SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2219. [1997: 110] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Show that there are an infinite number of solutions of the simultaneous equations:

$$\begin{aligned} x^2 - 1 &= (u+1)(v-1) \\ y^2 - 1 &= (u-1)(v+1) \end{aligned}$$

with x, y, u, v positive integers and $x \neq y$.

I. *Solution by Charles Ashbacher, Cedar Rapids, Iowa, USA; Edward J. Barbeau, University of Toronto, Toronto, Ontario; Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; Florian Herzig, student, Perchtoldsdorf, Austria; Cyrus Hsia, student, University of Toronto, Toronto, Ontario; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and Václav Konečný, Ferris State University, Big Rapids, Michigan, USA.*

For positive integers n, the quadruples

$$(x, y, u, v) = (1, 2n+1, 2n^2+2n+1, 1)$$

give an infinite set of solutions in which $x \neq y$, since

$$x^2 - 1 = (u+1)(v-1) = 0$$

and

$$y^2 - 1 = 4n^2 + 4n = (u-1)(v+1).$$

II. *Solution by Edward J. Barbeau, University of Toronto, Toronto, Ontario; Digby Smith, Mount Royal College, Calgary, Alberta; and the proposer.*

For any positive integer n, the quadruple

$$(x, y, u, v) = (2n^2 - n, 2n^2 + n, 4n^3 + n, n)$$

is a solution in which $x \neq y$, since

$$x^2 - 1 = (u+1)(v-1) = 4n^4 - 4n^3 + n^2 - 1$$

and

$$y^2 - 1 = (u-1)(v+1) = 4n^4 + 4n^3 + n^2 - 1.$$

III. *Solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA; and Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

It is well known that the Pell equation $s^2 - 2t^2 = 1$ has infinitely many solutions in positive integers s, t. Clearly, $s > t > 1$.

If we set $u = s + t$, $v = s - t$, $x = t - 1$ and $y = t + 1$, then

$(u+1)(v-1) = s^2 - (t+1)^2 = t^2 - 2t = (t-1)^2 - 1 = x^2 - 1$, and
$(u-1)(v+1) = s^2 - (t-1)^2 = t^2 + 2t = (t+1)^2 - 1 = y^2 - 1$.

Also solved by MANSUR BOASE, student, St. Paul's School, London, England; FLORIAN HERZIG, student, Perchtoldsdorf, Austria (a second solution); RICHARD I. HESS, Rancho Palos Verdes, California, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece (two solutions); GERRY LEVERSHA, St Paul's School, London, England; and PANOS E. TSAOUSSOGLOU, Athens, Greece.

The families given by I, II *and* III *above, do not exhaust all the possible solutions. It is interesting to note that the "smallest" solution produced by all these families is* $(1, 3, 5, 1)$. *The next smallest ones are* $(1, 5, 13, 1)$, $(6, 10, 34, 2)$ *and* $(11, 13, 29, 5)$, *respectively. Both Herzig and Leversha obtained another infinite set of solutions in which* $v = 2$, *by considering the Pell equation* $3x^2 - y^2 = 8$. *Their "smallest" solution is* $(6, 10, 34, 2)$ *listed above. However, the next solution,* $(22, 38, 482, 2)$ *is not obtainable from any of the families given in* I, II *and* III.

2220. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Let V be the set of an icosahedron's twelve vertices, which can be partitioned into four classes of three vertices, each one in such a way that the three selected vertices of each class belong to the same face.

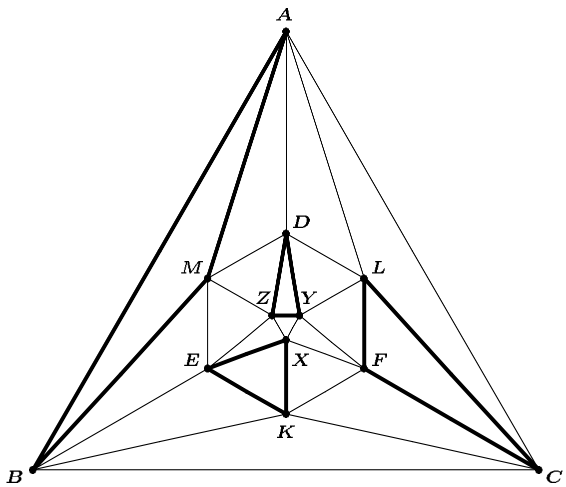
How many ways can this be done?

Solution by Florian Herzig, student, Perchtoldsdorf, Austria.

I will prove that there are 10 different ways of partitioning the vertices of an icosahedron in the described manner. Since only the topological properties of the icosahedron are important, consider the graph of the vertices: see figure on next page.

We want to find four triangles in this graph such that each vertex is used exactly once. Consider the vertex A. There are five triangles having A as vertex. Because of (spatial) symmetry we may assume that triangle ABM is chosen. Thus for the triangle containing point C only two choices remain: $\triangle CFL$ and $\triangle CFK$. Without loss of generality we may assume that triangle CFL is chosen. Now notice that the only "free" triangle containing D is $\triangle DZY$ and finally $\triangle EKX$ remains.

We have covered all possibilities already. For this case there are two different ways of finding "disjoint" triangles because we may choose from two equivalent triangles for vertex C. If all five possible triangles at vertex



A are considered, we get a total of 10 different configurations as claimed.

Also solved by MANSUR BOASE, student, St. Paul's School, London, England; JORDI DOU, Barcelona, Spain; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; and the proposer. There were two incorrect solutions.

Several solvers noted that the centres of the faces used in the partition are the vertices of a tetrahedron.

2221. [1997: 111] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Find all members of the sequence $a_n = 3^{2n-1} + 2^{n-1}$ $(n \in \mathbb{N})$ which are the squares of any positive integer.

Solution by David Doster, Choate Rosemary Hall, Wallingford, Connecticut, USA.

We have $a_1 = 4$ and $a_2 = 29$. For $n \geq 3$, $3^{2n-1} \equiv 3 \pmod 4$ and $2^{n-1} \equiv 0 \pmod 4$. Thus, $a_n \equiv 3 \pmod 4$. But a positive integer is a square only if it is congruent to 0 or $1 \pmod 4$. Hence, $a_1 = 4$ is the only square in the sequence.

Also solved by SAM BAETHGE, Nordheim, Texas, USA; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; GORAN CONAR, student, Gymnasium Varaždin, Varaždin, Croatia; YEO KENG HEE, Hwa Chong Junior College, Singapore;

FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; ROBERT B. ISRAEL, University of British Columbia, Vancouver, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St Paul's School, London, England; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; BOB PRIELIPP, University of Wisconsin–Oshkosh, Wisconsin, USA; ISTVÁN REIMAN, Budapest, Hungary; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; ZUN SHAN and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; and the proposer.

2222. [1997: 111] *Proposed by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.*

Find the value of the continued root:

$$\sqrt{4+27\sqrt{4+29\sqrt{4+31\sqrt{4+33\sqrt{\cdots}}}}}.$$

NOTE: This was inspired by the problems in chapter 26 "*Ramanujan, Infinity and the Majesty of the Quattuordecillion*", pp. 193–195, in "*Keys to Infinity*" by Clifford A. Pickover, John Wiley and Sons, 1995.

I. *Solution by Robert B. Israel, University of British Columbia, Vancouver, BC.*

The answer is 29. More generally, for any positive integer n, we claim that

$$\sqrt{4+n\sqrt{4+(n+2)\sqrt{4+(n+4)\sqrt{\cdots}}}} \;=\; n+2\,,$$

where the left side is defined as the limit of

$$F(n,m) \;=\; \sqrt{4+n\sqrt{4+(n+2)\sqrt{4+(n+4)\sqrt{\cdots\sqrt{4+m\sqrt{4}}}}}}$$

as $m \to \infty$ (where m is an integer and $(m-n)$ is even).

If $g(n,m) = F(n,m) - (n+2)$, we have

$$\begin{aligned} F(n,m)^2 - (n+2)^2 &= (4+nF(n+2,m)) - (4+n(n+4)) \\ &= n(F(n+2,m) - (n+4))\,, \end{aligned}$$

so

$$g(n,m) \;=\; \frac{n}{F(n,m)+n+2}\,g(n+2,m)\,.$$

Clearly $F(n,m) > 2$, so

$$|g(n,m)| \;<\; \frac{n}{n+4}\,|g(n+2,m)|\,.$$

By iterating this, we obtain

$$|g(n,m)| \;<\; \frac{n(n+2)}{m(m+2)}|g(m,m)| \;<\; \frac{n(n+2)}{m}\,.$$

Therefore $g(n,m) \to 0$ as $m \to \infty$.

II. *Solution by Efstratios Rappos, Girton College, University of Cambridge, England*

Let

$$S_n \;=\; \sqrt{4+(2n-1)\sqrt{4+(2n+1)\sqrt{4+(2n+3)\sqrt{\cdots}}}}$$

S_n satisfies the recurrence relation

$$S_n \;=\; \sqrt{4+(2n-1)S_{n+1}}$$

if and only if

$$(S_n-2)(S_n+2) \;=\; (2n-1)S_{n+1}\,.$$

By inspection, this admits $S_n = 2n+1$ as a solution. We only have to prove that $S_1 = 3$ to make this induction complete. Let

$$T_n \;=\; \sqrt{4+\sqrt{4+3\sqrt{\cdots(2n-3)\sqrt{4+(2n-1)\sqrt{2n+3)}}}}}$$

and

$$U_n \;=\; \sqrt{4+\sqrt{4+3\sqrt{\cdots(2n-3)\sqrt{4+(2n-1)(2n+3)}}}} = 3\,.$$

Clearly $T_n \le U_n$ and the latter is identically equal to 3. Therefore, using the fact that $B \ge A > 0$ implies that $\sqrt{(4+A)/(4+B)} \ge \sqrt{A/B}$,

$$\begin{aligned}
1 \;&\ge\; \frac{T_n}{3} \;=\; \frac{T_n}{U_n} \;=\; \frac{\sqrt{4+\sqrt{\cdots+(2n-1)\sqrt{2n+3}}}}{\sqrt{4+\sqrt{\cdots(2n-1))(2n+3)}}}\\
&\ge\; \frac{\sqrt{\sqrt{\cdots+(2n-1)\sqrt{2n+3}}}}{\sqrt{\sqrt{\cdots+(2n-1)(2n+3)}}} \;\ge\; \cdots \;\ge\; \sqrt[2^{n+1}]{\frac{1}{2n+3}}\\
&=\; \frac{1}{(2n+3)^{(\frac{1}{2})^{n+1}}} \;\longrightarrow\; 1
\end{aligned}$$

as $n \to \infty$ [for example, by rewriting as $\exp\{-\ln(2n+3)/2^{n+1}\}$ and using L'Hôpital's rule]. This proves that $S_1 = \lim_{n\to\infty} T_n = 3$. The required expression is precisely S_{14} and hence its value is 29.

Also solved or answered by MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St Paul's School, London, England; J.A. MCCALLUM, Medicine Hat, Alberta; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.

Several readers made reference to the July 1996 issue of the Mathematical Gazette, which contained a similar problem, Problem 80E posed by Tony Ward: evaluate:

$$\sqrt{1+1\sqrt{1+2\sqrt{1+3\sqrt{\cdots}}}}.$$

Bradley notes that the solution appeared in the March 1997 issue where the value of the continued root was proved to be 2, *and to be "well-defined". He also adds that the editor of the Problems section of the Gazette concludes "Clearly, the problem raises some deep questions about the meaning of 'well-defined'." Along the same lines as this editorial comment, Murray S. Klamkin, University of Alberta, Edmonton, Alberta adds that "the value can be anything since there is no definition regarding the continuation of the root". Klamkin refers to A. Herschfeld, On Infinite Radicals, Amer. Math. Monthly, 42 (1935) 419-429 where Herschfeld notes that Ramanujan's solution for*

$$\sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}} = 3$$

is incomplete since one may write similarly that

$$\begin{aligned} 4 &= \sqrt{1+2\cdot(15/2)} = \sqrt{1+2\sqrt{1+3\cdot(221/12)}} \\ &= \sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}}. \end{aligned}$$

Despite these comments most solvers expressed no difficulty in understanding the meaning of the continued root, and for that reason we have decided to print the above "solutions". Another reference to similar problems given by several readers was to J.M. Borwein, G. de Barra, Nested Radicals, Amer. Math. Monthly, 98 (1991) 735-739.

2224. [1997: 111] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Point P lies inside triangle ABC. Triangle BCD is erected outwardly on side BC such that $\angle BCD = \angle ACP$ and $\angle CBD = \angle ABC$. Prove that if the area of quadrilateral $PBDC$ is equal to the area of triangle ABC, then triangles ACP and BCD are similar.

Solution by Ian June L. Graces, Manila, The Philippines; and Giovanni Mazzarello, Firenze, Italy.

Let A' and P' be the respective images of A and P under reflection in the line BC. Note that B, D, A' are collinear (by the definition of D). Denoting by $[XYZ]$ the area of $\triangle XYZ$, we have

$$[P'CB] = \frac{1}{2}P'C \cdot BC \sin \angle P'CB,$$

and

$$[A'CD] = \frac{1}{2}A'C \cdot CD \sin \angle A'CD.$$

As a consequence of the given conditions ($[PBDC] = [ABC]$) and the effect of the reflection, $[P'CB] = [A'CD]$ (these are the complements of $\triangle BDC$ in $PBDC$ and in $\triangle A'CB$) and $\angle A'CD = \angle P'CB$. Thus

$$A'C \cdot CD = P'C \cdot BC.$$

In other words, (by SAS) we have $\triangle A'CP' \sim \triangle BCD$.

From the reflection we have $\triangle A'CP' \cong ACP$, and the desired result follows.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St Paul's School, London, England; ISTVÁN REIMAN, Budapest, Hungary; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2225. [1997: 111] *Proposed by Kenneth Kam Chiu Ko, Mississauga, Ontario.*

(a) For any positive integer n, prove that there exists a unique n-digit number N such that:

(i) N is formed with only digits 1 and 2; and

(ii) N is divisible by 2^n.

(b) Can digits "1" and "2" in (a) be replaced by any other digits?

Solution by Robert B. Israel, University of British Columbia, Vancouver, BC.

We can use any two non-zero digits whose difference is odd. Let the digits be a and b, where $a, b \in \{1, 2, \ldots, 9\}$ and $a - b$ is odd.

There are 2^n different n-digit numbers formed with these two digits, and 2^n residue classes modulo 2^n. I claim that the 2^n numbers are all in distinct residue classes. By the Pigeonhole Principle, exactly one of these numbers must be in the residue class 0.

Consider two distinct n-digit numbers, N_1 and N_2, formed with the digits a and b. Suppose that the first digit, counting from right to left, where they differ, is in the 10^k position, $0 \leq k \leq n-1$, where N_1 has a and N_2 has b. Then $N_1 - N_2 \equiv (a-b)10^k = (a-b)5^k 2^k \pmod{10^{k+1}}$, and thus modulo 2^{k+1}. Since $(a-b)5^k$ is odd, we have $N_1 \not\equiv N_2 \pmod{2^n}$.

Also solved by SAM BAETHGE, Nordheim, Texas, USA; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CURTIS COOPER, Central Missouri State University, Warrensburg, Missouri, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; GERRY LEVERSHA, St Paul's School, London, England; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, NY, USA; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; JOEL SCHLOSBERG, student, Hunter College High School, New York, NY, USA; ZUN SHAN and EDWARD T.H. WANG, Wilfrid Laurier University, Waterloo, Ontario; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; KENNETH M. WILKE, Topeka, Kansas, USA; YEO KENG HEE, student, Hwa Chong Junior College, Singapore; and the proposer.

Besides Israel, only Lambrou, Leversha and Schlosberg proved the more general result presented above. Shan and Wang pointed out that if $\{a, b\} = \{2, 4\}, \{2, 8\}, \{4, 6\}, \{6, 8\}$ *or* $\{4, 8\}$*, then the "existence" claim is still true and the "uniqueness" claim would be true if one strengthens condition* (ii) *to* $2^{n+1}|N$ *for the first four pairs, and to* $2^{n+2}|N$ *for the pair* $\{4, 8\}$.

Diminnie asked whether any similar results are possible if 2^n is replaced by k^n for $3 \le k \le 9$.

2226. [1997: 166] *Proposed by K.R.S. Sastry, Dodballapur, India.*

An old man willed that, upon his death, his three sons would receive the u^{th}, v^{th}, w^{th} parts of his herd of camels respectively. He had $uvw - 1$ camels in the herd when he died. Obviously, their sophisticated calculator could not divide $uvw - 1$ exactly into u, v or w parts. They approached a distinguished ***CRUX*** problem solver for help, who rode over on his camel, which he added to the herd and then fulfilled the old man's wishes, and took the one camel that remained, which was, of course, his own.

Dear ***CRUX*** reader, how many camels were there in the herd?

I. *Solution by Robert B. Israel, University of British Columbia, Vancouver, BC.*

There were 41 camels. We may assume $u \le v \le w$. Of course $u \ge 2$, and $vw + uw + uv = uvw - 1$.

If $u = 2$, the equation becomes $(v - 2)(w - 2) = 5$. Since the only factorization of 5 is 1×5, this means $v = 3$, $w = 7$, and $uvw - 1 = 41$.

If $u = 3$, the equation becomes $(2v - 3)(2w - 3) = 11$. Again there is only one factorization, and $v = 2$ (which violates $u \le v$).

Finally, if $u \ge 4$ we have $1 - 1/(uvw) = 1/u + 1/v + 1/w \le 3/4$ so $uvw \le 4$, which is impossible.

II. *Solution by Michael Lambrou, University of Crete, Crete, Greece. This solution is so much in keeping with the spirit of the problem that the editor felt a need to share it with all **CRUX** readers.*

"In the holy name of the Almighty," said the distinguished problem solver who saw the ***CRUX*** of the matter, "if you add my humble camel to your herd then there will be uvw camels to share. Your kind selves, whose renowned hospitality offered me your well to quench my thirst, will receive uv, vw, and wu camels respectively. This amply fulfils the deceased's will, whose soul may rest in tranquility."

"So," interrupted the sophisticated calculator, a student of logistics (the art of the practical arithmetician), perpetuator of the Pythagorean doctrine that whole numbers are the essence of nature, "we must then have that $uv + vw + wu + 1$ equals uvw, since numbers are the balance of ideas, the epitome of fairness, and we must take into account the camel of our distinguished guest. We cannot let him leave our oasis for his long journey to redeem his pilgrimage pledge, without a fair chance to cross the desert."

The sophisticated calculator, well versed in the new art of Al-jabr, could easily re-write the condition as

$$u + v + w = (u - 1)(v - 1)(w - 1).$$

"What?" he exclaimed. "If any of u, v, w is 1, then the right hand side is a product of numbers, one of which is nothing!" He was perplexed although he had seen this 'nothing' number in Ptolemy's Almagest, the monumental astronomical work, in the Table of Chords in Chapter One. "I do not understand," he continued. "How can nothing exist? It is contrary to nature. Nature abhors void because it would make motion impossible, as falling bodies would have to have infinite speed."

"On the contrary," said the distinguished problem solver, "in my long travels I have heard that the wise men of the east have discovered a nothing number. They call it 'as-sifr', and it comes from the Sanskrit 'sunya'. It has the property that when multiplied by anything it gives as-sifr. So, mathematically speaking, we have to exclude equality of any of u, v, w to 1 because this would contradict the equation: $u + v + w = (u - 1)(v - 1)(w - 1)$."

"Perhaps mathematically we have to exclude this case but we have an inheritance problem," answered the calculator, trying to gain time, " and this nothing is not, philosophically speaking, on solid ground." He then turned to the respectful cadi, the assessor of values and conservator of culture, whose judgment had a Rhadamanthean wisdom. "What do you say, esteemed reverend?"

The cadi replied that "if any of u, v, w was 1, then one of the sons would take the entire herd, which is contrary to our sacred traditions. Surely it was not the intention of their late father to incite hatred in the thoughts of the two losers. Surely he did not want to upset the good values and bonds of his family." This answer satisfied the calculator.

"Fine! May your shadow never be less, but let us continue the analysis. We may assume that $w \geq v \geq u > 1$ since birth rites allow shares to be larger or lesser. But could u be 4 or more?"

"I hope not," continued the calculator, "because the u^{th} part would be too small, unworthy of the respect showed to his late father. Ah yes, if $u \geq 4$ then

$$3w \geq u + v + w = (u - 1)(v - 1)(w - 1) \geq 3 \cdot 3(w - 1)$$

giving $9 \geq 6w$, which cannot be, since then $w = 1$, but then he would take the entire herd, inciting hatred in the thoughts of the other two, as forewarned by the incontestable cadi."

"So we must have $u \leq 3$. Let us then see what happens if $u = 3$. Here $w \geq v \geq u = 3$ gives

$$3 + 2w \geq u + v + w = (u - 1)(v - 1)(w - 1) \geq 2 \cdot 2(w - 1);$$

that is, $7 \geq 2w$, giving $w = 1$, 2, or 3. The cases $w = 1, 2$ are excluded since $w \geq u = 3$, leaving $3 = w \geq u = 3$; that is, all shares equal, $u = v = w = 3$. But this does not satisfy the original equation and must be dropped."

The dropping of the equal shares possibility came as a relief to all. It must have been Almighty's wish since not all three sons deserved an equal share. One of the three was certainly more praiseworthy spiritually, as he attended prayers and consulted often the holy book.

"Last but not least we have to analyze the possibility $u = 2$." Everybody listened carefully, especially potential brides, because $u = 2$ meant that one son would take half the herd. That is, as much as the other two together. Wise is the Lord!

"So we have

$$2 + v + w = 1(v - 1)(w - 1)$$

which, after some Al-jabr, gives

$$(v - 2)w = 2v + 1."$$

At this point the problem solver interrupted again. "Observe," he said, "if $v = 2$ then the left hand side gives as-sifr, which is incompatible with the right, so this case must be dropped."

"I will do as you say," replied the calculator, "although I think there is a deeper reason for that. Harmony with nature does not allow $u = v = 2$ because the original equation then becomes

$$4 + w = w - 1$$

and, if anything is added to 4, be it something of substance or void, you get at least 4 more than the addend, and not one less than the addend."

"We, therefore, have," he continued:

$$\text{``}w = \frac{2v + 1}{v - 2} = 2 + \frac{5}{v - 2}$$

This is a difficult situation. How can an integer equal a fraction? Only when what seems a fraction is not really a fraction but an integer concealed. We are rescued from this difficult situation by appealing to the ideas of Diophantus. I am so glad I have a recent manuscript with a translation of his eternal book, because the original Greek is too difficult."

"This is how he approaches such problems: The denominator $v - 2$ must be a divisor of 5, a sacred number, the number of Platonic solids and the length of the hypotenuse of the eternal triangle. Divine wisdom arranged that 5 has the prime property that it possesses precisely two divisors, unity and itself. So v is either 3 or 7. It cannot be 7 because w would then be 3, a smaller number. This leaves $v = 3$ and $w = 7$."

"Thus the total number of camels in the original herd is 41. One camel is left over for our guest, which he can have back, as it is not counted in the 41. The shares are 21, 14, and 6 respectively," concluded the calculator boastfully.

Everybody applauded the sagacity of this artful manipulator of numbers whose eurhythmic mind interpreted the inheritance laws with the infallible ways of the mathematician.

The distinguished problem solver smiled to himself. He had succeeded again. In a true Socratic manner he led the dialogue by giving imperceptible hints, the ***CRUX*** of the matter, to his counterpart who then discovered the truth that had been known to the problem solver from the very beginning. He saddled his camel, thanked for the hospitality and the knowledge he acquired, savoured a sip of water and left for the next stage of his Promethean journey.

Also solved or answered by SAM BAETHGE, Nordheim, Texas, USA; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARÍA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines and GIOVANNI MAZZARELLO, Ferrovie dello Stato, Florence, Italy; DAVID HANKIN, Hunter College Campus Schools, New York, NY, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; D. KIPP JOHNSON, Beaverton, Oregon, USA; GERRY LEVERSHA, St Paul's School, London, England; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; REZA SHAHIDI, student, University of Waterloo, Waterloo, Ontario; D.J. SMEENK, Zaltbommel, the Netherlands; DAVID STONE and VREJ ZARIKIAN, Georgia Southern University, Statesboro, Georgia; PANOS E. TSAOUSSOGLOU, Athens, Greece; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposer. There were two incomplete solutions.

Lambrou also considers the general problem where the number of camels is $p-1$, *and* u, v, w *all divide evenly into* p. *In addition to the solution given above he finds* 13 *other solutions* (u,v,w,p), *where* $w \le v \le w$: $(2,3,8,24)$, $(2,3,9,18)$, $(2,3,10,15)$, $(2,3,12,12)$, $(2,4,5,20)$, $(2,4,6,12)$, $(2,4,8,8)$, $(2,5,5,10)$, $(2,6,6,6)$, $(3,3,4,12)$, $(3,3,6,6)$, $(3,4,4,6)$, $(4,4,4,4)$.

2229. [1997: 167] *Proposed by Kenneth Kam Chiu Ko, Mississauga, Ontario.*

(a) Let m be any positive integer greater than 2, such that $x^2 \equiv 1 \pmod{m}$ whenever $(x, m) = 1$.

Let n be a positive integer. If $m|n+1$, prove that the sum of all divisors of n is divisible by m.

(b)* Find all possible values of m.

Solution by Kee-Wai Lau, Hong Kong (modified by the editor).

(a) We first show that n cannot be a perfect square.

Suppose that $n = k^2$. Then $k^2 \equiv -1 \pmod{m}$. But $k|n$, $m|n+1$ and $(n, n+1) = 1$ together imply that $(k, m) = 1$, and so, $k^2 \equiv 1 \pmod{m}$. Thus $1 \equiv -1 \pmod{m}$, which is false since $m > 2$. Therefore, all the divisors of n can be grouped into pairs (s, t), where $st = n$ and $s \neq t$. It then suffices to show that $m|(s+t)$. As above, $(s, m) = 1$ implies that $s^2 \equiv 1 \pmod{m}$. Adding $st \equiv -1 \pmod{m}$, we have that $s(s+t) \equiv 0 \pmod{m}$, or $s + t \equiv 0 \pmod{m}$.

(b) We show that the possible values of m are precisely 3, 4, 6, 8, 12, and 24.

For each m such that $3 \leq m \leq 24$, direct checking of those x with $1 < x < m$ and $(x, m) = 1$ reveals that these values are indeed the only ones that satisfy the described condition.

Assume then that $m > 24$ and let p_1, p_2, p_3, $\ldots$, denote the sequence of prime numbers. Then $2|m$, for otherwise $(2, m) = 1$ implies that $2^2 \equiv 1 \pmod{m}$, which is false. Similarly, $3|m$ and $5|m$. If $(7, m) = 1$, then $7^2 \equiv 1 \pmod{m}$, or $m|48$, which is impossible since $5|m$. Thus $7|m$.

Suppose that $p_i|m$ for all $i = 1, 2, \ldots, k$, for some $k \geq 4$. If $(m, p_{k+1}) = 1$, then $p_{k+1}^2 \equiv 1 \pmod{m}$, which implies that $p_{k+1}^2 > \prod_{i=1}^{k} p_i$. However, this contradicts the Bonse Inequality, which states that for all $k \geq 4$, $p_{k+1}^2 < \prod_{i=1}^{k} p_i$. (See, for example, chapter 27 of *The Enjoyment of Mathematics* by H. Rademacher and O. Toeplitz; Dover, 1990.)

It follows that $p_{k+1}|m$ and so m is divisible by any prime, which is clearly impossible. This shows that if $m > 24$, we cannot have $x^2 \equiv 1 \pmod{m}$ whenever $(x, m) = 1$, and the proof is complete.

Also solved by ADRIAN BIRKA, student, Lakeshore Catholic High School, Port Colbourne, Ontario; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; D. KIPP JOHNSON, Beaverton, Oregon, USA; MICHAEL LAMBROU, University of Crete, Crete, Greece; and HEINZ-JÜRGEN SEIFFERT, Berlin, Germany.

Part (a) *only was solved by JIMMY CHUI, student, Earl Haig Secondary School, North York, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; SEAN MCILROY, student, University of British Columbia, Vancouver, BC; JOEL SCHLOSBERG, student, Hunter College High School, New York, NY, USA; and the proposer.*

Regarding the solution to (b), *the Bonse Inequality was also used, explicitly or implicitly, by Boase, Bradley, Herzig and Seiffert. This inequality is an easy consequence of Bertrand's Postulate as shown by Herzig and Seiffert. Johnson gave a solution using Bertrand's Postulate directly.*

2230. [1997: 167] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Triangles BCD and ACE are constructed outwardly on sides BC and CA of triangle ABC such that $AE = BD$ and $\angle BDC + \angle AEC = 180°$. The point F is chosen to lie on the segment AB so that

$$\frac{AF}{FB} = \frac{DC}{CE}.$$

Prove that

$$\frac{DE}{CD + CE} = \frac{EF}{BC} = \frac{FD}{AC}.$$

Solution by Toshio Seimiya, Kawasaki, Japan.

Let G be a point on AE produced beyond E such that $\angle ECG = \angle DCB$. Since $\angle CEG = 180° - \angle AEC = \angle CDB$, we have $\triangle CEG \sim \triangle CDB$, (directly similar) from which we have $\triangle CBG \sim \triangle CDE$. Thus

$$\angle BGC = \angle DEC. \tag{1}$$

Since $\frac{AF}{FB} = \frac{CD}{CE} = \frac{BD}{EG} = \frac{AE}{EG}$, we get $FE \parallel BG$, so that

$$\angle AGB = \angle AEF. \tag{2}$$

Hence we have from (1) and (2)

$$\begin{aligned} \angle FED &= \angle AEC - (\angle AEF + \angle DEC) \\ &= \angle AEC - (\angle AGB + \angle BGC) \\ &= \angle AEC - \angle AGC \\ &= \angle ECG \\ &= \angle BCD. \end{aligned} \tag{3}$$

Similarly we have

$$\angle FDE = \angle ACE. \tag{4}$$

Let H be a point on CD produced beyond D such that $DH = EC$. Since $BD = AE$ and $\angle BDH = 180° - \angle BDC = \angle AEC$, we have

$$\triangle BDH = \triangle AEC,$$

so that $BH = AC$, and $\angle BHD = \angle ACE$.

As $\angle FED = \angle BCD = \angle BCH$, and $\angle FDE = \angle ACE = \angle BHD = \angle BHC$, we have $\triangle FDE \sim \triangle BHC$.

Thus we get

$$\frac{DE}{HC} = \frac{EF}{BC} = \frac{FD}{BH}.$$

Since $HC = CD + DH = CD + CE$, and $BH = AC$, we have

$$\frac{DE}{CD + CE} = \frac{EF}{BC} = \frac{FD}{AC}.$$

Also solved by FLORIAN HERZIG, student, Perchtoldsdorf, Austria; ISTVÁN REIMAN, Budapest, Hungary; and the proposer.