Sum of powers of a finite sequence: a geometric approach

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In this note we give a geometric-combinatoric derivation of a formula for the sum of r^{th} powers of a finite positive integer sequence:

$$a_1^r + a_2^r + \cdots + a_n^r = \sum_{\ell=1}^{r} \sum_{i=1}^{n} \binom{a_i}{\ell} \mu(r, \ell), \qquad (1)$$

where

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!}, & \text{if } 0 \le k \le n\,, \\ 0, & \text{if } k > n \ge 1\,, \end{cases}$$

and the numbers $\mu(r, \ell)$ are determined by the recurrence

$$\begin{aligned} \mu(r,1) &= 1, \quad r = 1,2,3,\cdots; \quad \mu(1,\ell) = 0, \quad \ell = 2,3,\cdots; \\ \mu(r,\ell) &= \ell(\mu(r-1,\ell-1) + \mu(r-1,\ell)), \ r \ge 2, \ \ell \ge 2\,. \end{aligned} \qquad (2)$$

The derivation parallels, simplifies and generalizes the proof given in [1], and seems to be more elementary than proofs given in [2] and [3].

For fixed (but arbitrary) positive integers m, r, let $L_m^{(r)}$ denote the r-dimensional integer lattice points

$$\{(x_1, x_2, \cdots, x_r) \mid x_i \text{ integers}, \ 0 \le x_1, x_2, \cdots, x_r \le m\}\,,$$

and for $w \ge 1$ let $S_m^{(r)}(w)$ denote the set of r-dimensional "cubes" with faces parallel to the coordinate planes, vertices in $L_m^{(r)}$ and width w. Of course $S_m^{(r)}(w) = \emptyset$ if $w > m$. A cube in $S_m^{(r)}(w)$ is identified by its vertex $(\alpha) = (\alpha_1, \alpha_2, \cdots, \alpha_r)$ closest to $(0, \cdots, 0)$. Clearly

$$w + \alpha_1 \le m, \quad w + \alpha_2 \le m, \quad \cdots, \quad w + \alpha_r \le m, \qquad \alpha_1, \alpha_2, \cdots, \alpha_r \ge 0\,,$$

or

$$0 \le \alpha_1 \le m - w, \quad 0 \le \alpha_2 \le m - w, \quad \cdots,$$
$$\cdots \ 0 \le \alpha_r \le m - w, \qquad 1 \le w \le m\,, \qquad (3)$$

so the number of cubes in $S_m^{(r)}(w)$ is

$$|S_m^{(r)}(w)| = \begin{cases} (m - w + 1)^r\,, & \text{if } 1 \le w \le m\,, \\ 0\,, & \text{if } w > m \ge 1\,. \end{cases}$$

For given $1 \leq w \leq m$ we can also find the number of (α)'s satisfying (3) as follows. Any r-tuple (α) satisfying (3) determines the sequence (β) consisting of the distinct values, in increasing order, of the entries in (α):

$$(\beta) : 0 \leq \beta_1 < \beta_2 < \cdots < \beta_\ell \leq m - w, \quad \ell \leq r\,. \tag{4}$$

Given a sequence (β) satisfying (4) there are many r-tuples (α) satisfying (3) which have entries with values (β). How many? Let $\mu(r,\ell)$ denote this number. It depends on ℓ (and not on the particular numbers $\beta_1, \cdots, \beta_\ell$) and satisfies the recurrence (2). For, given (β), there are ℓ choices (namely $\beta_1, \beta_2, \cdots, \beta_\ell$) for the first entry α_1 of (α); (α) can then be completed in $\mu(r-1,\ell)$ ways if the integer chosen for α_1 appears among the entries $\alpha_2, \alpha_3, \cdots, \alpha_r$, and in $\mu(r-1,\ell-1)$ ways otherwise; (2) then follows.

The numbers $\mu(r,\ell)$ are exhibited in the display below:

$r\backslash\ell$	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	2!	0	0	0	0
3	1	6	3!	0	0	0
4	1	14	36	4!	0	0
5	1	30	150	240	5!	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Since there are $\binom{m-w+1}{\ell}$ ways of choosing the sequence (β) satisfying (4) we find that

$$(m-w+1)^r = \sum_{\ell=1}^{r} \binom{m-w+1}{\ell} \mu(r,\ell), \qquad 1 \leq w \leq m, \qquad 1 \leq \ell \leq r,$$

and setting $a = m - w + 1 \geq 1$

$$a^r = \sum_{\ell=1}^{r} \binom{a}{\ell} \mu(r,\ell), \quad a \geq 1, \quad r \geq 1\,.$$

Summing over a set of a's we have (1). Taking $a_i = a + (i-1)d$, $i = 1, 2, \cdots, n$ in (1), we have

$$a^r + (a+d)^r + \cdots + (a+(n-1)d)^r = \sum_{\ell=1}^{r} \sum_{i=1}^{n} \binom{a+(i-1)d}{\ell} \mu(r,\ell)\,,$$

the sum of the r^{th} powers of an arithmetic progression. When $a = d = 1$

$$s^{(r)}(n) = 1^r + 2^r + \cdots + n^r = \sum_{\ell=1}^{r} \binom{n+1}{\ell+1} \mu(r,\ell)\,,$$

since $\sum_{i=1}^{n} \binom{i}{\ell} = \binom{n+1}{\ell+1}$.

The display below exhibits $s^{(r)}(n)$ for $r = 1, 2, 3, 4, 5$.

$$\begin{array}{lllllllllll}
s^{(1)}(n) & & & & & 1\binom{n+1}{2} \\
s^{(2)}(n) & & & & 1\binom{n+1}{2} & + & 2!\binom{n+1}{3} \\
s^{(3)}(n) & & & 1\binom{n+1}{2} & + & 6\binom{n+1}{3} & + & 3!\binom{n+1}{4} \\
s^{(4)}(n) & & 1\binom{n+1}{2} & + & 14\binom{n+1}{3} & + & 36\binom{n+1}{4} & + & 4!\binom{n+1}{5} \\
s^{(5)}(n) 1\binom{n+1}{2} & + & 30\binom{n+1}{3} & + & 150\binom{n+1}{4} & + & 240\binom{n+1}{5} & + & 5!\binom{n+1}{6}
\end{array}$$

References

1. [1] Moser, W., Sums of d^{th} powers. *Math. Gazette* **75** (1991) 332.
2. [2] Paul, J.L., On the sum of the k^{th} powers of the first n integers. *Amer. Math. Monthly* **78** (1971) 271–273. MR 43 #4092.
3. [3] Wagner, C., Combinatorial proofs of formulas for power sums. *Arch. Math.* (Basel) **68** (1997), no. 6, 464–467.