MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, Yale University, PO Box 208283 Yale Station, New Haven, CT 06520–8283 USA. The electronic address is still

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Mayhem Problems

The Mayhem Problems editors are:

Richard Hoshino	*Mayhem High School Problems Editor,*
Cyrus Hsia	*Mayhem Advanced Problems Editor,*
David Savitt	*Mayhem Challenge Board Problems Editor.*

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. We request that solutions from this issue be submitted by 1 February 1999, for publication in issue 4 of 1999. Also, starting with this issue, we would like to re-open the problems to all ***CRUX with MAYHEM*** readers, not just students, so now all solutions will be considered for publication.

Erratum

We regret to report that the High School porblems in volume 24, issue 1 [1998: 42], were mislabelled. Instead of **H223**, **H224**, **H225** and **H226**, they should have been labelled **H233**, **H234**, **H235** and **H236** respectively. We kindly ask readers to respect the new labelling when submitting solutions.

High School Problems

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H237. The letters of the word MATHEMATICAL are arranged at random. What is the probability that the resulting arrangement contains no adjacent A's?

H238. Johnny is dazed and confused. Starting at $A(0,0)$ in the Cartesian grid, he moves 1 unit to the right, then r units up, r^2 units left, r^3 units down, r^4 units right, r^5 units up, and continues the same pattern indefinitely. If r is a positive number less than 1, he will be approaching a point $B(x,y)$. Show that the length of the line segment AB is greater than $\frac{7}{10}$.

H239. Find all pairs of integers (x,y) which satisfy the equation $y^2(x^2+1)+x^2(y^2+16)=448$.

H240. *Proposed by Alexandre Trichtchenko, Brookfield High School, Ottawa, Ontario.*

A Pythagorean triple (a,b,c) is a triple of integers satisfying the equation $a^2+b^2=c^2$. We say that such a triple is *primitive* if $\gcd(a,b,c)=1$. Let p be an odd integer with exactly n prime divisors. Show that there exist exactly 2^{n-1} primitive Pythagorean triples where p is the first element of the triple. For example, if $p=15$, then $(15,8,17)$ and $(15,112,113)$ are the primitive Pythagorean triples with first element 15.

Advanced Problems

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A213. Show that the number of non-negative integer solutions to the equation $a+b+c+d=98$, where $a\le b\le c\le d$, is equal to the number of non-negative integer solutions to the equation $p+2q+3r+4s=98$.

A214. Show that any rational number can be written as the sum of a finite number of distinct unit fractions. A unit fraction is of the form $\frac{1}{n}$, where n is an integer.

A215. For a fixed integer $n\ge 2$, determine the maximum value of $k_1+\cdots+k_n$, where $k_1,\ldots,k_n$ are positive integers with $k_1^3+\cdots+k_n^3\le 7n$.

(Polish Mathematical Olympiad)

A216. Given a continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying the conditions:

$$f(1000) = 999,$$

$$f(x) \cdot f(f(x)) = 1 \text{ for all } x \in \mathbb{R}.$$

Determine $f(500)$.

(Polish Mathematical Olympiad)

Challenge Board Problems

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Editorial Notes: Part (b) is the interesting part of C77. Part (a) is a commonly asked problem, but I think it is better to ask it again than simply to state it for use in (b).

C77. Let F_n denote the n^{th} Fibonacci number, with $F_0 = 1$ and $F_1 = 1$. (Then $F_2 = 2$, $F_3 = 3$, $F_4 = 5$, etc.)

(a) Prove that each positive integer is uniquely expressible in the form $F_{a_1} + \cdots + F_{a_k}$, where the subscripts form a strictly increasing sequence of positive integers, no pair of which are consecutive.

(b) Let $\tau = \frac{1}{2}(1 + \sqrt{5})$, and for any positive integer n, let $f(n)$ equal the integer nearest to τn. If $n = F_{a_1} + \cdots + F_{a_k}$ is the expression for n from part (a), prove that $f(n) = F_{a_1+1} + \cdots + F_{a_k+1}$.

C78. Let n be a positive integer. An $n \times n$ matrix A is a *magic matrix* of order m if each entry is a non-negative integer and each row and column sum is m. (That is, for all i and j, $\sum_k A_{ik} = \sum_k A_{kj} = m$.)

Let A be a magic matrix of order m. Show that A can be expressed as the sum of m magic matrices of order 1.

Tips on Inequalities

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Inequalities can be difficult to solve because there are few systematic methods for tackling even the most simple formulations. Indeed, solving usually involves a trial and error of different approaches, before one hits the right combination of estimations and manipulations. In this article, we expose some useful standard approaches and techniques. We recall two basic and fundamental inequalities:

AM-GM Inequality. For all $x_1, x_2, \ldots, x_n \geq 0$,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Cauchy-Schwarz Inequality (CSB). For all real x_i, y_i, $i = 1, 2, \ldots, n$,

$$(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^2 \leq (x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2),$$

with equality if and only if the vectors $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are proportional.

Manipulating the Expressions

A typical approach in proving an inequality of the form $A \geq B$ is to find intermediate expressions, so we have a chain

$$A \geq P_1 \geq P_2 \geq \cdots \geq P_n \geq B.$$

These are usually found through using the classic inequalities, and by manipulating terms until we get what we want. As anyone who has worked with inequalities knows, this takes great care; one constantly has to make sure that the estimates are not too crude, and that the inequality signs are going the right way. In this kind of approach, there are several things one should keep in mind:

1. Is the inequality sharp or strict?

An inequality is sharp if equality occurs at a point, and strict if equality never occurs. It is always a good idea to check which type it is, though it is usually given or obvious. A strict inequality may allow for generous estimates, but not always. A sharp inequality leaves no such allowance.

2. If equality does occur, when/where does it occur?

The points where equality occurs are points you must work around. In the chain above, each intermediate inequality must become an equality at these points. This is a good check of whether your intermediate expressions are the right ones.

Pairing and Grouping

In inequalities where several terms are involved, it might be possible to group terms together and prove "smaller" inequalities.

Problem 1.

(a) Prove that $x^2 + y^2 + z^2 \geq xy + yz + zx$ for all $x, y, z \in \mathbb{R}$.

(b) Prove that $x^3 + y^3 + z^3 \geq x^2y + y^2z + z^2x$ for all $x, y, z \geq 0$.

Solution. (a) No grouping is immediately obvious. We know $x^2 + y^2 \geq 2xy$, but how can we incorporate this? By adding $x^2 + y^2 \geq 2xy$, $y^2 + z^2 \geq 2yz$, and $z^2 + x^2 \geq 2zx$, and dividing by 2, we obtain the desired inequality.

(b) Here, we know $2x^3 + y^3 = x^3 + x^3 + y^3 \geq 3x^2y$ by AM-GM. We then add the other two corresponding inequalities, and then divide by 3.

Problem 2. For all $a, b, c, d > 0$, show that

$$\frac{a^3+b^3+c^3}{a+b+c} + \frac{a^3+b^3+d^3}{a+b+d} + \frac{a^3+c^3+d^3}{a+c+d} + \frac{b^3+c^3+d^3}{b+c+d} \geq a^2+b^2+c^2+d^2 .$$

Solution. We claim that

$$\frac{a^3+b^3+c^3}{a+b+c} \geq \frac{a^2+b^2+c^2}{3} .$$

Then, the problem follows by symmetry. Our inequality is equivalent to

$$3a^3 + 3b^3 + 3c^3 \geq (a^2+b^2+c^2)(a+b+c) ,$$

which, in turn, is equivalent to

$$\begin{aligned} &2a^3 + 2b^3 + 2c^3 - a^2b - ab^2 - a^2c - ac^2 - b^2c - bc^2 \\ &\quad = (a^3 - a^2b - ab^2 + b^3) + (a^3 - a^2c - ac^2 + c^3) + (b^3 - b^2c - bc^2 + c^3) \\ &\quad = (a-b)^2(a+b) + (a-c)^2(a+c) + (b-c)^2(b+c) \geq 0 , \end{aligned}$$

which is true. Note that this inequality also follows from Chebyshev's inequality.

The Cauchy-Schwarz Inequality

The Cauchy-Schwarz inequality is a good way to deal with squares and especially fractions.

Problem 3. For $x_1, x_2, \ldots , x_n > 0$, show that

$$\frac{x_1^2}{x_1+x_2} + \frac{x_2^2}{x_2+x_3} + \cdots + \frac{x_n^2}{x_n+x_1} \geq \frac{x_1+x_2+\cdots+x_n}{2} .$$

Solution. By CSB,

$$\begin{aligned}&[(x_1+x_2)+(x_2+x_3)+\cdots+(x_n+x_1)]\\ &\quad\times\left[\frac{x_1^2}{x_1+x_2}+\frac{x_2^2}{x_2+x_3}+\cdots+\frac{x_n^2}{x_n+x_1}\right]\\ &\quad\geq\ (x_1+x_2+\cdots+x_n)^2\,.\end{aligned}$$

The result then follows by dividing each side by $2\,(x_1+x_2+\cdots+x_n)$.

Problem 4. Prove that for $a_1,\ a_2,\ \ldots\ ,\ a_n>0$,

$$\frac{(a_1+a_2+\cdots+a_n)^2}{2(a_1^2+a_2^2+\cdots+a_n^2)}\ \leq\ \frac{a_1}{a_2+a_3}+\frac{a_2}{a_3+a_4}+\cdots+\frac{a_n}{a_1+a_2}\,.$$

(1990–1991 IMO Correspondence)

Solution. Recall that CSB states that for all real $x_i,\ y_i,\ i=1,\,2,\,\ldots\ ,\,n$,

$$(x_1y_1+x_2y_2+\cdots+x_ny_n)^2\ \leq\ \left(x_1^2+x_2^2+\cdots+x_n^2\right)\left(y_1^2+y_2^2+\cdots+y_n^2\right)\,.$$

Setting $x_i=\sqrt[4]{a_i},\ y_i=\sqrt[4]{a_i^3}$, we obtain

$$\begin{aligned}&(a_1+a_2+\cdots+a_n)^2\\ &\quad\leq\ \left(\sqrt{a_1}+\sqrt{a_2}+\cdots+\sqrt{a_n}\right)\left(a_1\sqrt{a_1}+a_2\sqrt{a_2}+\cdots+a_n\sqrt{a_n}\right)\,.\end{aligned}$$

Setting $x_i=\sqrt{a_i},\ y_i=a_i$, we obtain

$$\begin{aligned}&\left(a_1\sqrt{a_1}+a_2\sqrt{a_2}+\cdots+a_n\sqrt{a_n}\right)^2\\ &\quad\leq\ (a_1+a_2+\cdots+a_n)\left(a_1^2+a_2^2+\cdots+a_n^2\right)\,.\end{aligned}$$

Combining these two, we obtain

$$\begin{aligned}&(a_1+a_2+\cdots+a_n)^3\\ &\quad\leq\ \left(\sqrt{a_1}+\sqrt{a_2}+\cdots+\sqrt{a_n}\right)^2\left(a_1^2+a_2^2+\cdots+a_n^2\right)\,.\end{aligned}$$

Finally, setting $x_i=\sqrt{a_{i+1}+a_{i+2}},\ y_i=\sqrt{\frac{a_i}{a_{i+1}+a_{i+2}}}$, we obtain

$$\begin{aligned}&\left(\sqrt{a_1}+\sqrt{a_2}+\cdots+\sqrt{a_n}\right)^2\\ &\quad\leq\ 2\,(a_1+a_2+\cdots+a_n)\left(\frac{a_1}{a_2+a_3}+\frac{a_2}{a_3+a_4}+\cdots+\frac{a_n}{a_1+a_2}\right)\,.\end{aligned}$$

The last two inequalities give the desired result.

Elementary Symmetric Polynomials

Given a set of variables X, such as $X=\{x,y,z\}$, a polynomial is **symmetric** in X if it is invariant under any permutation of the variables under X, and **homogeneous of degree** k if every term in the polynomial has degree k.

For example, $x^2 + y^2 + z^2 - xyz$ is symmetric but not homogeneous in X, and $x^3 + x^2y - xyz$ is homogeneous of degree 3 but not symmetric in X.

The elementary symmetric polynomials in X are the polynomials obtained as the sum of the products of the variables in X, taken k at a time. For $X = \{x, y, z\}$, these would be $x + y + z$, $xy + xz + yz$, and xyz. Note that each of these is homogeneous. A theorem of Gauss states that any symmetric, homogeneous polynomial in X can be expressed as a polynomial in these elementary polynomials.

After all these tedious definitions, we finally get to the point that it can be useful to use these elementary polynomials.

Problem 5. For non-negative reals x, y, and z satisfying $x + y + z = 1$, show that

$$\left(\frac{1}{x} + 1\right)\left(\frac{1}{y} + 1\right)\left(\frac{1}{z} + 1\right) \;\geq\; 64\,.$$

Proof. Expanding, we first must show

$$1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xy} + \frac{1}{xz} + \frac{1}{yz} + \frac{1}{xyz} \;\geq\; 64\,.$$

By AM-GM,

$$xyz \;\leq\; \left(\frac{x+y+z}{3}\right)^3 \;=\; \frac{1}{27} \qquad \text{so that} \qquad \frac{1}{xyz} \;\geq\; 27\,.$$

Therefore,

$$\begin{aligned} & 1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xy} + \frac{1}{xz} + \frac{1}{yz} + \frac{1}{xyz} \\ &\geq\; 1 + \frac{3}{\sqrt[3]{xyz}} + \frac{3}{\sqrt[3]{x^2y^2z^2}} + \frac{1}{xyz} \\ &=\; \left(1 + \frac{1}{\sqrt[3]{xyz}}\right)^3 \\ &\geq\; 4^3 \;=\; 64\,. \end{aligned}$$

Problem 6. Prove that

$$0 \;\leq\; yz + zx + xy - 2xyz \;\leq\; \frac{7}{27},$$

where x, y, and z are non-negative real numbers for which $x + y + z = 1$.
(1984 IMO, #1)

Proof. Let $S = xy + xz + yz - 2xyz$, $P = (1 - 2x)(1 - 2y)(1 - 2z)$. Then

$$P \;=\; 1 - 2(x + y + z) + 4(xy + xz + yz) - 8xyz \;=\; 4S - 1\,.$$

We must show that $0 \le S \le \frac{7}{27}$, or equivalently, $-1 \le P \le \frac{1}{27}$. Since $0 \le x, y, z \le 1$, $-1 \le 2x - 1, 2y - 1, 2z - 1 \le 1$, so $-1 \le P$. Now, if one of the variables, say x, was greater than $1/2$, then the other two would be less than $1/2$, and we would have $1 - 2x < 0$, $1 - 2y$, $1 - 2z > 0$, and $P < 0 < \frac{1}{27}$. Otherwise, x, y, and z are at most $1/2$, and all factors of P are non-negative, so by AM-GM,

$$P \le \left(\frac{1 - 2x + 1 - 2y + 1 - 2z}{3}\right)^3 = \frac{1}{27}.$$

Introducing and Removing Constraints

Inequalities often come with constraints on the variables. Removing these constraints can simplify the problem. Alternately, introducing them may help as well. The most common way of removing a constraint is to "homogenize" the given inequality. For example, suppose we are given the expression $x^3 + xy - 2$, where $xyz = 1$. Then

$$x^3 + xy - 2 = x^3 + xy\sqrt[3]{xyz} - 2xyz.$$

This new expression is homogeneous of degree 3. It's not pretty, but for the expressions given in problems, most of the time it will be nice and much easier to work with.

At this point, we introduce an inequality that is not well-known, but that seems to pop up from time to time.

Schur's Inequality. For all x, y, $z \ge 0$ and non-negative integers n,

$$x^n(x - y)(x - z) + y^n(y - z)(y - x) + z^n(z - x)(z - y) \ge 0.$$

For $n = 1$, this becomes

$$x^3 + y^3 + z^3 + 3xyz \ge x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2,$$

or in shorthand, $\sum x^3 + 3xyz \ge \sum x^2y$. This is a useful inequality to know, with which we present another solution.

Problem 6. Prove that

$$0 \le yz + zx + xy - 2xyz \le \frac{7}{27},$$

where x, y, z are non-negative real numbers for which $x + y + z = 1$.
(1984 IMO, #1)

Solution. This is equivalent to the following problem: For x, y, $z \ge 0$, prove that

$$0 \le (yz + zx + xy)(x + y + z) - 2xyz \le \frac{7}{27}(x + y + z)^3.$$

This is the homogeneous version of the original inequality. The expression in the middle expands to $\sum x^2y + xyz$, which is clearly non-negative. We focus on the right inequality, which becomes

$$\sum x^2y + xyz \ \le \ \frac{7}{27}\sum x^3 + \frac{7}{9}\sum x^2y + \frac{14}{9}\sum xyz\,,$$

which implies

$$6\sum x^2y \ \le \ 7\sum x^3 + 15xyz\,.$$

This is where we must think backwards. What results do we know that we can use to prove this? By Schur's, $5\sum x^2y \le 5\sum x^3 + 15xyz$ (we try to eliminate the xyz term). Hence, to prove the above inequality, we must show that $\sum x^2y \le 2\sum x^3$, which is left as an exercise for the reader (it has been virtually done already in this article).

We stated early in the solution that the modified problem was equivalent to the original problem. It is easy to see that the modified problem implies the original problem (which is the only direction we actually needed), but what about the converse? What if $x + y + z \ne 1$? In such a case, we can normalize.

A property of homogeneous polynomials, and an alternate definition, is the following: $p(x_1, x_2, \ldots, x_n)$ is homogeneous of degree k if

$$p(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) \ = \ \lambda^k p(x_1, x_2, \ldots, x_n)$$

for all $\lambda \in \mathbb{R}$.

Going back to the original problem,

$$p(x,y,z) \ = \ (yz + zx + xy)(x + y + z) - 2xyz\,.$$

If $x + y + z = 0$, then all three variables must be 0, and the inequality follows. Otherwise, we can set $\lambda = \frac{1}{x+y+z}$, and so we must show

$$0 \ \le \ p\left(\frac{x}{x+y+z}, \frac{y}{x+y+z}, \frac{z}{x+y+z}\right) \ \le \ \frac{7}{27}\,.$$

This is the original problem. Thus, we can set $x + y + z$ equal to 1, or indeed anything we want to (except 0). It can be vital to exploit this degree of freedom. We perform a similar setting in the next problem.

Problem 7. Given $0 < a \le b$, and $x_1, x_2, \ldots, x_n \ge 0$, show that

$$(x_1^a + x_2^a + \cdots + x_n^a)^{1/a} \ \ge \ (x_1^b + x_2^b + \cdots + x_n^b)^{1/b}.$$

Solution. If $x_1^b + x_2^b + \cdots + x_n^b = 0$, then the problem is solved. Otherwise, by the arguments above, we can assume $x_1^b + x_2^b + \cdots + x_n^b = 1$. Then

$$\begin{aligned} x_i^b \le 1 \ &\Longrightarrow\ x_i \le 1 \Longrightarrow x_i^{b-a} \le 1 \Longrightarrow x_i^b \le x_i^a \\ &\Longrightarrow\ x_1^a + x_2^a + \cdots + x_n^a \ge x_1^b + x_2^b + \cdots + x_n^b = 1 \\ &\Longrightarrow\ (x_1^a + x_2^a + \cdots + x_n^a)^{1/a} \ge 1 = (x_1^b + x_2^b + \cdots + x_n^b)^{1/b}. \end{aligned}$$

Problems

1. Prove that if x, y, and z are non-negative real numbers such that $x + y + z = 1$, then
$$2(x^2 + y^2 + z^2) + 9xyz \;\geq\; 1\,.$$

2. For any real numbers a, b, and c, show that
$$\min\left[(a-b)^2, (b-c)^2, (c-a)^2\right] \;\leq\; \frac{a^2+b^2+c^2}{2}\,.$$

3. Let a, b, and c be the sides of a triangle with perimeter 2. Prove that $a^2 + b^2 + c^2 + 2abc < 2$.

4. For non-negative reals x, y, and z satisfying $2xyz + xy + xz + yz = 1$, prove that $x + y + z \geq \frac{3}{2}$.

5. For all positive integers n, show that
$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n} \;\leq\; \frac{1}{\sqrt{3n+1}}\,.$$
(Hint: The inequality is almost certainly not sharp, so there is some room for approximation. The RHS suggests squaring.)

6. Show that if x, y, and z are non-negative reals such that $x+y+z = 1$, then
$$\left(\frac{1}{x}-1\right)\left(\frac{1}{y}-1\right)\left(\frac{1}{z}-1\right) \;\geq\; 8\,.$$
(Note: The solution in Problem 5 does not work!)

7. Given a, b, c, d, $e > 0$, $abcde = 1$, show that
$$a^4 + b^4 + c^4 + d^4 + e^4 \;\geq\; a + b + c + d + e\,.$$
(A favourite of Ravi Vakil's. Find as many different solutions as you can.)

8. Show that if non-negative reals a, b, and c satisfy
$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \;=\; 1\,,$$
then $abc \geq 8$.

Riveting Properties of Pascal's Triangle

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Consider the following table of integers, known as **Pascal's Triangle**:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}$$

This table is named after the French mathematician Blaise Pascal, who conceived of it at the age of thirteen (although the Chinese discovered some properties of this triangle in the early 14^{th} century, centuries before Pascal was even born!).

To generate a row of Pascal's Triangle, look at the row immediately above it. Each element of the triangle is the sum of the two elements above it (for example, the second element of the fourth row is 4 because $4 = 1 + 3$). By convention, we denote the top row as the 0^{th} row, and we denote the left most entry of each row as the 0^{th} entry, even though it may seem a little awkward at first.

Let us show that this table is the same as the following table:

$$\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\
 & & & & & \vdots & & & & &
\end{array}$$

In this table, $\binom{n}{k}$ is determined by the formula $\frac{n!}{k!(n-k)!}$. Thus, we will show that the k^{th} element of the n^{th} row of Pascal's Triangle equals $\binom{n}{k}$ for all integers n and k. Each of these elements in the triangle, namely $\binom{0}{0}$, $\binom{1}{0}$, $\binom{1}{1}$, $\binom{2}{0}$, $\binom{2}{1}$, $\ldots$, is called a **binomial coefficient**, since the coefficients of the expansion of the binomial $(1 + x)^n$ correspond to the entries of the n^{th} row of Pascal's Triangle (for example, $(1+x)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$). Let us prove these statements, as they will be fundamental to our analysis of the properties of Pascal's Triangle.

Now $\binom{n}{k}$ denotes the number of ways we may select k objects from a set of n objects. By convention, we let $\binom{n}{0} = 1$, since there is technically only "one" way we may select nothing from a set of n objects. Let us show that

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

for all n and k. Note that the right side denotes the number of ways we may select a k-member committee from a class of n girls and one boy (we choose k members from a set of $(n+1)$ people). Now, if the boy is on the committee, then we have $\binom{n}{k-1}$ ways of selecting the remaining $k-1$ members. And if he is not on the committee, then there are $\binom{n}{k}$ ways of selecting the committee. Hence,

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

This is known as **Pascal's Identity**.

Since $\binom{0}{0} = 1$, the table of binomial coefficients corresponds directly to Pascal's Triangle – since the initial element is the same and, like Pascal's Triangle, each element is the sum of the two directly above it. Thus, we can determine any element in Pascal's Triangle with this formula. For example, the thirty-fifth element in the seventy-ninth row of Pascal's Triangle is $\binom{79}{35}$.

We now prove that the entries in the n^{th} row of Pascal's Triangle are the coefficients in the expansion of $(1+x)^n$. We proceed by induction. The case is trivial for $n = 1$. Suppose that

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{k}x^k + \cdots + \binom{n}{n}x^n$$

for some n. Then

$$\begin{aligned}(1+x)^{n+1} &= (1+x)^n(1+x) \\ &= \left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right](1+x) \\ &= \binom{n}{0} + \left[\binom{n}{1} + \binom{n}{0}\right]x + \cdots + \left[\binom{n}{n} + \binom{n}{n-1}\right]x^n + \binom{n}{n}x^{n+1}.\end{aligned}$$

Since $\binom{n}{0} = \binom{n+1}{0} = \binom{n}{n} = \binom{n+1}{n+1} = 1$, and using Pascal's Identity for all the other terms, we immediately arrive at the case for $n+1$. Hence, the coefficient of x^k in $(1+x)^n$ is equal to $\binom{n}{k}$.

Now we illustrate some of the really neat properties of Pascal's Triangle.

Theorem 1.

(i) The sum of the coefficients in the n^{th} row of Pascal's Triangle is 2^n.

(ii) If we alternately add and subtract the digits in the n^{th} row of Pascal's Triangle, we always arrive at zero. For example, $1 - 4 + 6 - 4 + 1 = 0$ for $n = 4$.

Proof. (i) Since

$$(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{k}x^k + \cdots + \binom{n}{n}x^n,$$

substituting $x = 1$ into this expression yields

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}.$$

(ii) The second part of the theorem is left as an exercise. It is the same technique as above; just substitute a different value for x. Can you guess which value of x we ought to substitute?

Theorem 2. If the binary representation of n contains p ones, then there are 2^p odd numbers in the n^{th} row of Pascal's Triangle. For example, since $9 = 1001_2$, there are $2^2 = 4$ odd numbers in the ninth row.

Proof. We shall analyze everything in modulo 2. For those of you not familiar with modular arithmetic, when we say that x is congruent to y modulo m, which is written $x \equiv y \pmod{m}$, we mean that x and y are numbers such that when both are divided by m, they give the same remainder. For example, 1998 is congruent to 2 modulo 4. Also, if $r \equiv 0 \pmod{2}$, then r is even.

We first show that

$$(1 + x)^{2^n} \equiv 1 + x^{2^n} \pmod{2}$$

for all non-negative integers n. We proceed by induction. If $n = 0$, the claim is immediate. Suppose the claim is true for some $n = k$. Then

$$(1+x)^{2^{k+1}} \equiv ((1+x)^{2^k})^2 \equiv (1+x^{2^k})^2 \equiv 1+2x^{2^k}+x^{2^{k+1}} \equiv 1+x^{2^{k+1}} \pmod{2},$$

and so it is also true for $n = k + 1$. Hence, by induction, the claim has been verified.

Here is an example for a specific case. Let $n = 50$. Then $50 = 2^1 + 2^4 + 2^5 = 2 + 16 + 32$. Hence,

$$\begin{aligned}
(1 + x)^{50} &= (1 + x)^2(1 + x)^{16}(1 + x)^{32} \\
&\equiv (1 + x^2)(1 + x^{16})(1 + x^{32}) \\
&\equiv 1 + x^2 + x^{16} + x^{18} + x^{32} + x^{34} + x^{48} + x^{50} \pmod{2}.
\end{aligned}$$

Hence, there are eight odd entries in the 50^{th} row of Pascal's Triangle, namely $\binom{50}{0}$, $\binom{50}{2}$, $\binom{50}{16}$, $\binom{50}{18}$, $\binom{50}{32}$, $\binom{50}{34}$, $\binom{50}{48}$, and $\binom{50}{50}$. Since $50 = 110010_2$, this verifies that there are exactly $2^3 = 8$ odd terms in this row.

For a general n, suppose there are p ones in the binary representation of n. Then

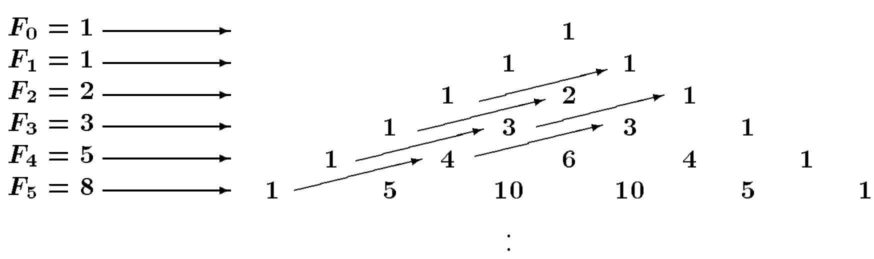
$$(1+x)^n = (1+x)^{2^{a_1}}(1+x)^{2^{a_2}}\cdots(1+x)^{2^{a_p}},$$

where $0 \le a_1 < a_2 < \cdots < a_p$ and the a_i^{th} digit of the binary representation of n is 1, starting from 0 at the right. The right side is congruent to

$$(1+x^{2^{a_1}})(1+x^{2^{a_2}})\cdots(1+x^{2^{a_p}})$$

modulo 2, and when we expand the right side, we will arrive at an expression with 2^p terms. Note that there must be exactly 2^p terms, since each exponent x^k can be formed in only one way by multiplying coefficients from the set $\{x^{2^{a_1}}, x^{2^{a_2}}, \ldots, x^{2^{a_p}}\}$. This is a direct result from the binary representation of k. Hence, if n has p ones in its binary representation, $(1+x)^n$ modulo 2 has 2^p terms, and hence there are 2^p odd entries in the n^{th} row of Pascal's Triangle.

Theorem 3. (The Fibonacci sequence is the sequence, 1, 1, 2, 3, 5, 8, 13, 21, 34, ... , where each element in the sequence is the sum of the two before it. More formally, we say that the Fibonacci sequence $\{F_n\}$ satisfies the conditions $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n > 1$.) We can derive the Fibonacci sequence from Pascal's Triangle in the following manner.



In other words, for all n, we have

$$F_{2n} = \binom{2n}{0} + \binom{2n-1}{1} + \binom{2n-2}{2} + \cdots + \binom{n}{n}$$

and

$$F_{2n+1} = \binom{2n+1}{0} + \binom{2n}{1} + \binom{2n-1}{2} + \cdots + \binom{n+1}{n}.$$

For example,

$$F_7 = \binom{7}{0} + \binom{6}{1} + \binom{5}{2} + \binom{4}{3} = 1 + 6 + 10 + 4 = 21.$$

Proof. We prove this theorem using double induction: we know that the claim is true for F_0 and F_1. Suppose the claim is true for some F_{2k} and F_{2k+1}. Then we want to show that

$$F_{2k+2} = \binom{2k+2}{0} + \binom{2k+1}{1} + \binom{2k}{2} + \cdots + \binom{k+1}{k+1};$$

that is, the right side is equal to the $(2k+2)^{\text{th}}$ Fibonacci number. But

$$\begin{aligned} F_{2k+2} &= F_{2k} + F_{2k+1} \\ &= \binom{2k+1}{0} + \left[\binom{2k}{0} + \binom{2k}{1}\right] + \cdots + \left[\binom{k+1}{k-1} + \binom{k+1}{k}\right] + \binom{k}{k} \\ &= \binom{2k+1}{0} + \binom{2k+1}{1} + \binom{2k}{2} + \cdots + \binom{k+2}{k} + \binom{k}{k}, \end{aligned}$$

by Pascal's Identity. Since $\binom{2k+1}{0} = 1 = \binom{2k+2}{0}$ and $\binom{k}{k} = 1 = \binom{k+1}{k+1}$, we have

$$F_{2k+2} = \binom{2k+2}{0} + \binom{2k+1}{1} + \binom{2k}{2} + \cdots + \binom{k+1}{k+1},$$

as required.

The second part of the induction follows similarly – assume the claim is true for F_{2k-1} and F_{2k}, and show that the claim is also true for F_{2k+1}. The proof is left as an exercise. Now we are done – since the proposition is true for $k = 0$ and $k = 1$, it is true for $k = 2$. Since it is true for $k = 1$ and $k = 2$, it is true for $k = 3$, etc. Then by double induction, the claim is true for all non-negative integers n.

Exercises

1. Show that $\binom{n}{k} = \binom{n}{n-k}$, and use this fact to show that Pascal's Triangle is symmetric about the vertical line that separates the triangle into two equal halves.

2. For which n is $\binom{2n}{n}$ odd?

3. Let a_n represent the number of elements in the n^{th} row of Pascal's Triangle that are congruent to 1 modulo 3. Let b_n represent the number of elements in the n^{th} row of Pascal's Triangle that are congruent to 2 modulo 3. Prove that for all n, $a_n - b_n$ is a power of two.

   (This problem was on the IMO short list one year – it is very tough!)

Swedish Mathematics Olympiad

1985 Qualifying Round

1. The real numbers a, b, and c satisfy the equations

$$\begin{aligned} ab + b &= -1 \\ bc + c &= -1 \\ ca + a &= -1 \end{aligned}$$

Calculate the product abc.

2. 1985 runners reported for a marathon. They were assigned the numbers 1, 2, $\ldots$, 1985. However, a number of runners dropped out before the race. In fact, among the starting runners, there were no two for whom one runner's number was 10 times the other. What is the greatest number of runners that could have participated?

3. In the system of equations

$$\begin{aligned} a_1x + b_1y + c_1z + d_1u &= 0 \\ a_2x + b_2y + c_2z + d_2u &= 0 \\ a_3x + b_3y + c_3z + d_3u &= 0 \\ a_4x + b_4y + c_4z + d_4u &= 0 \end{aligned}$$

the coefficients a_1, b_2, c_3, and d_4 are even integers and the other coefficients are odd integers. Prove that the only solution in integers is $x = y = z = u = 0$.

4. The non-negative integers p, q, r, and s satisfy the equality

$$(p + q)^2 + p = (r + s)^2 + r.$$

Show that $p = r$ and $q = s$.

5. Let f be defined by

$$f(x) = \frac{4x^2 \sin^2 x + 9}{x \sin x}\,.$$

Find the least value of f over the interval $0 < x < \pi$.

6. The point P lies on the perimeter or inside a given triangle T. The point P', in the plane of the triangle, lies at a distance d from P. Let r and r' be the radii of the smallest circles, with centres P and P' respectively, which contain T. Show that

$$r + d \leq 3r'.$$

Give an example where equality holds.

1985 Final Round

1. Let $a > b > 0$. Prove that

$$\frac{(a-b)^2}{8a} < \frac{a+b}{2} - \sqrt{ab} < \frac{(a-b)^2}{8b}.$$

2. Find the least natural number such that if the first digit is placed last, the new number is $7/2$ times as large as the original number. (The numbers are written in the decimal system.)

3. A, B, and C are three points on a circle with radius r, and $AB = BC$. D is a point inside the circle such that the triangle BCD is equilateral. The line through A and D meets the circle at the point E. Show that $DE = r$.

4. The polynomial $p(x)$ of degree n has real coefficients, and $p(x) \geq 0$ for all x. Show that

$$p(x) + p'(x) + p''(x) + \cdots + p^{(n)}(x) \geq 0.$$

5. In a right-angled coordinate system, a triangle has vertices $A(a, 0)$, $B(0, b)$, and $C(c, d)$, where the numbers a, b, c, and d are positive. Show that if we denote the origin by O,

$$AB + BC + CA \geq 2CO.$$

6. X-wich has a vibrant club-life. For every pair of inhabitants there is exactly one club to which they both belong. For every pair of clubs there is exactly one person who is a member of both. No club has fewer than 3 members. At least one club has 17 members. How many people live in X-wich?