problem:

> "4.2.28 Show that, for any integer n, there exists a multiple of n that contains only the digits 7 and 0."

Plainly the author is referring to the representation of the multiple as a numeral in base 10. Thus the reader of the problem is encouraged to believe that the validity of the statement depends in some way on the numbers 7 and 10 and their mutual relationship. Our analysis of the problem shows that, in fact, we may replace 10 by any base $b \geq 2$, and 7 by any digit t in base b. (Of course, we assume $t \neq 0$ to avoid absurd triviality.) Moreover, we prefer to restrict ourselves to numerals consisting of a sequence of k t's followed by l 0's; and, if we assume, as we may without real loss of generality, that $n > 0$, then we have $k \geq 1$, $l \geq 0$. We then give two arguments for the conclusion. The first does employ the Pigeonhole Principle but does not give us a means of calculating all pairs (k, l) from the data. Our second approach yields the minimal pair (k, l) and shows, as expected, that an arbitrary pair (K, L) in the set of solutions is obtained from the minimal pair by taking K to be an arbitrary multiple of k, and L an arbitrary integer satisfying $L \geq l$.

We prefer to replace the modulus n of the problem as stated by m, thus freeing n to stand for the multiple we are seeking. Also we point out that, of course, the analysis we carry out does not depend on the condition $t < b$ imposed by the requirement that t be a digit in base b. Thus the number n we seek may be represented, for arbitrary t and base b, by the expression

$$n = t\,\frac{(b^k - 1)b^l}{b - 1}. \tag{1.1}$$

2 The two main arguments

Let b be an arbitrary base and let t, $0 < t \leq b - 1$, be an arbitrary non-zero digit in base b.

Theorem 1 *For any integer $m > 0$, there exists a positive integer n, written as a numeral in base b consisting of a sequence of t's followed by a (possibly empty) sequence of 0's, such that $n \equiv 0 \bmod m$.*

We give two proofs, the first uses the Pigeonhole Principle, the second being an exercise in modular arithmetic.

Proof A.

Consider the sequence of numerals 0, t, tt, ttt, $\ldots$. The remainders $\bmod\ m$ of these integers lie in a set containing m elements. Thus after at most $(m+1)$ such numerals, some remainder must have been repeated. If $ttt\ldots$ ($k+l$ occurrences), and $ttt\ldots$, (l occurrences), $k > 0$ yield the same remainders, then the numeral $ttt\ldots000\ldots$ (k occurrences of t, l zeros) fulfils the conclusion of the theorem.

This elegant proof has the single defect that it gives no clue as to how one finds the minimal values of k and l as functions of b, t and m. Our second proof is not so neat, but gives more information.

Proof B.

Let $z = b - 1$. We show first how to find a smallest number $n = n_z$, whose numeral in base b consists of a sequence of z's followed by a sequence of 0's, such that $n \equiv 0 \bmod m$.

Write $m = vw$, where v is prime to b and, for every prime p, $p|w$ implies that $p|b$. Note that this factorization of m is unique. Let k be the order of $b \bmod v$ and let l be the smallest non-negative integer such that $w|b^l$. Then the number n, represented in base b by a sequence of k z's followed by a sequence of l 0's, is obviously a positive integer n_z of the form

$$n_z = zzz\ldots000\ldots \tag{2.1}$$

divisible by m. We will show below that n_z, given by (2.1), is minimal for this property.

It is now a trivial matter to complete the proof of the theorem. For if we replace m by mz in the argument above, we find a number $n_z = zzz\ldots000\ldots$ such that $n_z \equiv 0 \bmod mz$, so that n_1, given by $zn_1 = n_z$ gives us

$$n_1 = 111\ldots000\ldots \equiv 0 \bmod m. \tag{2.2}$$

But then

$$n = tn_1 = ttt\ldots000\ldots \equiv 0 \bmod m. \tag{2.3}$$

However, if we want to make sure we have the *smallest* number n of the required form, we should proceed more cautiously.

We first prove that if $n = n_z$ is chosen as in (2.1), then it is the smallest integer of the given form to satisfy $n \equiv 0 \bmod m$.

Now if n is of the form $zzz\ldots 000\ldots$, with K z's and L 0's, then [see (1.1)] $n = (b^K - 1)b^L$. Moreover, since v, w are coprime, we have

$$n \equiv 0 \bmod m \iff n \equiv 0 \bmod v \quad \text{and} \quad n \equiv 0 \bmod w.$$

Since v is prime to b,

$$n \equiv 0 \bmod v \iff b^K - 1 \equiv 0 \bmod v \iff k|K.$$

Further, since, for all primes p, $p|w$ implies that $p|b$, we have that w is prime to $b^K - 1$, so that

$$n \equiv 0 \bmod w \iff b^L \equiv 0 \bmod w \iff L \geq l.$$

Now suppose that n is chosen to be the smallest integer of the form $zzz\ldots 000\ldots$ such that $n \equiv 0 \bmod mz$, and let $n = zn_1$. Then since

$$zn_1 \equiv 0 \bmod mz \iff n_1 \equiv 0 \bmod m, \tag{2.4}$$

it follows that n_1, given by (2.2), is the smallest integer of the form $111\ldots 000\ldots$ to satisfy $n_1 \equiv 0 \bmod m$.

We want now to find the smallest integer n_t represented in base b by a sequence of t's followed by a sequence of 0's [compare (2.3)] to satisfy $n_t \equiv 0 \bmod m$.

Our recipe for constructing n_t is as follows. Let $d = \gcd(m, t)$, $m = m'd$, $t = t'd$. Then if n_1 has the form $111\ldots 000\ldots$,

$$tn_1 \equiv 0 \bmod m \iff t'n_1 \equiv 0 \bmod m' \iff n_1 \equiv 0 \bmod m'. \tag{2.5}$$

Thus we construct n as in (2.1) to be minimal satisfying $n \equiv 0 \bmod m'z$. Then $n = zn_1$, and, by (2.4), we have that n_1 is minimal of the required form to satisfy $n_1 \equiv 0 \bmod m'$, so that finally, by (2.5), we have that $n_t = tn_1$ is minimal of the form $ttt\ldots 000\ldots$ to satisfy $n_t \equiv 0 \bmod m$.

Example 1.

Let us work in base $b = 10$ and look for the smallest number n_6 of the form

$$n_6 = 666\ldots 000\ldots$$

divisible by 99. We have $b = 10$, $z = 9$, $m = 99$, $t = 6$, so $d = 3$, $m' = 33$, $m'z = 297$. To find n, minimal of the form $999\ldots 000\ldots$, to satisfy $n \equiv 0 \bmod 297$, we factorize 297, as in the construction above, as $297 = 297 \times 1$, since 297 is prime to 10. We find $k = 6$ (that is, the order of $10 \bmod 297$ is 6) and, of course, $l = 0$. Thus $n = 999999$, so $n_1 = 111111$, $n_6 = 666666$.

Example 1 brings out the important practical point that, to find n_t, we simply find the minimal n of the form $zzz\ldots 000\ldots$ to satisfy $n \equiv 0 \bmod m'z$, and then replace z by t in the numeral for n. Indeed, all we have to do is to find the minimal values of k and l which yield n (see Section 3).

It is also plain that, having obtained our minimal n_t, involving k t's followed by l 0's, we obtain *all* solutions of the congruence $N \equiv 0 \bmod m$ of the required form by taking K t's followed by L 0's, where $k|K$ and $L \geq l$. Notice that this implies that every solution of the congruence $N \equiv 0 \bmod m$, of the required form, is a multiple of the minimum solution.

3 The algorithm

We extract from the analysis in Section 2 the algorithm for finding the minimal pair (k, l) as functions of b, t and m.

Given: b (base), t (digit) and m (modulus).

- **Set** $\gcd(m, t) = d$, and $m = m'd$.
- **Write** $m'(b-1) = vw$ where v is prime to b and, for all primes p,

$$p|w \quad \Longrightarrow \quad p|b.$$

- **Finally**, let k be the order of $b \bmod v$ (that is, $b^k \equiv 1 \bmod v$, with k positive minimal) and let l be minimal such that $w|b^l$. Then

$$n_t = \overbrace{ttt\ldots}^{k \text{ times}}\overbrace{000\ldots}^{l \text{ times}}$$

  is the minimal numeral n, in base b, consisting of a sequence of k t's followed by a sequence of l 0's, such that $m|n$.

Example 2.

Given: $b = 7$, $t = 5$ and $m = 2499$.

- $\gcd(2499, 5) = 1$, so that $m = m'$.
- **Write** $2499 \times 6 = 306 \times 49$ (thus, $v = 306$, $w = 49$).
- **Finally**, let k be the order of $7 \bmod 306$ (that is, $7^k \equiv 1 \bmod 306$, with k positive minimal), and let l be minimal such that $49|7^l$. Using modular arithmetic[1] we see that $k = 48$, and it is clear that $l = 2$. Thus

$$n_5 = \overbrace{555\ldots}^{48 \text{ times}}00$$

[1] $306 = 2 \times 9 \times 17$. The order of $7 \bmod 2$ is 1; the order of $7 \bmod 9$ is 3; the order of $7 \bmod 17$ is 16. To see the last, observe that, by Fermat's Theorem, $7^{16} \equiv 1 \bmod 17$; but $7^2 \equiv -2 \bmod 17$, so $7^8 \equiv -1 \bmod 17$. Thus the order of $7 \bmod 306$ is 48.

is the minimal numeral, in base 7, consisting of a sequence of 5's followed by a sequence of 0's such that $2499|n_5$.

Example 2 shows that it may sometimes be very tedious to apply the Pigeonhole Principle to obtain k and l.

Notice that, in the special case $t = b-1$, we may simplify the algorithm by cutting out the first step and replacing $m'(b-1)$ by m in the second step.

Acknowledgment The authors would like to thank Professor Robert Bekes for bringing the original problem in [1] to their attention.

Reference [1] Zeitz, Paul, *The Art and Craft of Problem Solving*, John Wiley & Sons, 1998.

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THE SKOLIAD CORNER

No. 30

R.E. Woodrow

In the last number, we gave the problems of Part I of the Alberta High School Mathematics Competition. Here are the "official" solutions.

ALBERTA HIGH SCHOOL MATHEMATICS COMPETITION Part I — November, 1996

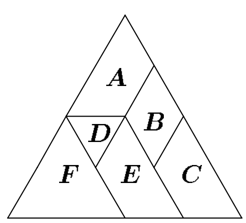
1. An eight-inch pizza is cut into three equal slices. A ten-inch pizza is cut into four equal slices. A twelve-inch pizza is cut into six equal slices. A fourteen inch pizza is cut into eight equal slices. From which pizza would you take a slice if you want as much pizza as possible?

Solution. **(b)** The area of a circle is proportional to the square of its radius. It follows that we are comparing $16/3$, $25/4$, $36/6$ and $49/8$.

2. One store sold red plums at four for a dollar and yellow plums at three for a dollar. A second store sold red plums at four for a dollar and yellow plums at six for a dollar. You bought m red plums and n yellow plums from each store, spending a total of ten dollars. How many plums in all did you buy?

Solution. **(d)** The total expenditure is $10 = \frac{m}{4} + \frac{n}{3} + \frac{m}{4} + \frac{n}{6} = \frac{(m+n)}{2}$.

3.



Six identical cardboard pieces are piled on top of one another, and the result is shown in the diagram.

The third piece to be placed is:

Solution. **(b)** Clearly, F, E, C, B and A are in that order from top to bottom. If D is pointing up, it is under A. If it is pointing down, it is under B.

4. A store offered triple the GST in savings. A sales clerk calculated the selling price by first reducing the original price by 21% and then adding the 7% GST based on the reduced price. A customer protested, saying that the store should first add the 7% GST and then reduce that total by 21%. They agreed on a compromise: the clerk just reduced the original price by the 14% difference. How do the three ways compare with one another from the customer's point of view?

Solution. **(e)** Since 1.07 times 0.79 is 0.8103, both the customer's way and the clerk's way yield a discount approximately 19%.

5. If m and n are integers such that $2m - n = 3$, then what will $m - 2n$ equal?

Solution. **(c)** We have $m - 2n = m - 2n + 2m - n - 3 = 3(m - n - 1)$.

6. If x is $x\%$ of y, and y is $y\%$ of z, where x, y and z are positive real numbers, what is z?

Solution. **(a)** Since y is $y\%$ of z, $z = 100$. The situation is possible if and only if $y = 100$ also.

7. About how many lines can one rotate a regular hexagon through some angle x, $0° < x < 360°$, so that the hexagon again occupies its original position?

Solution. **(e)** The axes of rotational symmetry are the three lines joining opposite vertices, the three lines joining the midpoints of opposite sides, and the line through the centre and perpendicular to the hexagon.

8. AB is a diameter of a circle of radius 1 unit. CD is a chord perpendicular to AB that cuts AB at E. If the arc CAD is $2/3$ of the circumference of the circle, what is the length of the segment AE?

Solution. **(b)** By symmetry, ACD is an equilateral triangle. Hence its centroid is the centre O of the circle. Since $AO = 1$, $AE = AO + OE = 3/2$.

9. One of Kerry and Kelly lies on Mondays, Tuesdays and Wednesdays, and tells the truth on the other days of the week. The other lies on Thursdays, Fridays and Saturdays, and tells the truth on the other days of the week. At noon, the two had the following conversation:

Kerry : I lie on Saturdays.

Kelly : I will lie tomorrow.

Kerry : I lie on Sundays.

On which day of the week did this conversation take place?

Solution. **(b)** Kerry is clearly lying, and is the one who tells the truth on Saturday. Hence the conversation takes place Monday, Tuesday or Wednesday, and Kelly's statement is true only on the last day.

10. How many integer pairs (m, n) satisfy the equation:
$m(m + 1) = 2n$?

Solution. **(c)** Either both m and $m + 1$ are powers of 2, or both are negatives of powers of 2. The two solutions are $(m, n) = (1, 1)$ and $(-2, 1)$.

11. Of the following triangles given by the lengths of their sides, which one has the greatest area?

Solution. **(b)** Note that $5^2 + 12^2 = 13^2$. With two sides of length 5 and 12, the area is maximum when there is a right angle between them.

12. If $x < y$ and $x < 0$, which of the following numbers is never greater than any of the others?

Solution. **(d)** If $y > 0$ or $y = 0$, then $x - y = x - |y|$ is the minimum. If $y < 0$, then $x + y = -|x + y| = x - |y|$ is the minimum.

13. An x by y flag, with $x < y$, consists of two perpendicular white stripes of equal width and four congruent blue rectangles at the corners. If the total area of the blue rectangles is half that of the flag, what is the length of the shorter side of each blue rectangle?

Solution. **(a)** Let z be the length of the shorter side of a blue rectangle. Then the longer side has length $z + \frac{(y-x)}{2}$. From $8z(z + (y - x)) = xy$, we have $z = \frac{1}{4}(x - y + \{x^2 + y^2\}^{1/2})$ or $z = \frac{1}{4}(x - y - \{x^2 + y^2\}^{1/2})$. Clearly, the negative square root is to be rejected.

14. A game is played with a deck of ten cards numbered from 1 to 10. Shuffle the deck thoroughly.

(i) Take the top card. If it is numbered 1, you win. If it is numbered k, where $k > 1$, go to (ii).

(ii) If this is the third time you have taken a card, you lose. Otherwise, put the card back into the deck at the k^{th} position from the top and go to (i).

What is the probability of winning?

Solution. **(c)** If card number 1 is initially in the first or second position from the top, you will win. You can also win if it is in the third, and card number 2 is not in the first. Hence the winning probability is $\frac{1}{10} + \frac{1}{10} + \frac{1}{10}\left(\frac{8}{9}\right)$.

15. Five of the angles of a convex polygon are each equal to 108°. In which of the following five intervals does the maximum angle of all such polygons lie?

Solution. **(a)** The sum of the exterior angles of the five given angles is $5(180° - 108°) = 360°$. Hence these five angles are the only angles of the convex polygon.

16. Which one of the following numbers cannot be expressed as the difference of the squares of two integers?

Solution. **(b)** Suppose $k = m^2 - n^2 = (m - n)(m + n)$. If k is odd, we can set $m - n = 1$ and $m + n = k$. If $k = 4l$, we can set $m - n = 2$ and $m + n = 2l$. However, if $k = 4l + 2$, then $m - n$ and $m + n$ cannot have the same parity.

As a contest this number we give the Fifteenth W.J. Blundon Contest, which was taken by students in Newfoundland and Labrador. The contest is supported in part by the Canadian Mathematical Society. My thanks go to Bruce Shawyer for forwarding me a copy.

15^{th} W.J. BLUNDON CONTEST
February 18, 1998

1. (a) Find the exact value of

$$\frac{1}{\log_2 36}+\frac{1}{\log_3 36}\,.$$

(b) If $\log_{15} 5 = a$, find $\log_{15} 9$ in terms of a.

2. (a) If the radius of a right circular cylinder is increased by 50% and the height is decreased by 20%, what is the change in the volume?

(b) How many digits are there in the number $2^{1998}\cdot 5^{1988}$?

3. Solve: $3^{2+x}+3^{2-x}=82$.

4. Find all ordered pairs of integers such that $x^6 = y^2+53$.

5. When one-fifth of the adults left a neighbourhood picnic, the ratio of adults to children was $2:3$. Later, when 44 children left, the ratio of children to adults was $2:5$. How many people remained at the picnic?

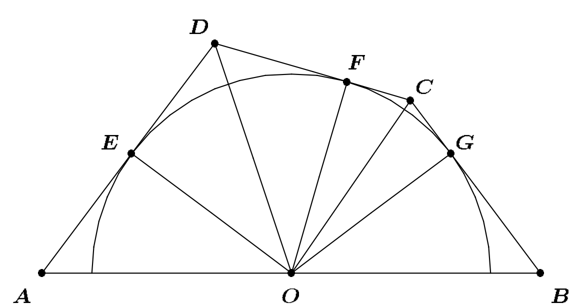
6. Find the area of a rhombus for which one side has length 10 and the diagonals differ by 4.

7. In how many ways can 10 dollars be changed into dimes and quarters, with at least one of each coin being used?

8. Solve: $\sqrt{2+10}+\sqrt[4]{x+10}=12$.

9. Find the remainder when the polynomial $x^{135}+x^{125}-x^{115}+x^5+1$ is divided by the polynomial x^3-x.

10. Quadrilateral $ABCD$ below has the following properties: (1) The midpoint O of side AB is the centre of a semicircle; (2) sides AD, DC and CB are tangent to this semicircle. Prove that $AB^2 = 4AD\times BC$.



Now a correction to a solution given in the October number of the Skoliad Corner.

4. [1997:347] *1995 P.E.I. Mathematics Competition.*
An *autobiographical number* is a natural number with ten digits or less in which the first digit of the number (reading from left to right) tells you how many zeros are in the number, the second digit tells you how many 1's, the third digit tells you how many 2's, and so on. For example, $6,210,001,000$ is an autobiographical number. Find the smallest autobiographical number and prove that it is the smallest.

Correction by Vedula N. Murty, Visakhapatnam, India. The answer given is 1201, but the smallest example is 1210.

Editor's Note: Dyslexia strikes again!

That is all for this issue of the Skoliad Corner. I need your suitable materials and your suggestions for directions for this feature.

Advance Announcement

The 1999 Summer Meeting of the Canadian Mathematical Society will take place at Memorial University in St. John's, Newfoundland, from Saturday, 29 May 1999 to Tuesday, 1 June 1999.

The Special Session on Mathematics Education will feature the topic
What Mathematics Competitions do for Mathematics.
The invited speakers are

Ed Barbeau (University of Toronto),
Ron Dunkley (University of Waterloo),
Tony Gardiner (University of Birmingham, UK), and
Rita Janes (Newfoundland and Labrador Senior Mathematics League).

Requests for further information, or to speak in this session, as well as suggestions for further speakers, should be sent to the session organizers:

Bruce Shawyer and Ed Williams
CMS Summer 1999 Meeting, Education Session
Department of Mathematics and Statistics, Memorial University
St. John's, Newfoundland, Canada A1C 5S7

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Haven, CT 06520–8283 USA. The electronic address is still

mayhem@math.toronto.edu

The Assistant Mayhem Editor is Cyrus Hsia (University of Toronto). The rest of the staff consists of Adrian Chan (Upper Canada College), Jimmy Chui (Earl Haig Secondary School), Richard Hoshino (University of Waterloo), David Savitt (Harvard University) and Wai Ling Yee (University of Waterloo).

Shreds and Slices

From the Archives

An excerpt from "Wilson's Theorem", by Oliver Johnson, Volume 5, Issue 2, *Mathematical Mayhem*:

> One of the more beautiful results in number theory is Wilson's Theorem. [So named because Sir John Wilson didn't discover it, and Leibniz who did discover it didn't prove it—Lagrange did. This is of course an example of the lawyers doing the best out of everything; Wilson was a judge, and it is worth considering how many of Fermat's theorems weren't just won in lawsuits rather than proved by himself... So sue me Pierre.]

Primitive Roots and Quadratic Residues, Part 1

For the following we will need some definitions, none of which should be too abstruse or unfamiliar. For a positive integer n, the polynomial $x^n - 1$ has n distinct roots in the field of complex numbers $\mathbb{C}$. These roots are called the n^{th} *roots of unity*, and denoted by μ_n. An element ζ in μ_n is called *primitive* if $\zeta^k \neq 1$ for $1 \leq k \leq n-1$; in other words, n is the smallest positive exponent k we must raise ζ to the power of to obtain 1. It can be shown that there are exactly $\phi(n)$ primitive n^{th} roots of unity. If n is a prime p, then for ζ to be primitive, it suffices that $\zeta \neq 1$; in other words, any p^{th} root of unity not equal to 1 is primitive. This turns out to be the case we are interested in.

Now, recall quadratic residues modulo n: these are the set of non-zero squares in modulo n. For example, the non-zero squares modulo 10 are 1, 4, 9, $16 \equiv 6$, $25 \equiv 5$, $36 \equiv 6$, and so on, so $\{1, 4, 5, 6, 9\}$ is the set of quadratic residues modulo 10. Let A_n denote the quadratic residues modulo n, and B_n the remaining numbers, so $B_{10} = \{2, 3, 7, 8\}$. If n is a prime p, then both A_p and B_p contain exactly $(p-1)/2$ elements, and this is where things finally get interesting.

Let p be a prime, let ζ be a primitive p^{th} root of unity, and let

$$x = \sum_{a \in A_p} \zeta^a, \quad y = \sum_{b \in B_p} \zeta^b.$$

Then both $x + y$ and xy are always integers. Actually $x + y = -1$, and this is not hard to see. Since $\zeta^p = 1$, or

$$\zeta^p - 1 = (\zeta - 1)\left(\zeta^{p-1} + \zeta^{p-2} + \ldots + \zeta + 1\right) = 0$$

and $\zeta \neq 1$, the latter factor must be 0. Also, A_p and B_p form a partition of $\{1, 2, \ldots, p-1\}$. Hence,

$$x + y = \zeta + \zeta^2 + \ldots + \zeta^{p-1} = -1.$$

But why xy is an integer is a little deeper. Let us consider an example. For $p = 7$, $x = \zeta + \zeta^2 + \zeta^4$ and $y = \zeta^3 + \zeta^5 + \zeta^6$. Then

$$\begin{aligned} xy &= \left(\zeta + \zeta^2 + \zeta^4\right)\left(\zeta^3 + \zeta^5 + \zeta^6\right) \\ &= \zeta^4 + \zeta^6 + \zeta^7 + \zeta^5 + \zeta^7 + \zeta^8 + \zeta^7 + \zeta^9 + \zeta^{10} \\ &= 3 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^6 \ = \ 2. \end{aligned}$$

In particular, this shows that to calculate xy, we do not need to resort to any messy cis notation, and that it is quite accessible by just pencil and paper. Playing around with other primes reveals the following values:

p	3	5	7	11	13
xy	1	-1	2	3	-3

The absolute value of xy always seems to be $(p \pm 1)/4$ (the sign is the one which makes the resulting value an integer). Why is this? And what determines the sign of xy? We encourage readers to play around with this, and to send in any interesting results. We will divulge the reasons behind this phenomenon, as well as some of the deeper theory it will lead to, in the next issue.

Hint: If indeed $x + y$ and xy are integers, then x and y are the roots of a quadratic equation with integer coefficients. What is this quadratic? What does this quadratic say about x and y?

Mayhem Problems

The Mayhem Problems editors are:

Richard Hoshino	*Mayhem High School Problems Editor,*
Cyrus Hsia	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only solutions — the next issue will feature only problems.

We warmly welcome proposals for problems and solutions. We request that solutions from the previous issue be submitted by 1 February 1999, for publication in issue 4 of 1999. Also, starting with this issue, we would like to re-open the problems to all ***CRUX with MAYHEM*** readers, not just students, so now all solutions will be considered for publication.

High School Solutions

Editor: Richard Hoshino, 17 Norman Ross Drive, Markham, Ontario, Canada. L3S 3E8 <rhoshino@undergrad.math.uwaterloo.ca>

H221. Let $P = 19^5 + 660^5 + 1316^5$. It is known that 25 is one of the forty-eight positive divisors of P. Determine the largest divisor of P that is less than $10,000$.

Solution.

If k divides $a+b$, then k divides a^5+b^5 since the latter is a multiple of $a+b$. If k also divides c, then k divides $a^5+b^5+c^5$, since k divides c^5. Notice that $19+660+1316 = 1995 = 3\cdot5\cdot7\cdot19$. Since 3 divides 660 and 3 also divides $19+1316 = 1335$, by our result, 3 divides $660^5+19^5+1316^5 = P$. Similarly, since 5 divides 660, 7 divides 1316 and 19 divides 19, we can show that P is also divisible by 5, 7 and 19. Therefore, P can be expressed in the form $3\cdot5^2\cdot7\cdot19\cdot R$ for some positive integer R, since we are given that 25 is a divisor of P.

If R is prime, we know that P will have $2\cdot3\cdot2\cdot2\cdot2 = 48$ divisors. (Recall that if $p_1, p_2, \ldots, p_n$ are distinct primes, then ${p_1}^{a_1}{p_2}^{a_2}\cdots{p_n}^{a_n}$ has $(a_1+1)(a_2+1)\cdots(a_n+1)$ positive divisors.)

Thus, R must be prime; otherwise P will have more than 48 divisors, and so $P = 3\cdot5^2\cdot7\cdot19\cdot R = 9975\cdot R$, for some large prime R. Since $1316^5 > 1000^5 = 10^{15}$, R will certainly be larger than $10,000$. (Using **Maple**, one can show that R is $408,255,160,853$.) Hence, it follows that the largest divisor of P less than $10,000$ is 9975.

H222. McGregor becomes very bored one day and decides to write down a three digit number ABC, and the six permutations of its digits. To his surprise, he finds that ABC is divisible by 2, ACB is divisible by 3, BAC is divisible by 4, BCA is divisible by 5, CAB is divisible by 6 and CBA is a divisor of 1995. Determine ABC.

Solution by Evan Borenstein, student, Woodward Academy, College Park, Georgia.

Since BCA is divisible by 5, A must be 0 or 5. But we are given that ABC is a three-digit number, so $A = 5$. Since ABC is divisible by 2 and CAB is divisible by 6, both B and C must be even. Since BAC is divisible by 4, $10A + C = 50 + C$ must be a multiple of 4. So C can be 2 or 6. If $C = 2$, B must be 2 or 8, since ACB is divisible by 3 and B is even. If $C = 6$, B must be 4. Hence, there are only three possibilities for ABC: 522, 582 and 546. And of these three, only the second one will make CBA a divisor of 1995, since $1995 = 285 \cdot 7$. Thus we conclude that ABC is 582.

Also solved by Joel Schlosberg, student, Robert Louis Stevenson School, New York, NY, USA.

H223. There are n black marbles and two red marbles in a jar. One by one, marbles are drawn at random out of the jar. Jeanette wins as soon as two black marbles are drawn, and Fraserette wins as soon as two red marbles are drawn. The game continues until one of the two wins. Let $J(n)$ and $F(n)$ be the two probabilities that Jeanette and Fraserette win, respectively.

1. Determine the value of $F(1) + F(2) + \cdots + F(3992)$.

2. As n approaches infinity, what does $J(2) \times J(3) \times J(4) \times \cdots \times J(n)$ approach?

Solution.

1. If Fraserette wins, the balls must be drawn in one of the following three ways: red, red; red, black, red; or black, red, red. This must be the case, as otherwise two black balls will be drawn and Jeanette will win. Hence, the probability that Fraserette wins is the sum of the probabilities of each of three cases above.

Thus,

$$\begin{aligned} F(n) &= \frac{2}{n+2} \cdot \frac{1}{n+1} + \frac{2}{n+2} \cdot \frac{n}{n+1} \cdot \frac{1}{n} + \frac{n}{n+2} \cdot \frac{2}{n+1} \cdot \frac{1}{n} \\ &= \frac{2}{(n+1)(n+2)} + \frac{2}{(n+1)(n+2)} + \frac{2}{(n+1)(n+2)} \\ &= \frac{6}{(n+1)(n+2)}. \end{aligned}$$

Note that $F(n) = \frac{6}{n+1} - \frac{6}{n+2}$, so we can use a telescoping series to calculate the desired sum. We have

$$\begin{aligned} F(1) + F(2) + \cdots + F(3992) &= \left(\frac{6}{2} - \frac{6}{3}\right) + \left(\frac{6}{3} - \frac{6}{4}\right) + \left(\frac{6}{4} - \frac{6}{5}\right) \\ &\quad + \cdots + \left(\frac{6}{3993} - \frac{6}{3994}\right) \\ &= \frac{6}{2} - \frac{6}{3994} \\ &= 3 - \frac{3}{1997} \\ &= \frac{5988}{1997}. \end{aligned}$$

2. Now $J(n) = 1 - F(n) = 1 - \frac{6}{(n+1)(n+2)} = \frac{n^2+3n-4}{(n+1)(n+2)} = \frac{(n-1)(n+4)}{(n+1)(n+2)}$. Thus,

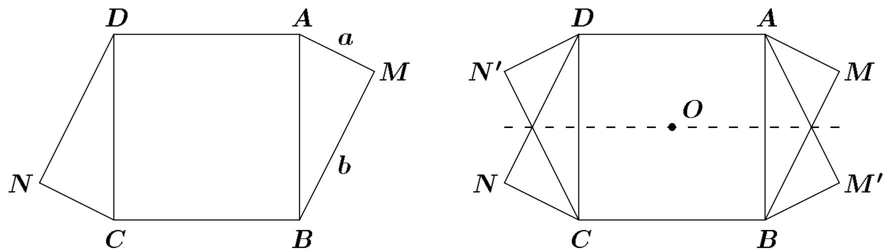
$$\begin{aligned} & J(2) \times J(3) \times J(4) \times \cdots \times J(n) \\ &= \frac{1\cdot 6}{3\cdot 4}\cdot\frac{2\cdot 7}{4\cdot 5}\cdot\frac{3\cdot 8}{5\cdot 6}\cdot\frac{4\cdot 9}{6\cdot 7}\cdots\frac{(n-1)\cdot(n+4)}{(n+1)\cdot(n+2)} \\ &= \frac{1\cdot 2\cdot 3\cdot 4\cdots(n-2)\cdot(n-1)}{3\cdot 4\cdot 5\cdot 6\cdots n\cdot(n+1)} \\ &\quad \cdot\frac{6\cdot 7\cdot 8\cdot 9\cdot(n+3)\cdot(n+4)}{4\cdot 5\cdot 6\cdot 7\cdots(n+1)\cdot(n+2)} \\ &= \frac{1\cdot 2}{n\cdot(n+1)}\cdot\frac{(n+3)\cdot(n+4)}{4\cdot 5} \\ &= \frac{1}{10}\cdot\frac{(n+3)(n+4)}{n(n+1)} \\ &= \frac{1}{10}\cdot\frac{n+3}{n}\cdot\frac{n+4}{n+1}. \end{aligned}$$

Thus, as n approaches infinity, $\frac{n+3}{n} = 1 + \frac{3}{n}$ and $\frac{n+4}{n+1} = 1 + \frac{3}{n+1}$ both approach 1, and so $J(2) \times J(3) \times J(4) \times \cdots \times J(n)$ approaches $\frac{1}{10}$.

H224. Consider square $ABCD$ with side length 1. Select a point M exterior to the square so that $\angle AMB$ is $90°$. Let $a = AM$ and $b = BM$. Now, determine the point N exterior to the square so that $CN = a$ and $DN = b$. Find, as a function of a and b, the length of the line segment MN.

I. *Solution by Adrian Chan, student, Upper Canada College, Toronto, Ontario.*

Let O be the center of the square. Consider a horizontal reflection through the line parallel to DA and passing through O. Let the image of M and N about this line be M' and N', respectively. There exists a point where



MN intersects $M'N'$. Let this point be K. Since K lies on both lines, this point must lie on this horizontal line of reflection.

Now, the diagram is also symmetrical about the line parallel to DC passing through O. Then K must lie on this line as well. This leaves us with K coinciding with O, since the two lines intersect at O.

Now, because of the symmetry, $NO = OM$. That is, $OM = \frac{MN}{2}$. Now consider quadrilateral $OAMB$. Since $\angle AOB$ is right and so is $\angle AMB$, then the quadrilateral is cyclic.

By Ptolemy's Theorem on this quadrilateral, we have $OM \cdot AB = AM \cdot OB + MB \cdot OA = a \cdot \frac{1}{\sqrt{2}} + b \cdot \frac{1}{\sqrt{2}}$, since $OA = OB = \frac{1}{\sqrt{2}}$. Since $OB = 1$, we have $OM = \frac{\sqrt{2}}{2}(a+b)$, and so $MN = \sqrt{2}(a+b)$, since $MN = 2OM$. Thus the length of line segment MN is $\sqrt{2}(a+b)$.

II. *Solution by Joel Schlosberg, student, Robert Louis Stevenson School, New York, NY, USA.*

Let $\angle BAM = x$. Then $a = \cos x$ and $b = \sin x$. Let O be the centre of $ABCD$ and R be the midpoint of AB. Then $\angle BAM = \angle RAM = \angle RMA = x$, and so $\angle ORM = \angle ARO + \angle ARM = \frac{\pi}{2} + (\pi - 2x) = \frac{3\pi}{2} - 2x$. Now $OR = RA = RM = \frac{1}{2}$. By the cosine law on triangle ROM, we have:

$$\begin{aligned}
OM^2 &= OR^2 + RM^2 - 2OR \cdot OM \cos \angle ORM \\
&= \frac{1}{4} + \frac{1}{4} - 2 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \cos\left(\frac{3\pi}{2} - 2x\right) \\
&= \frac{1}{2} - \frac{1}{2} \cdot (-\sin 2x) \\
&= \frac{1}{2} - \frac{1}{2} \cdot 2\sin x \cos x = \frac{1}{2} + ab.
\end{aligned}$$

Thus using the fact that $a^2 + b^2 = 1$, we have $4OM^2 = 2(1 + 2ab) = 2(a^2 + b^2 + 2ab) = 2(a+b)^2$. Hence, $2OM = \sqrt{2}(a+b)$, by taking the square root of both sides (Note: a, b and OM are all positive).

Since $2OM = MN$, we have $MN = \sqrt{2}(a+b)$.

Advanced Solutions

Editor: Cyrus Hsia, 21 Van Allan Road, Scarborough, Ontario, Canada. M1G 1C3 <hsia@math.toronto.edu>

A197. Calculate

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin(2N+1)\theta}{\sin\theta}\, d\theta,$$

where N is a non-negative integer.

Solution by D.J. Smeenk, Zaltbommel, the Netherlands, with minor modifications.

Let $f(N)$ denote $\frac{\sin(2N+1)\theta}{\sin\theta}$. Then

$$f(N+1) - f(N) = \frac{\sin(2N+3)\theta - \sin(2N+1)\theta}{\sin\theta}.$$

Now

$$\begin{aligned}
&\sin(2N+3)\theta - \sin(2N+1)\theta \\
&\quad= \sin(2N+1)\theta\cos 2\theta + \cos(2N+1)\theta\sin 2\theta - \sin(2N+1)\theta \\
&\quad= \sin(2N+1)\theta(\cos 2\theta - 1) + \cos(2N+1)\theta\sin 2\theta \\
&\quad= -2\sin(2N+1)\theta\sin^2\theta + 2\cos(2N+1)\theta\sin\theta\cos\theta \\
&\quad= 2\sin\theta(\cos(2N+1)\theta\cos\theta - \sin(2N+1)\theta\sin\theta) \\
&\quad= 2\sin\theta\cos(2N+2)\theta.
\end{aligned}$$

These two equations imply $f(N+1) - f(N) = 2\cos(2N+2)\theta$. So,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(N+1)\, d\theta - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(N)\, d\theta = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2N+2)\theta\, d\theta = 0.$$

This means

$$f(N) = f(0) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin\theta}{\sin\theta}\, d\theta = \pi.$$

A198. Given positive real numbers a, b, and c such that $a+b+c=1$, show that $a^a b^b c^c + a^b b^c c^a + a^c b^a c^b \le 1$.

Solution.

Using the weighted AM-GM inequality three times, we have the following:

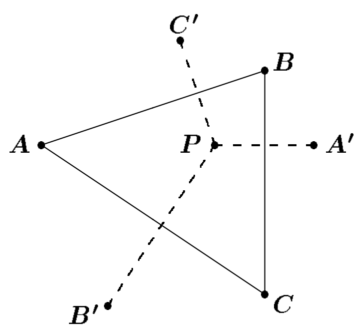
$$\begin{aligned}
\frac{c\cdot a + a\cdot b + b\cdot c}{c+a+b} &\ge (a^c b^a c^b)^{\frac{1}{a+b+c}}, \\
\frac{b\cdot a + c\cdot b + a\cdot c}{b+c+a} &\ge (a^b b^c c^a)^{\frac{1}{a+b+c}}, \\
\frac{a\cdot a + b\cdot b + c\cdot c}{a+b+c} &\ge (a^a b^b c^c)^{\frac{1}{a+b+c}}.
\end{aligned}$$

Adding these inequalities together gives

$$\begin{aligned}1 = a+b+c &= \frac{(a+b+c)^2}{a+b+c} \\ &= \frac{a^2+b^2+c^2+2ab+2ac+2bc}{a+b+c} \\ &\geq a^a b^b c^c + a^b b^c c^a + a^c b^a c^b.\end{aligned}$$

A199. Let P be a point inside triangle ABC. Let A', B', and C' be the reflections of P through the sides BC, AC, and AB respectively. For what points P are the six points A, B, C, A', B', and C' concyclic?

Solution.



We show that AP must be perpendicular to BC. Similar arguments will show the same for BP and CP. Thus, P must be the orthocentre of triangle ABC.

To see that AP is perpendicular to BC, consider angles $C'A'A$ and $C'A'P$ as shown in the diagram. Then $\angle C'A'A = \angle C'BA = \angle PBA = \angle PBF = \angle FDP = \angle C'A'P$ from the quadrilaterals $AC'BA'$ and $BFPD$ being concyclic and the last equality from similar triangles PFD and $PC'A'$. Thus, P lies on AA' so AP is perpendicular to BC.

A200. Given positive integers n and k, for $0 \leq i \leq k-1$, let

$$S_{n,k,i} = \sum_{j \equiv i \pmod{k}} \binom{n}{j}.$$

Do there exist positive integers n, $k > 2$, such that $S_{n,k,0}$, $S_{n,k,1}$, ..., $S_{n,k,k-1}$ are all equal?

Solution.

The answer is NO. To see this, consider the k^{th} roots of unity. In particular, since $k > 2$, there is a k such that $\omega^k = 1$, $\omega \neq -1$. Now consider the expansion of $(1+\omega)^n$:

$$\begin{aligned}(1+\omega)^n &= \sum_{j=0}^{n}\binom{n}{j}\omega^j \\ &= \sum_{j\equiv 0 \pmod{k}}\binom{n}{j} + \omega \sum_{j\equiv 1 \pmod{k}}\binom{n}{j} \\ &\qquad + \cdots + \omega^{k-1}\sum_{j\equiv k-1 \pmod{k}}\binom{n}{j} \\ &= S_{n,k,0} + S_{n,k,1}\omega + \cdots + S_{n,k,k-1}\omega^{k-1}.\end{aligned}$$

Now if all the $S_{n,k,i}$ are equal, say to A, then we have $(1+\omega)^n = A(1+\omega+\cdots+\omega^{k-1}) = A\cdot 0 = 0$. Thus $\omega = -1$, contradiction.

Challenge Board Solutions

Editor: David Savitt, Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA, USA 02138 <dsavitt@math.harvard.edu>

C74. Prove that the k-dimensional volume of a parallelepiped in $\mathbb{R}^n$ spanned by vectors $\vec{v}_1, \ldots, \vec{v}_k$ is the square root of the determinant of the $k \times k$ matrix $\{\vec{v}_i \cdot \vec{v}_j\}_{i,j}$.

Solution.

First, note that by restricting to any k-dimensional subspace of the n-space which contains $\vec{v}_1, \ldots, \vec{v}_k$, we may assume without loss of generality that $k = n$. Let M be the $n \times n$ matrix whose i^{th} column is $\vec{v}_i$, and let P be the parallelepiped spanned by the $\vec{v}_i$. Under the coordinate transformation ϕ sending $\vec{x}$ to $M\vec{x}$, the i^{th} elementary basis vector $\vec{e}_i = (0, \ldots, 1, \ldots, 0)$ is sent to $\vec{v}_i$, and so ϕ transforms the unit cube $[0,1]^n$ onto P. We find then, that

$$\text{Volume}(P) = \int_P 1\, dV = \int_{[0,1]^n} |\det \phi'|\, dV.$$

Since $\phi' = M$, it follows that $\text{Volume}(P) = |\det M| = (\det M^T M)^{1/2}$, and since the i,j^{th}-entry of $M^T M$ is indeed $\vec{v}_i \cdot \vec{v}_j$, we are done.

Remark. The above proof that $\text{Volume}(P) = |\det M|$ is actually a bit bogus, since use is usually made of that result when deriving the change-of-variables formula for integration. So, for those of you who are dissatisfied: let $V(\vec{v}_1, \ldots, \vec{v}_n)$ be the oriented volume of the parallelepiped spanned by $\vec{v}_1, \ldots, \vec{v}_n$ in that order. One can then verify that:

(1) $V(\vec{e}_1, \ldots, \vec{e}_n) = 1$,

(2) $V(\vec{v}_1, \ldots, \vec{v}_i, \ldots, \vec{v}_j, \ldots, \vec{v}_n) = -V(\vec{v}_1, \ldots, \vec{v}_j, \ldots, \vec{v}_i, \ldots, \vec{v}_n)$, and

(3) $V(\vec{v}_1, \ldots, \vec{v}_i + c\vec{v}_i', \ldots, \vec{v}_n) = V(\vec{v}_1, \ldots, \vec{v}_i, \ldots, \vec{v}_n) + cV(\vec{v}_1, \ldots, \vec{v}_i', \ldots, \vec{v}_n)$,

where the last assertion follows because $V(\vec{v}_1, \ldots, \vec{w}, \ldots, \vec{v}_n)$ is directly proportional to the component of $\vec{w}$ which lies perpendicular to the hyperplane spanned by $\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_n$. However, we know from linear algebra that the determinant is the unique function satisfying conditions (1)-(3), so we conclude that $V = \det$, the oriented volume of P is $\det M$, and indeed $\text{Volume}(P) = |\det M|$.

In clearing out the ***Mayhem*** archives, we also dug up this solution to an old problem.

S21. *Proposed by Colin Springer.*

There are n houses situated around a certain lake, in a circle. Each is painted one of k colours, chosen at random. Find the probability that no two neighbouring houses are of the same colour.

Solution by Philip Oppenheimer, South Norwalk, CT.

For convenience, we use the following notation: Let $P(n,k)$ denote the desired probability. Let $h(a)$ denote the colour of the a^{th} house, starting from a fixed house, going around the circle.

Without loss of generality, we may fix $h(1) = 1$. Under this condition, let $p(n,a,b)$ denote the probability that $h(a) = b$, if indeed no two adjacent houses are painted the same colour.

With $h(1) = 1$, there are $k-1$ ways to colour the second house (out of k colours), the third house, etc., until the $(n-1)^{\text{th}}$ house. If $h(n-1) = 1$, then there are $k-1$ ways to colour the n^{th} house. However, if $h(n-1) \neq 1$, then there are $k-2$ ways to colour the n^{th} house. Hence,

$$P(n,k) = \left(\frac{k-1}{k}\right)^{n-2}\left(\frac{k-1}{k}\,p(n,n-1,1) + \frac{k-2}{k}\,(1 - p(n,n-1,1))\right).$$

But for all a,

$$\begin{aligned} p(n,a,1) &= \frac{1}{k-1}\,(1 - p(n,a-1,1)) \\ &= \frac{1}{k-1} - \frac{1}{k-1}\,p(n,a-1,1) \end{aligned}$$

so that

$$\begin{aligned} p(n,a,1) - \frac{1}{k} &= \frac{1}{k(k-1)} - \frac{1}{k-1}\, p(n,a-1,1) \\ &= \frac{-1}{k-1}\left(p(n,a-1,1) - \frac{1}{k}\right) \\ &= \left(\frac{-1}{k-1}\right)^2 \left(p(n,a-2,1) - \frac{1}{k}\right) \\ &= \vdots \\ &= \left(\frac{-1}{k-1}\right)^{a-1} \left(p(n,1,1) - \frac{1}{k}\right) \\ &= \frac{(-1)^{a-1}}{(k-1)^{a-1}} \cdot \frac{k-1}{k} \\ &= \frac{(-1)^{a-1}}{k(k-1)^{a-2}}\,. \end{aligned}$$

Therefore,

$$p(n,n-1,1) = \frac{(-1)^n}{k(k-1)^{n-3}}\,,$$

and

$$\begin{aligned} P(n,k) &= \left(\frac{k-1}{k}\right)^{n-2} \left[\frac{k-1}{k}\left(\frac{1}{k} + \frac{(-1)^n}{k(k-1)^{n-3}}\right) \right. \\ &\qquad \left. + \frac{k-2}{k}\left(1 - \frac{1}{k} - \frac{(-1)^n}{k(k-1)^{n-3}}\right)\right] \\ &= \left(\frac{k-1}{k}\right)^{n-2} \left(\frac{k-1}{k^2} + \frac{(-1)^n(k-1)}{k^2(k-1)^{n-3}} \right. \\ &\qquad \left. + \frac{(k-1)(k-2)}{k^2} - \frac{(-1)^n(k-2)}{k^2(k-1)^{n-3}}\right) \\ &= \left(\frac{k-1}{k}\right)^{n-2} \left(\frac{(k-1)^2}{k^2} + \frac{(-1)^n}{k^2(k-1)^{n-3}}\right) \\ &= \left(\frac{k-1}{k}\right)^{n} + \frac{(-1)^n(k-1)}{k^n}\,. \end{aligned}$$

Swedish Mathematics Olympiad

1986 Qualifying Round

1. Show that
$$(1986!)^{1/1986} < (1987!)^{1/1987}.$$

2. Show that for $t > 0$,
$$t^2 + \frac{1}{t^2} - 3\left(t + \frac{1}{t}\right) + 4 \geq 0.$$

3. A circle C_1 with radius 1 is internally tangent to a circle C_2 with radius 2. Let ℓ be a line through the centres of the circles C_1 and C_2. A circle C_3 is tangent to C_1, C_2, and ℓ. Find the radius of C_3.

4. In how many ways can 11 apples and 9 pears be shared among 4 children, so that every child gets 5 fruit? (The apples are identical, as are the pears.)

5. P is a polynomial of degree greater than 2 with integer coefficients and such that $P(2) = 13$ and $P(10) = 5$. It is known that P has a root which is an integer. Find it.

6. The numbers 1, 2, $\ldots$, n are placed in some order at different points on the circumference of a circle. Form the product of each pair of neighbouring numbers. How should the numbers be placed in order for the sum of these products to be as large as possible?

1986 Final Round

1. Show that the polynomial
$$x^6 - x^5 + x^4 - x^3 + x^2 - x + \tfrac{3}{4}$$
has no real roots.

2. $ABCD$ is a quadrilateral, and O is the intersection of the diagonals AC and BD. The triangles AOB and COD have areas S_1 and S_2 respectively, and the area of $ABCD$ is S. Show that
$$\sqrt{S_1} + \sqrt{S_2} \leq \sqrt{S}.$$
Show also that equality holds if and only if the lines AB and CD are parallel.

3. Let N be a positive integer, $N \geq 3$. Form all pairs (a, b) of consecutive integers such that $1 \leq a < b \leq N$ and consider the quotient $q = \frac{b}{a}$ for every such pair. Remove all pairs with $q = 2$. Show that of the remaining pairs, there are as many with $q < 2$ as there are with $q > 2$.

4. Show that the only positive solution of

$$\begin{aligned} x + y^2 + z^3 &= 3 \\ y + z^2 + x^3 &= 3 \\ z + x^2 + y^3 &= 3 \end{aligned}$$

is $x = y = z = 1$.

5. In the arrangement of pn real numbers below, the difference between the greatest and least numbers in every row is at most d, where $d > 0$:

$$\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pn} \end{matrix}$$

In each column, the numbers are now ordered by size, so that the greatest appears in the first row, the next greatest in the second row, and so on. Show that the difference between the greatest and least numbers in each of the rows is still at most d.

6. A finite number of intervals on the real line together cover the interval $[0, 1]$. Show that one can choose a number of these intervals such that no two have any points in common and whose total length is at least $1/2$.

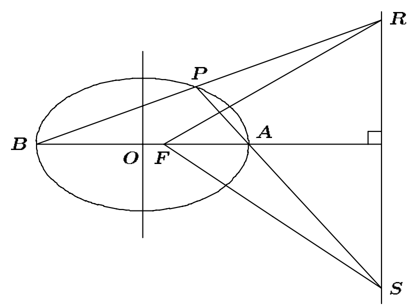
J.I.R. McKnight Problems Contest 1982

1. (a) Given the equal positive rationals $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, prove that

$$\left(\frac{ma^4 + nc^4 + pe^4}{mb^4 + nd^4 + pf^4}\right)^{1/4} \text{ is equal to each of the given rationals.}$$

(b) Given that $a^4 + b^4 = 7^c$ and that a and b are the roots of $x^2 - 5x + 3$, find c.

2. Consider AB, the major axis of an ellipse centred at the origin with focus F as shown. Let P be any point on the ellipse. Draw the lines BP and AP and extend them so that they cross the directrix of F at R and S respectively. Prove that $\angle RFS$ is a right angle.



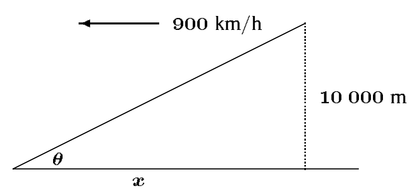
3. Solve the system of equations:

$$\begin{aligned} xy + yz + zx &= -4 \\ y + z - yz &= 4 \\ x - y - z &= 3 \end{aligned}$$

4. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, prove that $x^2 + y^2 + z^2 + 2xyz = 1$.

5. A shopkeeper orders 19 large and 3 small packets of marbles, all alike. When they arrive at the shop it is discovered that all the packets have come open with the marbles loose in the container. If the total number of marbles is 224, can you help the shopkeeper put up the packets with the proper number of marbles in each?

6. A radar tracking station is located at ground level vertically below the path of an approaching aircraft flying at 900 km/h at a constant height of 10000 m. Find the rate in degree/s at which the radar beam to the aircraft is turning at the instant when the aircraft is at a horizontal distance of 3 km from the station.



PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 December 1998**. *They may also be sent by email to* **crux-editors@cms.math.ca**. *(It would be appreciated if email proposals and solutions were written in* LaTeX*). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

2338. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose $ABCD$ is a convex cyclic quadrilateral, and P is the intersection of the diagonals AC and BD. Let I_1, I_2, I_3 and I_4 be the incentres of triangles PAB, PBC, PCD and PDA respectively. Suppose that I_1, I_2, I_3 and I_4 are concyclic.

Prove that $ABCD$ has an incircle.

2339. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

A rhombus $ABCD$ has incircle Γ, and Γ touches AB at T. A tangent to Γ meets sides AB, AD at P, S respectively, and the line PS meets BC, CD at Q, R respectively. Prove that

(a) $\dfrac{1}{PQ}+\dfrac{1}{RS}=\dfrac{1}{BT}$,

and

(b) $\dfrac{1}{PS}-\dfrac{1}{QR}=\dfrac{1}{AT}$.

2340. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $\lambda > 0$ be a real number and a, b, c be the sides of a triangle. Prove that

$$\prod_{\text{cyclic}} \frac{s+\lambda a}{s-a} \geq (2\lambda+3)^3.$$

[As usual s denotes the semiperimeter.]

2341. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let a, b, c be the sides of a triangle. For real $\lambda > 0$, put

$$s(\lambda) := \left|\sum_{\text{cyclic}} \left[\left(\frac{a}{b}\right)^{\lambda} - \left(\frac{b}{a}\right)^{\lambda}\right]\right|$$

and let $\triangle(\lambda)$ be the supremum of $s(\lambda)$ over all triangles.

1. Show that $\triangle(\lambda)$ is finite if $\lambda \in (0,1]$ and $\triangle(\lambda)$ is infinite for $\lambda > 1$.
2. ⋆ What is the exact value of $\triangle(\lambda)$ for $\lambda \in (0,1)$?

2342. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Given A and B are fixed points of circle Γ. The point C moves on Γ, on one side of AB. D and E are points outside $\triangle ABC$ such that $\triangle ACD$ and $\triangle BCE$ are both equilateral.

(a) Show that CD and CE each pass through a fixed point of Γ when C moves on Γ.

(b) Determine the locus of the midpoint of DE.

2343. *Proposed by Doru Popescu Anastasiu, Liceul "Radu Greceanu", Slatina, Olt, Romania.*

For positive numbers sequences $\{x_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1}$, $\{z_n\}_{n\geq 1}$ with conditions: for $n \geq 1$, we have

$$(n+1)x_n^2 + (n^2+1)y_n^2 + (n^2+n)z_n^2 = 2\sqrt{n}\left(nx_ny_n + \sqrt{n}x_nz_n + y_nz_n\right),$$

and for $n \geq 2$, we have

$$x_n + \sqrt{n}y_n - nz_n = x_{n-1} + y_{n-1} - \sqrt{n-1}z_{n-1}.$$

Find $\lim_{n\to\infty} x_n$, $\lim_{n\to\infty} y_n$ and $\lim_{n\to\infty} z_n$.

2344. *Proposed by Murali Vajapeyam, student, Campina Grande, Brazil and Florian Herzig, student, Perchtoldsdorf, Austria.*

Find all positive integers N that are quadratic residues modulo all primes greater than N.

2345. *Proposed by Vedula N. Murty, Visakhapatnam, India.*

Suppose that $x > 1$.

(a) Show that $\ln(x) > \dfrac{3(x^2-1)}{x^2+4x+1}$.

(b) Show that $\dfrac{a-b}{\ln(a)-\ln(b)} < \dfrac{1}{3}\left(2\sqrt{ab}+\dfrac{a+b}{2}\right)$,

where $a > 0$, $b > 0$ and $a \neq b$.

2346. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

The angles of $\triangle ABC$ satisfy $A > B \geq C$. Suppose that H is the foot of the perpendicular from A to BC, that D is the foot of the perpendicular from H to AB, that E is the foot of the perpendicular from H to AC, that P is the foot of the perpendicular form D to BC, and that Q is the foot of the perpendicular from E to AB.

Prove that A is acute, right or obtuse according as $\overline{AH} - \overline{DP} - \overline{EQ}$ is positive, zero or negative.

2347. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Prove that the equation $x^2 + y^2 = z^{1998}$ has infinitely many solutions in positive integers, x, y and z.

2348. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Without the use of trigonometrical formulae, prove that

$$\sin(54^\circ) = \tfrac{1}{2} + \sin(18^\circ)\ .$$

2349. *Proposed by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

Suppose that $\triangle ABC$ has acute angles such that $A < B < C$. Prove that

$$\sin^2 B \sin\left(\tfrac{A}{2}\right)\sin\left(A+\tfrac{B}{2}\right) > \sin^2 A \sin\left(\tfrac{B}{2}\right)\sin\left(B+\tfrac{A}{2}\right)\ .$$

2350. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Suppose that the centroid of $\triangle ABC$ is G, and that M and N are the mid-points of AC and AB respectively. Suppose that circles ANC and AMB meet at (A and) P, and that circle AMN meets AP again at T.

1. Determine $AT : AP$.
2. Prove that $\angle BAG = \angle CAT$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2223. [1997: 111] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

We are given a bag with n identical bolts and n identical nuts, which are to be used to secure the n holes of a gadget.

The $2n$ pieces are drawn from the bag at random one by one. Throughout the draw, bolts and nuts are screwed together in the holes, but if the number of bolts exceeds the number of available nuts, the bolt is put into a hole until one obtains a nut, whereas if the number of nuts exceeds the number of bolts, the nuts are piled up, one on top of the other, until one obtains a bolt.

Let L denote the discrete random variable which measures the height of the pile of nuts.

Find $E[L] + E[L^2]$.

Solution by Gerry Leversha, St. Paul's School, London, England.

[*Editor's comment: Leversha noticed the typographical error in the original statement of the problem and recognized correctly that what was required was $E[L] + E[L^2]$. He found the equivalent: $E[L^2 + L]$.*]

Denote the sequences of bolts and nuts by 0's and 1's respectively; thus, for $n = 2$ there are six possible sequences: 1100, 1010, 1001, 0110, 0101, and 0011, and, in general, there are $\binom{2n}{n}$ different sequences. Call the maximum height of the pile of nuts represented by such a sequence the **value** of the sequence. In the sequences above the heights are respectively $2, 1, 1, 1, 0, 0$.

We approach this problem by deriving a probability generating function $G_n(x)$ for the case of n nuts and n bolts. Define $g_n(x) = \binom{2n}{n} G_n(x)$; then $g_n(x) = \sum_{r=0}^{n} a_r^n x^r$, where a_r^n is the number of sequences of length $2n$ which have value r. The first few such polynomials are as follows:

$$\begin{aligned}
g_0(x) &= 1 \\
g_1(x) &= x + 1 \\
g_2(x) &= x^2 + 3x + 2 \\
g_3(x) &= x^3 + 5x^2 + 9x + 5 \\
g_4(x) &= x^4 + 7x^3 + 20x^2 + 28x + 14
\end{aligned}$$

Notice in each case that the sum of the coefficients is $g_n(1) = \binom{2n}{n}$, as required. In fact, the polynomials can be calculated from the following inductive definition:

$$xg_{n+1}(x) = (x+1)^2 g_n(x) - (x+1)g_n(0) \qquad (*)$$

starting with $g_0(x) = 1$. It is easy to check that $g_n(x)$ is indeed a polynomial of degree n.

$(*)$ follows by virtue of the following relationships between the coefficients:

$$\begin{aligned}
a_{n+1}^{n+1} &= a_n^n \\
a_n^{n+1} &= 2a_n^n + a_{n-1}^n \\
a_r^{n+1} &= a_{r+1}^n + 2a_r^n + a_{r-1}^n \qquad (1 \le r \le n-1) \\
a_0^{n+1} &= a_1^n + a_0^n
\end{aligned}$$

The first of these is immediate, since it merely states that the leading coefficient is 1, and there is obviously only one way to obtain a value of $n+1$ in an $n \pm 1$ sequence, namely by having all the 1's at the front.

The second statement follows because a value of n can be obtained either by placing 10 in front of the unique sequence of value n, or by placing 01 in front of the sequence of value n, or by placing 1 in a sequence of value $n-1$ in front of that part of the sequence which creates the value.

The third statement is the most difficult to see. It states that a sequence of value r, where r is at least 1 and at most $n-1$, can be made either by placing 01 or 10 in front of an existing r sequence, or by devaluing an existing $r+1$ sequence by placing an extra 0 in front of the part which matters, and an extra 1 at the end, or by adding 1 to an existing $r-1$ sequence.

The final statement says that a 0 sequence can be created either by putting 01 in front of an existing 0 sequence or by devaluing an existing 1 sequence by placing an extra 0 in front of the critical part.

These claims are best checked by looking at, say, the sequences for $n = 3$ and seeing how the sequences for $n = 4$ are constructed.

We can now proceed to a calculation of $E[L+L^2]$; this is done by finding the value of $G_n''(1) + 2G_n'(1) = \dfrac{g_n''(1) + 2g_n'(1)}{g_n(1)}$. In fact, we can now make the following claim: $E[L^2 + L] = n$.

This is proved using induction on n. It is trivially true for $n = 0$, so we assume that it is true for $n = k$; that is, we assume that

$$g_k''(1) + 2g_k'(1) = kg_k(1).$$

By differentiating twice the defining equation for $g_{k+1}(x)$, we have

$$\begin{aligned}
xg_{k+1}'(x) + g_{k+1}(x) &= (x+1)^2 g_k'(x) + 2(x+1)g_k(x) - g_k(0)\,, \\
xg_{k+1}''(x) + 2g_{k+1}'(x) &= (x+1)^2 g_k''(x) + 4(x+1)g_k'(x) + 2g_k(x)\,.
\end{aligned}$$

Hence, putting $x = 1$, we have

$$\begin{aligned} g''_{k+1}(1) + 2g'_{k+1}(1) &= 4g''_k(1) + 8g'_k(1) + 2g_k(1) \\ &= (4k+2)g_k(1) \quad \text{(by the inductive hypothesis)} \end{aligned}$$

Hence, for $n = k+1$, $\quad E[L^2 + L] = \dfrac{(4k+2)g_k(1)}{g_{k+1}(1)}$.

However, we know that $g_k(1) = \dbinom{2k}{k}$, and so

$$E[L^2 + L] = \frac{(4k+2)(2k)!(k+1)!(k+1)!}{k!k!(2k+2)!},$$

and it is straightforward to check that this reduces to $k+1$. This finishes the induction step and establishes the claim.

Also solved by the proposer (by a completely different method).

2227. [1997: 166] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Evaluate

$$\prod_p \left[\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2p)^{2k}} \right].$$

where the product is extended over all prime numbers.

Composite solution by David Doster, Choate Rosemary Hall, Wallingford, Connecticut, USA and Kee-Wai Lau, Hong Kong.

It is well known that for $|x| < \frac{1}{4}$, we have

$$\sum_{k=0}^{\infty} \binom{2k}{k} x^k = (1-4x)^{-1/2}, \text{ and hence, } \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2p)^{2k}} = \left(1 - \frac{1}{p^2}\right)^{-1/2}.$$

It is also known that $\zeta(s) = \prod_p \left(1 - \dfrac{1}{p^s}\right)^{-1}$ for $s > 1$, where ζ is the Riemann Zeta function, and that $\zeta(2) = \frac{\pi^2}{6}$. (See, for example, Hardy and Wright, *An Introduction to the Theory of Numbers*, 5$^{\text{th}}$ edition, p. 246.) Therefore,

$$\prod_p \left(\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2p)^{2k}} \right) = \prod_p \left(1 - \frac{1}{p^2}\right)^{-1/2} = (\zeta(2))^{1/2} = \frac{\pi}{\sqrt{6}}.$$

Also solved by MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CON AMORE

PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; RICHARD I. HESS, Rancho Palos Verdes, California, USA; ROBERT B. ISRAEL, University of British Columbia, Vancouver, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.

Most of the submitted solutions are similar to, or virtually the same as, the one given above. By using the same argument, one can easily show that the value of the slightly more general product $P(\lambda) = \prod_p \left(\frac{\binom{2k}{k}}{(4p^\lambda)^k}\right)$ *is* $(\zeta(\lambda))^{1/2}$. *This was pointed out by Janous and the proposer.*

2228. [1997: 167] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Let A be the set of all real numbers from the interval $(0,1)$ whose decimal representation consists only of 1's and 7's; that is, let

$$A = \left\{\sum_{k=1}^{\infty} \frac{a_k}{10^k} : a_k \in \{1,7\}\right\}.$$

Let B be the set of all reals that cannot be expressed as finite sums of members of A. Find $\sup B$.

Solution by the proposer.

We prove that $\sup B = 1$.

Fix $a \in [1,7]$ and consider $b = (a-1)/6$. Since $b \in [0,1]$, b can be easily expressed as a sum of nine reals from $[0,1]$ each consisting only of 0's and 1's in decimal representation. Now multiplying each by 6 and next adding $.1111\ldots$ to the result, we obtain (since $1 = 9 \times .1111\ldots$) an expression for a as a sum of nine reals, each consisting only of 1's and 7's.

Now take any $a > 7$ and subtract from a as many $.1111\ldots$'s as needed to get into the interval $[1,7]$. This procedure shows that any $a \geq 1$ is a finite sum of members of A; that is, $\sup B \leq 1$.

Now let $x \in (8/9, 1)$ be a finite sum of members of A (so that $x \notin B$). Fixing our attention, assume that x is a sum of k members of A; since $x \in (8/9,1)$ and each member of A is at least $.1111\ldots$, we know that $2 \leq k \leq 8$. Consider

$$y = \frac{x - k(.1111\ldots)}{6}.$$

Then y is a sum of k real numbers, each consisting only of 1's and 0's. This implies, since $k \leq 8$, that y does not contain a digit 9 in its decimal representation. Thus we have shown that a number $x \in (8/9,1)$ must be in B if the following property holds:

(∗): *all seven numbers* $\frac{1}{6}(x - k(.1111\ldots))$, $k = 2, 3, \ldots, 8$, *contain a* 9 *in their decimal representation.*

Denote the interval

$$\left(\frac{8-k}{54}, \frac{9-k}{54}\right)$$

by J_k ($k = 2, 3, \ldots, 8$). The set C_k of all numbers from J_k having a unique decimal representation and containing at least one 9 is easily seen to be open and dense in J_k. [*Editorial note.* For the benefit of readers not familiar with these terms, here are a couple of explanations. "C_k is open" means that for any $c \in C_k$, there is some interval $(c - \epsilon, c + \epsilon)$ ($\epsilon > 0$) which is completely contained in C_k; that is, all numbers close enough to c also have a 9 in their decimal representation. "C_k is dense in J_k" means that for any number $y \in J_k$, and for any $\epsilon > 0$, the interval $(c - \epsilon, c + \epsilon)$ must contain some member of C_k; that is, there are numbers with at least one digit 9 as close as you like to y, even if y itself does not have a 9 in it.]

For each $k = 2, 3, \ldots, 8$, the map

$$\varphi_k(x) = \frac{x - k(.1111\ldots)}{6} = \frac{9x - k}{54}$$

is increasing and continuous and takes $(8/9, 1)$ onto J_k. Therefore the sets $\varphi_k^{-1}(C_k)$, $k = 2, 3, \ldots, 8$ are open and dense in $(8/9, 1)$, and thus so is the intersection

$$\varphi_2^{-1}(C_2) \cap \varphi_3^{-1}(C_3) \cap \cdots \cap \varphi_8^{-1}(C_8).$$

[*Editorial note.* In other words, the intersection of finitely many open and dense sets is open and dense. Proof left to the reader! Or look at problem 16, Chapter 2 of Rudin's *Principles of Mathematical Analysis.*] But all elements x in this intersection have the property (∗), and thus lie in B, which means that B is dense in $(8/9, 1)$. This completes the proof that $\sup B = 1$.

Remark. In the last part of the proof we have used Baire's Theorem, in fact its easier version for the intersection of finitely many open and dense sets instead of countably many. Also, the argument used in the solution shows that there are plenty of elements of B in $(8/9, 1)$! So I believe that a reader not familiar with topology might find an explicit example of a sequence of elements of B which get arbitrarily close to 1.

Also solved by GERALD ALLEN, CHARLES DIMINNIE, TREY SMITH and ROGER ZARNOWSKI (jointly), Angelo State University, San Angelo, Texas, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; and DAVID STONE and VREJ ZARIKIAN, Georgia Southern University, Statesboro, Georgia, USA. Two other readers sent in incomplete solutions.

The other solutions to this problem actually contain the explicit examples that the proposer asks for in his remark above. Allen, Diminnie, Smith and Zarnowski give $b_n = .99\ldots944999$, *where there are* $(3n + 1)$ 9*'s before the two* 4*'s. Stone and Zarikian use* $b_n = .99\ldots9546$, *where there are* $(3n - 2)$ 9*'s before the* 546.

Lambrou uses somewhat more complicated b_n's. In all cases the solvers then must do some calculation to show that the b_n's all lie in B.

2231. [1997: 167] *Proposed by Herbert Gülicher, Westfalische Wilhelms-Universität, Münster, Germany.*

In quadrilateral $P_1P_2P_3P_4$, suppose that the diagonals intersect at the point $M \neq P_i$ $(i = 1, 2, 3, 4)$. Let $\angle MP_1P_4 = \alpha_1$, $\angle MP_3P_4 = \alpha_2$, $\angle MP_1P_2 = \beta_1$ and $\angle MP_3P_2 = \beta_2$.

Prove that

$$\lambda_{13} := \frac{|P_1M|}{|MP_3|} = \frac{\cot\alpha_1 \pm \cot\beta_1}{\cot\alpha_2 \pm \cot\beta_2},$$

where the $+(-)$ sign holds if the line segment P_1P_3 is located inside (outside) the quadrilateral.

Preliminary comment. No solver made use of directed angles and segments, which would have reduced the problem to a single case. The proposer's angles are assumed to be positive, and nobody explicitly went through all four resulting possibilities: M can lie between P_1 and P_3 or not, and between P_2 and P_4 or not. Since $P_1P_2P_3P_4$ need not have an interior, a more careful statement of the proposer's two cases would distinguish whether or not P_2 and P_4 lie on opposite sides of the line P_1P_3. We shall feature the opposite-side case, and leave the other to the reader.

Solution by Michael Lambrou, University of Crete, Crete, Greece (modified by the editor to explicitly distinguish the two subcases).

Assume M to be any point of the line P_1P_3 except P_1, P_3, and let P_2P_4 be any other line through M with P_2, P_4 on opposite sides of M. Let $\omega = \angle P_1MP_4$. Then $\sin\angle P_1P_4M = \sin(\pi-(\alpha_1+\omega)) = \sin(\alpha_1+\omega)$, while $\sin\angle P_3P_4M = \sin(\omega \pm \alpha_2)$ where '$+$' corresponds to the subcase where M is outside the segment P_1P_3, while '$-$' corresponds to where M is between P_1 and P_3. From the sine rule applied to triangles MP_1P_4 and MP_3P_4 we have

$$P_1M\frac{\sin\alpha_1}{\sin(\alpha_1+\omega)} = MP_4 = MP_3\frac{\sin\alpha_2}{\sin(\omega\pm\alpha_2)}.$$

Hence,

$$\begin{aligned}\frac{P_1M}{MP_3} &= \frac{\sin\alpha_2\sin(\alpha_1+\omega)}{\sin(\omega\pm\alpha_2)\sin\alpha_1}\\ &= \frac{\sin\alpha_2\sin\alpha_1\cos\omega+\sin\alpha_2\cos\alpha_1\sin\omega}{\sin\omega\cos\alpha_2\sin\alpha_1\pm\cos\omega\sin\alpha_2\sin\alpha_1}\\ &= \frac{\cot\omega+\cot\alpha_1}{\cot\alpha_2\pm\cot\omega} \qquad (1)\end{aligned}$$

(where we divided both the numerator and the denominator by the product of three sines to get the last equality).

Similarly from triangles MP_1P_2 and MP_3P_2 we have

$$\begin{aligned} \frac{P_1M}{MP_3} &= \frac{\sin\beta_2 \sin(\omega-\beta_1)}{\sin\beta_1 \sin(\omega \mp \beta_2)} \\ &= \frac{\cot\beta_1 + \cot\omega}{\cot\beta_2 \mp \cot\omega}. \end{aligned} \qquad (2)$$

The result follows by combining (1) and (2) using the following property of proportions: if $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ then $\frac{a}{b} = \frac{(c+e)}{(d+f)}$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GERRY LEVERSHA, St. Paul's School, London, England; ISTVÁN REIMAN, Budapest, Hungary; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

2232. [1997: 168] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Find all solutions of the inequality:

$$n^2+n-5 < \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+1}{3} \right\rfloor + \left\lfloor \frac{n+2}{3} \right\rfloor < n^2+2n-2, \ (n \in \mathbb{N}).$$

(Note: If x is a real number, then $\lfloor x \rfloor$ is the largest integer not exceeding x.)

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

The only solution is $n = 2$. Let

$$M = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+1}{3} \right\rfloor + \left\lfloor \frac{n+2}{3} \right\rfloor.$$

Then

$$M \le \frac{n}{3} + \frac{n+1}{3} + \frac{n+2}{3} = n+1.$$

Hence $n^2+n-5 < n+1$, or $n^2 < 6$, and so $n = 1$ or 2. But when $n = 1$, $M = 1 = n^2+2n-2$, and so $n = 1$ is not a solution. By inspection, $n = 2$ is a solution.

Also solved by CHARLES ASHBACHER, Cedar Rapids, Iowa, USA; MICHEL BATAILLE, Rouen, France; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, Oregon, USA; MICHAEL

LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; SEAM MCILROY, student, Vancouver, BC; ANNE MARTIN, Westmont College, Santa Barbara, California, USA; CAN ANH MINH, Berkeley, California, USA; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; CHRISTOS SARAGOTIS, student, Aristotle University, Thessaloniki, Greece; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; REZA SHAHIDI, student, University of Waterloo, Waterloo, Ontario, Canada; D.J. SMEENK, Zaltbommel, the Netherlands; DIGBY SMITH, Mount Royal College, Calgary, Alberta; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; and the proposer. There were also three incomplete and one incorrect solution submitted.

Some solvers initially prove the equality

$$\left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+1}{3} \right\rfloor + \left\lfloor \frac{n+2}{3} \right\rfloor = n,$$

or some more general form of it. The following general form is well-known.

$$\sum_{i=0}^{n-1} \left\lfloor x + \frac{i}{n} \right\rfloor = \lfloor nx \rfloor \text{ for } n \in N, n \neq 0, \text{ and } x \in R$$

(*See for example, D.O. Shklarsky, N.N. Chentzov, and I.M. Yaglom, The USSR Olympiad Problem Book, W.H. Freeman and Company, 1962, p. 24, problem 101,* (*3*).)

2233. [1997: 168] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let x, y, z be non-negative real numbers such that $x + y + z = 1$, and let p be a positive real number.

(a) If $0 < p \leq 1$, prove that

$$x^p + y^p + z^p \;\geq\; C_p\left((xy)^p + (yz)^p + (zx)^p\right),$$

where

$$C_p \;=\; \begin{cases} 3^p & \text{if} \quad p \leq \frac{\log 2}{\log 3 - \log 2}, \\ 2^{p+1} & \text{if} \quad p \geq \frac{\log 2}{\log 3 - \log 2}. \end{cases}$$

(b)* Prove the same inequality for $p > 1$.

Show that the constant C_p is best possible in all cases.

Solution by G.P. Henderson, Garden Hill, Campbellcroft, Ontario.

(a) We are to prove that $F(x, y, z) \geq 0$ where

$$F = \sum x^p - C \sum x^p y^p$$

(the sums here and below are cyclic over x, y, z). Here

$$C = \min(3^p, 2^{p+1}) = 3^p$$

for $0 < p \le 1$. For any real u, v, w,

$$uv + vw + wu \le \frac{1}{3}(u + v + w)^2.$$

[*Editorial note*. This follows from the identity

$$(u + v + w)^2 = 3(uv + vw + wu) + \tfrac{1}{2}\left[(u - v)^2 + (v - w)^2 + (w - u)^2\right],$$

as other solvers point out.] Therefore

$$F \ge \sum x^p - \frac{C}{3}\left(\sum x^p\right)^2 = \left(\sum x^p\right)\left(1 - 3^{p-1}\sum x^p\right) \ge 0,$$

the last inequality following since

$$\max \sum x^p = 3^{1-p}.$$

[For example, since $p \le 1$,

$$\left(\frac{x^p + y^p + z^p}{3}\right)^{1/p} \le \frac{x + y + z}{3} = \frac{1}{3}$$

by the power mean inequality. — *Ed*.] This is the best possible C because $F(1/3, 1/3, 1/3) = 0$.

(b) With no loss of generality we assume $x \ge y \ge z$, which implies

$$z \le \frac{1}{3} \qquad \text{and} \qquad x \ge \frac{1 - z}{2}.$$

Give z a fixed value in $[0, 1/3]$ and set $y = 1 - x - z$ [so that $dy/dx = -1$]. Then F is a function of x only and

$$\begin{aligned}\frac{dF}{dx} &= px^{p-1} - py^{p-1} - Cpx^{p-1}y^p + Cx^p py^{p-1} + Cpy^{p-1}z^p - Cpz^p x^{p-1} \\ &= p(x^{p-1} - y^{p-1})(1 - Cz^p) + pCx^{p-1}y^{p-1}(x - y).\end{aligned}$$

Since $C \le 3^p$ and $z^p \le 3^{-p}$ (and $p > 1$), we see that $dF/dx \ge 0$ and we only need to prove $F \ge 0$ when x has its minimum value, $(1 - z)/2$. That is, we are to prove $F(x, x, z) \ge 0$ where $x = (1 - z)/2 = y$ and $0 \le z \le 1/3$; that is,

$$2x^p + z^p - C(x^{2p} + 2x^p z^p) \ge 0.$$

This is equivalent to showing that

$$\min_{0 \le z \le 1/3} \frac{2x^p + z^p}{x^{2p} + 2x^p z^p} \ge C.$$

Calling the function to be minimized $G(z)$, we find [using $dx/dz = -1/2$]

$$\begin{aligned}\operatorname{sgn}\left(\frac{dG}{dz}\right) &= \operatorname{sgn}\Big((x^{2p}+2x^pz^p)(-px^{p-1}+pz^{p-1}) \\ &\qquad -(2x^p+z^p)(-px^{2p-1}-px^{p-1}z^p+2px^pz^{p-1})\Big) \\ &= \operatorname{sgn}\left(px^{p-1}z^{2p}+px^{2p-1}z^p-3px^{2p}z^{p-1}+px^{3p-1}\right) \\ &= \operatorname{sgn}\left[\left(\frac{z}{x}\right)^{2p}+\left(\frac{z}{x}\right)^{p}-3\left(\frac{z}{x}\right)^{p-1}+1\right].\end{aligned}$$

Set

$$t = \frac{z}{x} = \frac{2z}{1-z}\,;$$

that is,

$$z = \frac{t}{t+2}\,, \qquad 0 \le t \le 1.$$

As t increases from 0 to 1, z increases from 0 to $1/3$. Then $\operatorname{sgn}(dG/dz) = \operatorname{sgn} H(t)$ where

$$H(t) = t^{2p}+t^p-3t^{p-1}+1.$$

We have

$$\frac{dH}{dt} = t^{p-2}(2pt^{p+1}+pt-3p+3).$$

The expression in brackets is a continuous, increasing function. It is negative at $t=0$ and positive at $t=1$. Therefore it has a unique root, t_0, in $[0,1]$. It follows that H decreases to a minimum at t_0 then increases. Since $H(0)=1$ and $H(1)=0$, $H(t_0)<0$ and we see that H has a unique root t_1 in $(0,t_0)$. Therefore

$$\frac{dG}{dz} \ge 0 \quad \text{for} \quad 0 \le t \le t_1 \qquad \text{and} \qquad \frac{dG}{dz} \le 0 \quad \text{for} \quad t_1 \le t \le 1.$$

Hence, putting $z_1 = t_1/(t_1+2)$, G increases in $0 \le z \le z_1$, is a maximum at z_1 and decreases in $z_1 \le z \le 1/3$. Therefore

$$\min G = \min[G(0), G(1/3)] = \min(2^{p+1}, 3^p) = C.$$

Since the minimum is attained in both cases, C is best possible.

Both parts also solved by RICHARD I. HESS, Rancho Palos Verdes, California, USA; and MICHAEL LAMBROU, University of Crete, Crete, Greece. Part (a) *only solved by HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. One other reader sent in a comment.*

Seiffert and the proposer had the same proof for part (a) *as Henderson (and incidentally, the proof seems to hold for negative* p *as well). For part* (b) *both Hess and Lambrou used multivariable calculus.*

2234. [1997: 168] *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Given triangle ABC, its centroid G and its incentre I, construct, using only an unmarked straightedge, its orthocentre H.

Solution by the proposer edited to fit the rewording of the problem from his original submission.

We will first establish a lemma:

Lemma. Given a line segment and its midpoint, and any other point O in the plane, a line passing through O and parallel to the given line segment may be constructed using only an unmarked straightedge.

Proof: Let AD be the given line segment and let K be its midpoint. Let P be any point on AO extended. Connect P with D and with K. Draw segment OD. Let M be the intersection of OD and PK, and let N be the intersection of AM with PD. We will show that ON is parallel to AD. Suppose instead that ON_1 is parallel to AD with N_1 on PD. Let M_1 be the intersection of AN_1 and OD, let K_1 be the intersection of PM_1 and AD, and let L_1 be the intersection of ON_1 with PK_1. Then $\triangle AM_1K_1 \sim \triangle N_1M_1L_1$ and $\triangle OM_1L_1 \sim \triangle DM_1K_1$. Therefore,

$$\frac{AK_1}{L_1N_1} = \frac{K_1M_1}{M_1L_1} = \frac{K_1D}{OL_1}. \tag{1}$$

Also $\triangle APK_1 \sim \triangle OPL_1$ and $\triangle DK_1P \sim \triangle N_1L_1P$, which implies

$$\frac{AK_1}{OL_1} = \frac{K_1P}{L_1P} = \frac{K_1D}{L_1N_1}. \tag{2}$$

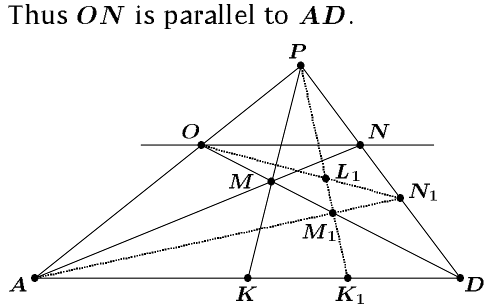
From (1) and (2) it follows that

$$AK_1 \cdot OL_1 = L_1N_1 \cdot K_1D, \qquad AK_1 \cdot L_1N_1 = OL_1 \cdot K_1D.$$

Consequently,

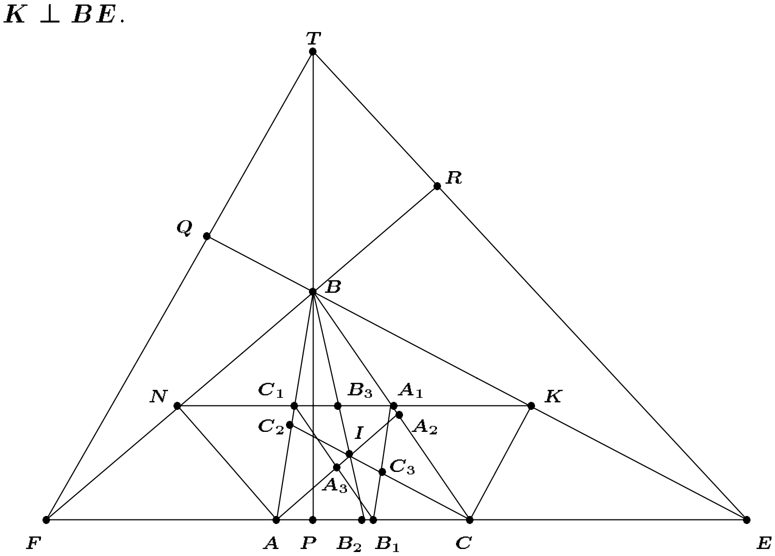
$$(AK_1)^2 \cdot OL_1 \cdot L_1N_1 = (K_1D)^2 \cdot OL_1 \cdot L_1N_1\,,$$

from which we have $(AK_1)^2 = (K_1D)^2$, and $AK_1 = K_1D$. Thus point K_1 coincides with point K, PK_1 coincides with PK, point M_1 with M and point N_1 with N. Thus ON is parallel to AD.



Using the centroid G draw the medians AA_1, BB_1, CC_1, where A_1, B_1, C_1 are the midpoints of BC, CA, AB, respectively. Similarly, using the incentre I draw the angle bisectors AA_2, BB_2, CC_2, where A_2, B_2, C_2 lie on BC, CA, AB, respectively. Draw triangle $A_1B_1C_1$ and denote $C_1A_1 \cap BB_2$ by B_3, $A_1B_1 \cap CC_2$ by C_3, $B_1C_1 \cap AA_2$ by A_3. Thus A_3, B_3, C_3 are the midpoints of AA_2, BB_2, CC_2, respectively.

By applying the lemma twice we may draw a straight line BE through B parallel to angle bisector CC_2 and draw a straight line BF parallel to angle bisector AA_2 (with points E and F located on the line CA). We see that triangles FAB and BCE are isosceles ($\angle FBA = \angle BFA$, $\angle EBC = \angle BEC$). Draw a straight line A_1K parallel to CA with K on BE. Thus K is the midpoint of BE. Similarly get the midpoint N of FB. Thus $AN \perp FB$ and $CK \perp BE$.



Now we may draw lines parallel to FB and BE through the points C_1 and A_1 respectively, and get the midpoints of AN and CK. Then applying the lemma we may draw FT parallel to CK and ET parallel to AN. Denote $ET \cap FB$ by R and $BE \cap FT$ by Q. Thus $FR \perp ET$, $EQ \perp FT$ and point B is the point of intersection of the altitudes of triangle FTE. Draw line TB, denote $TB \cap CA$ by P and get $BP \perp CA$; that is, BP is an altitude of triangle ABC. We can construct a second altitude similarly. Their intersection is H, the orthocentre.

Also solved or answered by JORDI DOU, Barcelona, Spain and TOSHIO SEIMIYA, Kawasaki, Japan. There was one incorrect solution.

2235. [1997: 168] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands, not Walther Janous, Ursulinengymnasium, Innsbruck, Austria as was printed.*

Triangle ABC has angle $\angle CAB = 90^\circ$. Let $\Gamma_1(O, R)$ be the circumcircle and $\Gamma_2(T, r)$ be the incircle. The tangent to Γ_1 at A and the polar line of A with respect to Γ_2 intersect at S. The distances from S to AC and AB are denoted by d_1 and d_2 respectively.

Show that

(a) $ST \| BC$,

(b) $|d_1 - d_2| = r$.

[For the benefit of readers who are not familiar with the term "polar line", we give the following definition as in, for example, *Modern Geometries*, 4th Edition, by James R. Smart, Brooks/Cole, 1994:

The line through an inverse point and perpendicular to the line joining the original point to the centre of the circle of inversion is called the polar of the original point, whereas the point itself is called the pole of the line.]

Solution by István Reiman, Budapest, Hungary.

Assume without loss of generality that $AB > AC$. [Should $AB = AC$ the polar line and tangent would be parallel so that S would be at infinity, while $AB < AC$ would involve only minor changes in notation.] Let P be the point where Γ_2 touches AB, and Q be where it touches AC. Thus $AQTP$ is a square whose sides have length r. Next, let U and V be the feet of the perpendiculars from S to AB and to AC respectively, so that $AUSV$ is a rectangle.

The polar of A is the line PQ. Since PQT is an isosceles right triangle, PQ makes a 45^o angle with AC, which implies that QSV is also an isosceles right triangle. Consequently $SV = QV = d_1$, and $AV - SV = d_2 - d_1 = AV - QV = AQ = r$ (which proves (b)).

Now, $\angle QAS = \angle CBA$, since both angles are subtended by the chord AC of the circle Γ_1. Moreover, $\angle QAS = \angle QTS$ because these angles are symmetric about the line PQ. Since the corresponding side vectors $\overrightarrow{TQ}$ and $\overrightarrow{BA}$ have the same direction, it follows that so do the vectors $\overrightarrow{TS}$ and $\overrightarrow{BC}$, and we conclude that the lines ST and BC are parallel as desired.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; ADRIAN BIRKA, student, Lakeshore Catholic High School, Niagara Falls, Ontario; MANSUR BOASE, student, St. Paul's School, London, England;; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; JIMMY CHUI, student, Earl Haig Secondary School, North York, Ontario; FILIP CRNOGORAC, student, Western Canada High School, Calgary, Alberta; JORDI

DOU, Barcelona, Spain; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece (2 solutions); JAMES LEE, student, Eric Hamber Secondary School, Vancouver, BC; GERRY LEVERSHA, St. Paul's School, London, England; DAVID NICHOLSON, student, Fenelon Falls Secondary School, Fenelon Falls, Ontario; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; CHRISTOPHER SO, student, Franch Liberman Catholic High School, Scarborough, Ontario; KAREN YEATS, student, St. Patrick's High School, Halifax, Nova Scotia; and the proposer.

2236. [1997: 169] *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Let ABC be an arbitrary triangle and let P be an arbitrary point in the interior of the circumcircle of $\triangle ABC$. Let K, L, M, denote the feet of the perpendiculars from P to the lines AB, BC, CA, respectively.

Prove that $[KLM] \leq \dfrac{[ABC]}{4}$.

Note: $[XYZ]$ denotes the area of $\triangle XYZ$.

Almost identical solutions were submitted by Niels Bejlegaard, Stavanger, Norway; Mansur Boase, student, St. Paul's School, London, England; István Reiman, Budapest, Hungary; Toshio Seimiya, Kawasaki, Japan; and Panos E. Tsaoussoglou, Athens, Greece.

For $\triangle KLM$, with O as circumcentre, we have

$$[KLM] \;=\; \frac{R^2 - OP^2}{4R^2}[ABC] \;\leq\; \frac{[ABC]}{4}\,.$$

[Various different references were given.]

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2237. [1997: 169] *Proposed by Meletis D. Vasiliou, Elefsis, Greece.*

$ABCD$ is a square with incircle Γ. Let ℓ be a tangent to Γ. Let A', B', C', D' be points on ℓ such that AA', BB', CC', DD' are all perpendicular to ℓ.

Prove that $AA' \cdot CC' = BB' \cdot DD'$.

I. *Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.*

We give a solution with coordinates. Clearly, without loss of generality, we can take $A(-1,1)$, $B(1,1)$, $C(1,-1)$, $D(-1,-1)$, and the equation of the incircle is $x^2+y^2=1$. Then, if $P(\cos t, \sin t)$ is any point of the incircle, the equation of the line ℓ, tangent to the incircle, is

$$(\cos t)x + (\sin t)y = 1.$$

(The cases in which ℓ is parallel to either of the axes are trivial.) Then, calculating

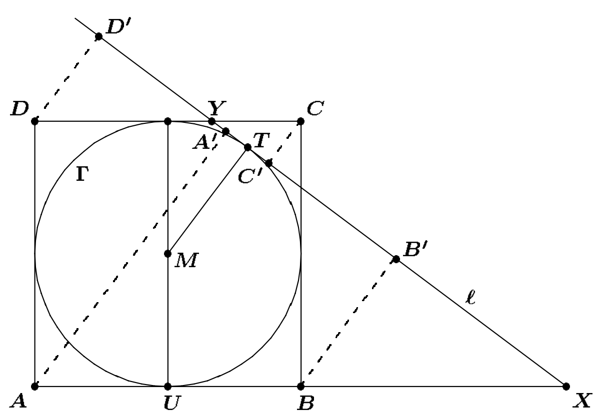
$$\begin{aligned} AA' &= d(A,\ell) = |-\cos t + \sin t - 1| \\ BB' &= d(B,\ell) = |\cos t + \sin t - 1| \\ CC' &= d(C,\ell) = |\cos t - \sin t - 1| \\ DD' &= d(D,\ell) = |-\cos t - \sin t - 1| \end{aligned}$$

we easily obtain

$$AA' \cdot CC' = |\sin 2t| = |-\sin 2t| = BB' \cdot DD',$$

and we are done.

II. *Solution by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.*



First we note that the equation holds obviously if the point of tangency of ℓ and Γ is the midpoint of a side of the square. Otherwise, by rotational symmetry about the centre M of $ABCD$, we may assume that, without loss of generality, ℓ touches the arc of Γ "next to C", say at T. Let X and Y be the points of intersection of l with AB produced and CD, respectively, and U and V the midpoints of AB and CD, respectively. Note that quadrangles $MUXT$ and $YVMT$ are similar, since their respective angles correspond, and $XU = XT$ and $MV = MT$. Hence it follows that

$$\begin{aligned}\frac{AA'}{BB'} = \frac{AX}{BX} &= \frac{AU+UX}{UX-UB} = \frac{MU+UX}{UX-MU} \\ &= \frac{YV+VM}{VM-YV} = \frac{YV+VD}{CV-YV} = \frac{YD}{CY} = \frac{DD'}{CC'}.\end{aligned}$$

III. *Solution by Heinz-Jürgen Seiffert, Berlin, Germany.*

There exist vectors $\mathbf{E}$ and $\mathbf{F}$ with $\mathbf{E}\cdot\mathbf{F} = 0$ and $\|\mathbf{F}\| = 1$, such that

$$\ell = \{\mathbf{E} + t\mathbf{F} \mid t \in \mathbb{R}\}.$$

We have $\mathbf{A} = x\mathbf{E} + y\mathbf{F}$ and $\mathbf{B} = u\mathbf{E} + v\mathbf{F}$ for some reals x, y, u, v. If $\mathbf{A}' = \mathbf{E} + t_A\mathbf{F}$, where $t_A \in \mathbb{R}$, then from $(\mathbf{A} - \mathbf{A}')\cdot\mathbf{F} = 0$, it follows that $t_A = y$, so that $\mathbf{A}' = \mathbf{E} + y\mathbf{F}$. Similarly, we find $\mathbf{B}' = \mathbf{E} + v\mathbf{F}$, and since $\mathbf{C} = -\mathbf{A}$ and $\mathbf{D} = -\mathbf{B}$, we have $\mathbf{C}' = \mathbf{E} - y\mathbf{F}$ and $\mathbf{D}' = \mathbf{E} - v\mathbf{F}$. From $2r^2 = \|\mathbf{A}\|^2 = \|\mathbf{B}\|^2$, where r denotes the radius of Γ, and $\mathbf{A}\cdot\mathbf{B} = 0$, we obtain

$$2r^2 = x^2r^2 + y^2 = u^2r^2 + v^2 \qquad \text{and} \qquad xur^2 + yv = 0.$$

The first equation gives $v^2 = r^2(2 - u^2)$, so that, using the second equation, we have

$$\begin{aligned} x^2u^2r^4 = y^2v^2 &= y^2r^2(2-u^2) \\ \text{or} \qquad u^2(x^2r^2 + y^2) &= 2y^2. \end{aligned}$$

Using the first equation again, we get $y^2 = u^2r^2$ and then $2 = x^2 + u^2$, which by $AA' = \|\mathbf{A} - \mathbf{A}'\| = |x - 1|r$, $BB' = \|\mathbf{B} - \mathbf{B}'\| = |u - 1|r$, $CC' = \|\mathbf{C} - \mathbf{C}'\| = |x + 1|r$, and $DD' = \|\mathbf{D} - \mathbf{D}'\| = |u + 1|r$, implies the desired equation.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; SAM BAETHGE, Nordheim, Texas, USA; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; DENISE CHEUNG, student, Albert Campbell Collegiate Institute, Scarborough, Ontario; JIMMY CHUI, student, Earl Haig Secondary School, North York, Ontario; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; JORDI DOU, Barcelona, Spain; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, Oregon, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece (4 methods); KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; ISTVÁN REIMAN, Budapest, Hungary; K.R.S. SASTRY, Dodballapur, India; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

Most of the solutions were similar to either I *or* II *above. Several solvers made generalizations in different directions.*

Janous considers a regular $2n$*-gon* $P_{2n} := A_1A_2\cdots A_{2n}$ *with incircle* Γ *and* ℓ*, a tangent line to* Γ*; then for the orthogonal projections* $A_1', A_2', \ldots, A_{2n}'$ *of the vertices of* P_{2n} *to* ℓ *we have*

$$\prod_{\substack{k=1 \\ k \text{ even}}}^{2n} A_kA_k' = \prod_{\substack{k=1 \\ k \text{ odd}}}^{2n} A_kA_k' .$$

Konečný observes that the condition "AA'*,* BB'*,* CC'*,* DD' *are all* perpendicular *to* ℓ*" is not necessary; it is sufficient to state that they make the same angle with* ℓ*; that is, they are parallel to one another.*

Sastry shows that if we start with $ABCD$ *being a rhombus with incircle* Γ*, and with* A'*,* B'*,* C'*,* D' *defined as before, we can prove that the equation holds for all lines* ℓ *tangent to* Γ *if and only if* $ABCD$ *is a square.*

Finally, Seimiya comments that if we start with $ABCD$ *any quadrilateral having an incircle* Γ *with centre* O*, and with* ℓ *being a line tangent to* Γ*, then when* A'*,* B'*,* C'*,* D' *are the feet of the perpendiculars from* A*,* B*,* C*,* D*, respectively, to* ℓ*, we have*

$$\frac{AA' \cdot CC'}{BB' \cdot DD'} = \frac{AO \cdot CO}{BO \cdot DO}, \quad \text{a constant.}$$

2238. [1997: 242] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

A four-digit number $\overline{abcd}$ is said to be *faulty* if it has the following property:

> The product of the two last digits c and d equals the two-digit number $\overline{ab}$, while the product of the digits $c-1$ and $d-1$ equals the two digit number $\overline{ba}$.

Determine all faulty numbers!

Solution by David R. Stone, Georgia Southern University, Statesboro, Georgia, USA.

Implicit in the defining properties is that $a \neq 0$, $c \geq 1$, $d \geq 1$, and $b \neq 0$. Translating the given conditions, if $\overline{abcd}$ is to be faulty we must have

$$c \cdot d = \overline{ab} = 10a + b, \tag{1}$$

and

$$(c-1)\cdot(d-1) = \overline{ba} = 10b + a.$$

These imply

$$10b + a = cd - d - c + 1 = 10a + b - d - c + 1,$$

so

$$9(a - b) = c + d - 1.$$

Thus $1 \leq 9(a - b) \leq 17$, forcing $9(a - b) = 9$, or $a = b + 1$. Hence $c + d = 10$. Substituting into (1), we get $c(10 - c) = 10(b + 1) + b$, or

$$c^2 - 10c + (11b + 10) = 0.$$

By the quadratic formula, $c = 5 \pm \sqrt{15 - 11b}$, which forces $15 - 11b = 4$, or $b = 1$. Thus $a = 2$ and $c = 7$ or $c = 3$, which forces $d = 3$ or $d = 7$, respectively. That is, the only two faulty numbers are

2137 and **2173.**

(Checking, $3 \times 7 = 21$ and $(3 - 1) \times (7 - 1) = 12$.)

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHARLES ASHBACHER, Cedar Rapids, Iowa, USA; SAM BAETHGE, Nordheim, Texas, USA; FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts, USA; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; CON AMORE PROBLEM GROUP, The Royal Danish School of Educational Studies, Copenhagen, Denmark; MAYUMI DUBREE, student, Angelo State University, San Angelo, Texas, USA; KEITH EKBLAW, Walla Walla, Washington, USA; JEFFREY K. FLOYD, Newnan, Georgia, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, Oregon, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St. Paul's School, London, England; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; J.A. MCCALLUM, Medicine Hat, Alberta; GRADY MYDLAK, student, University College of the Cariboo, Kamloops, BC; CHRISTOS SARAGIOTIS, student, Aristotle University, Thessaloniki, Greece; JOEL SCHLOSBERG, student, Robert Louis Stevenson School, New York, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; REZA SHAHIDI, student, University of Waterloo, Waterloo, Ontario; SKIDMORE COLLEGE PROBLEM GROUP, Saratoga Springs, New York, USA; D.J. SMEENK, Zaltbommel, the Netherlands; DIGBY SMITH, Mount Royal College, Calgary, Alberta; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

*Although nobody raised the issue, readers must have wondered why the proposer chose the term "faulty" for numbers satisfying this condition. He reveals his reason at the end of his solution with the following remark: "***Crux*** *problems 2137 and 2173 both have been corrected in* ***Crux****, 2137 even twice! They are not faulty anymore!" (See [1996: 317] and [1997: 48] for 2137, and [1997: 169] for 2173.)*

2239. [1997: 242] *Proposed by Kenneth Kam Chiu Ko, Mississauga, Ontario.*

Suppose that $1 \le r \le n$ and consider all subsets of r elements of the set $\{1, 2, 3, \ldots, n\}$. The elements of these subsets are arranged in ascending order of magnitude. For i from 1 to r, let t_i denote the i^{th} smallest element in the subset. Let $T(n, r, i)$ denote the arithmetic mean of the elements t_i.

Prove that $T(n, r, i) = i\,\dfrac{n+1}{r+1}$.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let $X(n) = \{1, 2, ..., n\}$. Since there are $\binom{n}{r}$ subsets of $X(n)$ with r elements,

$$T(n, r, i) = \frac{\sum t_i}{\binom{n}{r}},$$

where the summation is over all r-subsets. Thus we are to show that

$$\sum t_i = i\frac{n+1}{r+1}\binom{n}{r} = i\binom{n+1}{r+1}. \qquad (1)$$

For each $k \in X(n)$, let u_k denote the number of r-subsets

$$(a_1, a_2, ..., a_{i-1}, k, a_{i+1}, ..., a_r)$$

of $X(n)$ with k being the i^{th} smallest element. Clearly, we must have $k \ge i$. There are $\binom{k-1}{i-1}$ ways of choosing the $i-1$ elements before k, and $\binom{n-k}{r-i}$ ways of choosing the $r-i$ elements after k. Thus

$$u_k = \binom{k-1}{i-1}\binom{n-k}{r-i}$$

from which we get

$$\sum t_i = \sum_{k \ge i} k u_k = \sum_{k \ge i} k\binom{k-1}{i-1}\binom{n-k}{r-i}. \qquad (2)$$

From (1) and (2) we see that it remains to prove that

$$\sum_{k \ge i} \frac{k}{i}\binom{k-1}{i-1}\binom{n-k}{r-i} = \binom{n+1}{r+1},$$

or, equivalently,

$$\sum_{k \ge i} \binom{k}{i}\binom{n-k}{r-i} = \binom{n+1}{r+1}. \qquad (3)$$

The equation (3) is well-known (for example, formula (11) on page 207 of *Applied Combinatorics*, 2nd edition by Alan Tucker). For completeness we give a combinatorial proof of this identity. Note first that $\binom{n+1}{r+1}$ is the number of $(r+1)$-subsets of $X(n+1)$. Again, we partition the family of all these subsets according to their $(i+1)^{\text{th}}$ smallest element, where i is fixed, $1 \le i \le r$. Specifically, for each $k \in X(n+1)$, we count the number, v_k, of $(r+1)$-subsets with $k+1$ being the $(i+1)^{\text{th}}$ smallest element. Clearly, $k \ge i$, and, by the same argument as before, with n, r, i, k replaced by $n+1, r+1, i+1$, and $k+1$, respectively, we have

$$v_k = \binom{k}{i}\binom{n-k}{r-i}.$$

Summing over k, we get

$$\binom{n+1}{r+1} = \sum_{k \ge i} v_k = \sum_{k \ge i} \binom{k}{i}\binom{n-k}{r-i}$$

which establishes (3) and completes the proof.

Also solved by NIELS BEJLEGAARD, Stavanger, Norway; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; ADRIAN BIRKA, student, Lakeshore Catholic High School, Port Colbourne, Ontario; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; JIMMY CHUI, student, Earl Haig Secondary School, North York, Ontario; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; ALAN LING, student, University of Toronto, Toronto, Ontario; CHRISTOPHER SO, student, Francis Liberman Catholic High School, Scarborough, Ontario; and the proposer. There was also one incomplete solution submitted.

Francisco Bellot Rosado noted that this problem is a generalization of the second problem from the IMO 1981 (Washington), where the question was to find the arithmetic mean of the smallest elements of the r-subsets. He also pointed out that a solution to the general question is given in a Romanian Olympiad book: Cuculescu, I., Olimpiadele Internationale de Matematica ale elevilor, *Ed. Tehnica, Bucarest 1984, p. 315.*