THE OLYMPIAD CORNER

No. 189

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We begin this issue with an exam set sent in by one of our newer correspondents. My thanks go to Enrique Valeriano, National University of Engineering, Lima, Peru.

PERU'S SELECTION TEST FOR THE XII IBEROAMERICAN OLYMPIAD

August 1997 — 3.5 hours

1. Given an integer $a_0 > 2$, the sequence $a_0, a_1, a_2, \ldots$ is defined as follows:

$$\begin{array}{ll} a_{k+1} = a_k(1 + a_k), & \text{if } a_k \text{ is an odd number,} \\ a_{k+1} = \frac{a_k}{2}, & \text{if } a_k \text{ is an even number.} \end{array}$$

Prove that there is a nonnegative integer p such that $a_p > a_{p+1} > a_{p+2}$.

2. A positive integer is called "almost-triangular" if the number is itself triangular or is the sum of different triangular numbers. How many almost-triangular numbers are there in the set $\{1, 2, 3, \ldots, 1997\}$?

Note: The triangular numbers are $a_1, a_2, a_3, \ldots, a_k, \ldots$, where $a_1 = 1$, and $a_k = k + a_{k-1}$, for all $k \geq 2$.

3. An $n \times n$ chessboard ($n \geq 2$) is numbered with n^2 non-zero numbers. This chessboard is called an "Incaican Board" if, for each square the number written on the square is the difference between two of the numbers written on two of the neighbouring squares (sharing a common edge). For which values of n can one obtain Incaican Boards?

4. Let ABC be a given acute triangle. Give a ruler and compass construction of an equilateral triangle DEF with D on BC, E on AC, and F on AB such that the perpendiculars to BC at D, to AC at E, and to AB at F, respectively, are concurrent.

Next we give the Third and Fourth Grade problems of the 38^{th} Mathematics Competition for Secondary School Students of the Republic of Slovenia. My thanks go to Richard Nowakowski, Canadian Team Leader to the IMO in Hong Kong, for collecting this contest and forwarding it to me.

REPUBLIC OF SLOVENIA
38^{th} Mathematics Competition for Secondary School Students
April, 1994

Third Grade

1. Let n be a natural number. Prove: if $2n+1$ and $3n+1$ are perfect squares, then n is divisible by 40.

2. Show that $\cos(\sin x) > \sin(\cos x)$ holds for every real number x.

3. The polynomial $p(x) = x^3 + ax^2 + bx + c$ has only real roots. Show that the polynomial $q(x) = x^3 \Leftrightarrow bx^2 + acx \Leftrightarrow c^2$ has at least one nonnegative root.

4. Let the point D on the hypotenuse AC of the right triangle ABC be such that $|AB| = |CD|$. Prove that the bisector of $\angle BAC$, the median through B, and the altitude through D, of the triangle ABD have a common point.

Fourth Grade

1. Prove that there does not exist a function $f : \mathbb{Z} \to \mathbb{Z}$, for which $f(f(x)) = x + 1$ for every $x \in \mathbb{Z}$.

2. Put a natural number in every empty field of the table so that you get an arithmetic sequence in every row and every column.

	74			
				186
		103		
0				

3. Prove that every number of the sequence

$$49, \;\; 4489, \;\; 444889, \;\; 44448889, \;\; \ldots$$

is a perfect square (in every number there are n fours, $n \Leftrightarrow 1$ eights and a nine).

4. Let Q be the midpoint of the side AB of an inscribed quadrilateral $ABCD$ and S the intersection of its diagonals. Denote by P and R the orthogonal projections of S on AD and BC respectively. Prove that $|PQ| = |QR|$.

As a final contest for your puzzling pleasure in this number, we give the VIII Nordic Mathematical Contest. Again my thanks go to Richard Nowakowski, Canadian Team leader to the IMO in Hong Kong, for collecting this contest and forwarding it to me.

VIII NORDIC MATHEMATICAL CONTEST

March 17^{th}, 1994

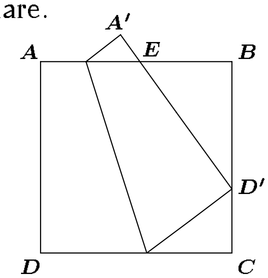
Time: 4 hours

1. Let O be a point in the interior of an equilateral triangle ABC with side length a. The lines AO, BO and CO intersect the sides of the triangle at the points A_1, B_1 and C_1 respectively. Prove that

$$|OA_1| + |OB_1| + |OC_1| < a.$$

2. A finite set S of points in the plane with integer coordinates is called a *two-neighbour set*, if for each (p,q) in S exactly two of the points $(p+1,q)$, $(p,q+1)$, $(p \Leftrightarrow 1, q)$, $(p, q \Leftrightarrow 1)$ are in S. For which n does there exist a two-neighbour set which contains exactly n points?

3. A square piece of paper $ABCD$ is folded by placing the corner D at some point D' on BC (see figure). Suppose AD is carried into $A'D'$, crossing AB at E. Prove that the perimeter of triangle EBD' is half as long as the perimeter of the square.



4. Determine all positive integers $n < 200$ such that $n^2 + (n+1)^2$ is a perfect square.

Turning now to comments and solutions related to the February 1997 number of the corner, we welcome two alternate solutions to problems of the Sixth Irish Mathematical Olympiad sent in by another new contributor whom we also welcome.

3. [1995: 151–152; 1997: 9–13] *Sixth Irish Mathematical Olympiad.*

For nonnegative integers n, r the binomial coefficient $\binom{n}{r}$ denotes the number of combinations of n objects chosen r at a time, with the convention that $\binom{n}{0} = 1$ and $\binom{n}{r} = 0$ if $n < r$. Prove the identity

$$\sum_{d=1}^{\infty} \binom{n \Leftrightarrow r + 1}{d} \binom{r \Leftrightarrow 1}{d \Leftrightarrow 1} = \binom{n}{r}$$

for all integers n, r with $1 \le r \le n$.

Alternate Solution by Mohammed Aassila, UFR de Mathématique et d'Informatique, L'Université Louis Pasteur, Strasbourg, France.

We have

$$\begin{aligned}(1+x)^n &= (1+x)^{n-r+1}(1+x)^{r-1} \\ &= \left[\sum_{i=0}^{m-r+1}\binom{n-r+1}{i}x^i\right]\left[\sum_{j=0}^{r-1}\binom{r-1}{j}x^j\right] \\ &= \sum_{i=0}^{m-r+1}\sum_{j=0}^{r-1}\binom{n-r+1}{i}\binom{r-1}{j}x^{i+j}\end{aligned}$$

The coefficient of x^r is

$$\sum_{d=0}^{r}\binom{n-r+1}{d}\binom{r-1}{r-d} = \sum_{d=1}^{r}\binom{n-r+1}{d}\binom{r-1}{d-1}$$

4. [1995:151–152; 1997: 9–13] *Sixth Irish Mathematical Olympiad*.

Let x be a real number with $0 < x < \pi$. Prove that, for all natural numbers n, the sum

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots + \frac{\sin(2n-1)x}{2n-1}$$

is positive.

Alternate solution by Mohammed Aassila, UFR de Mathématique et d'Informatique, L'Université Louis Pasteur, Strasbourg, France.

We know that

$$2\sin x\sin(2k-1)x = \cos(2k-2)x - \cos 2kx.$$

Hence

$$\begin{aligned}&2\sin x\left(\sin x + \frac{\sin 3x}{3} + \cdots + \frac{\sin(2n-1)x}{2n-1}\right) \\ &= 1 - \cos 2x + \frac{\cos 2x - \cos 4x}{3} + \cdots + \frac{\cos(2n-2)x - \cos 2nx}{2n-1} \\ &= 1 - \cos 2x\left(1 - \frac{1}{3}\right) - \cos 4x\left(\frac{1}{3} - \frac{1}{5}\right) - \cdots - \frac{\cos 2nx}{2n-1} \\ &\ge 1 - \left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \frac{1}{2n-1}\right] \;=\; 0.\end{aligned}$$

Next we turn to solutions to problems of the Latvian 44 Mathematical Olympiad given in the February number of the Corner last year [1997: 78].

LATVIAN 44 MATHEMATICAL OLYMPIAD
Final Grade, 3rd Round
Riga, 1994

1. It is given that $\cos x = \cos y$ and $\sin x = -\sin y$. Prove that $\sin 1994x + \sin 1994y = 0$.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Mansur Boase, student, St. Paul's School, London, England; by Panos E. Tsaoussoglou, Athens, Greece; and by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution of Covas.

More generally, we prove that $\sin mx + \sin my = 0$ where m is an integer.

Since $\sin mx + \sin my = 2\sin\left(m\frac{x+y}{2}\right)\cos\left(m\left(\frac{x-y}{2}\right)\right)$, it is sufficient to show that $\sin\left(\frac{x+y}{2}\right) = 0$ since $\sin\left(m\frac{x+y}{2}\right) = 0$ follows easily by induction from

$$\sin\left((m+1)\frac{x+y}{2}\right) = \sin\left(m\frac{(x+y)}{2}\right)\cos\left(\frac{x+y}{2}\right) + \cos\left(m\frac{(x+y)}{2}\right)\sin\left(\frac{x+y}{2}\right).$$

Now,

$$\begin{aligned}\cos x = \cos y &\iff \cos x - \cos y = 0\\ &\iff -2\sin\frac{x+y}{2}\sin\frac{x-y}{2} = 0 \qquad (1)\end{aligned}$$

and

$$\begin{aligned}\sin x = -\sin y &\iff \sin x + \sin y = 0\\ &\iff 2\sin\frac{x+y}{2}\cos\frac{x-y}{2} = 0\,. \qquad (2)\end{aligned}$$

Squaring each of (1) and (2) and adding, we find

$$4\sin^2\left(\frac{x+y}{2}\right)\underbrace{\left(\sin^2\frac{(x-y)}{2} + \cos^2\frac{(x-y)}{2}\right)}_{=1} = 0.$$

Hence $\sin\frac{x+y}{2} = 0$.

3. It is given that $a > 0$, $b > 0$, $c > 0$, $a + b + c = abc$. Prove that at least one of the numbers a, b, c exceeds $17/10$.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Heinz-Jürgen Seiffert, Berlin, Germany; by Panos E. Tsaoussoglou, Athens, Greece; and by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give two solutions of Shan and Wang.

First Solution.

We show, in general, that if $x_i > 0$ for $i = 1, 2, \ldots, n$, such that $\sum_{i=1}^{n} x_i = \prod_{i=1}^{n} x_i$, then $\max\{x_i : i = 1, 2, \ldots, n\} \geq n^{1/(n-1)}$. In particular, when $n = 3$ we get

$$\max\{x_1, x_2, x_3\} \geq \sqrt{3} > 1.7 = \frac{17}{10}.$$

By the arithmetic-geometric mean inequality we have

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^n \geq \prod_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i,$$

and thus $\sum_{i=1}^{n} x_i \geq n^{n/(n-1)}$.

Without loss of generality, we may assume that $\max\{x_i : i = 1, 2, \ldots, n\} = x_n$. Then $nx_n \geq \sum_{i=1}^{n} x_i \geq n^{n/(n-1)}$ from which $x_n \geq n^{1/(n-1)}$ follows.

Second Solution (without the AM-GM inequality).

Suppose $x_i > 0$ $(i = 1, 2, \ldots, n)$ such that $\sum_{i=1}^{n} x_i = \prod_{i=1}^{n} x_i$. Then dividing both sides by $\prod_{i=1}^{n} x_i$ we get

$$\sum_{i=1}^{n} \frac{1}{x_1 x_2 \ldots \hat{x}_i \ldots x_n} = 1,$$

where $\hat{x}_i$ indicates that the factor x_i is missing. Hence for some j we have

$$\frac{1}{x_1 x_2 \ldots \hat{x}_j \ldots x_n} \leq \frac{1}{n} \quad \text{or} \quad n \leq x_1 x_2 \ldots \hat{x}_j \ldots x_n.$$

Without loss we may assume that $x_n = \max\{x_i : i = 1, 2, \ldots, n\}$. Then $x_n^{n-1} \geq x_1 x_2 \ldots \hat{x}_j \ldots x_n \geq n$, from which $x_n \geq n^{1/(n-1)}$ follows. In particular, for $n = 3$, we get $x_3 \geq \sqrt{3} > 1.7 = \frac{17}{10}$.

4. Solve the equation $1! + 2! + 3! + \cdots + n! = m^3$ in natural numbers.

Solution by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We solve the more general problem of finding all solutions of the Diophantine equation $1! + 2! + 3! + \cdots + n! = m^k$ in natural numbers, n, m, and k. For convenience, let $S_n = 1! + 2! + 3! + \cdots + n!$.

When $k = 1$ clearly $m = S_n$ is the only solution for any n.

When $k = 2$ we claim that the equation $S_n = m^2$ has exactly two solutions: $n = m = 1$ and $n = m = 3$. Note first that $d! \equiv 0 \pmod{10}$

for all $d \geq 5$ and $S_4 = 1 + 2 + 6 + 24 = 33 \equiv 3 \pmod{10}$. Hence $S_n \equiv 3 \pmod{10}$ for all $n \geq 4$. However, it is easy to see that the last digit of a perfect square can never be 3 and so there are no solutions if $n \geq 4$. Checking the cases when $n = 1, 2, 3$ directly reveals that there are precisely two solutions, as given above.

When $k \geq 3$ we show that $n = m = 1$ is the only solution. If $n \geq 2$ then clearly $S_n \equiv 0 \pmod 3$. But $m^k \equiv 0 \pmod 3$ implies $m \equiv 0 \pmod 3$ and so $m^k \equiv 0 \pmod{27}$ as $k \geq 3$. Since $d! \equiv 0 \pmod{27}$ for all $d \geq 9$ and since

$$S_8 = 1 + 2 + 6 + 24 + 120 + 720 + 5040 + 40320 = 46233 \not\equiv 0 \pmod{27}$$

there are no solutions if $n \geq 8$. On the other hand, direct checking shows that for $n = 2, 3, 4, 5, 6, 7$, $S_n = 3, 9, 33, 153, 873$, and 5913, none of which is a perfect k^{th} power for any $k \geq 3$. Finally, it is trivial to see that $n = m = 1$ is a solution.

Remark: The special case of this problem when $k = 2$ was proposed by E.T.H. Wang as Quicky Q657 in the *Mathematics Magazine*, 52 (1979), p. 47. The general case was also proposed by him as problem #4203 in *Mathmedia* (in Chinese) 4(2), 1980; p. 64 with solution in 4(3), 1980, p. 49. This is a journal published by the Institute of Mathematics, Academia Sinica, Taipei, Taiwan.

5. There are 1994 employees in the office. Each of them knows 1600 others of them. Prove that we can find 6 employees, each of them knowing all 5 others.

Solution by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

Let E denote the set of these 1994 employees. For each $x \in E$, let $S(x)$ denote the set of all employees whom x does *not* know. Then by assumption, $|S(x)| = 393$ for all $x \in E$. Let a and b be any two employees who know each other. Since

$$|S(a) \cup S(b)| \leq 2 \times 393 = 786 < 1992,$$

$\exists c \in E$ such that a, b, and c form a triple of mutual acquaintances. Since

$$|S(a) \cup S(b) \cup S(c)| \leq 3 \times 393 = 1179 < 1991,$$

$\exists d \in E$ such that a, b, c, and d form a quadruple of mutual acquaintances. Since

$$|S(a) \cup S(b) \cup S(c) \cup S(d)| \leq 4 \times 393 = 1572 < 1990,$$

$\exists e \in E$ such that a, b, c, d, and e form a quintuple of mutual acquaintances. Finally, since

$$|S(a) \cup S(b) \cup S(c) \cup S(d) \cup S(e)| \leq 5 \times 393 = 1965 < 1989,$$

$\exists f \in E$ such that a, b, c, d, e, and f form a sextuple of mutual acquaintances.

1st SELECTION ROUND

1. It is given that x and y are positive integers and $3x^2 + x = 4y^2 + y$. Prove that $x \Leftrightarrow y$, $3x + 3y + 1$ and $4x + 4y + 1$ are squares of integers.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; by Panos E. Tsaoussoglou, Athens, Greece; and by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution by Shan and Wang and their comment.

Note first that the given equation implies the following two equations:

$$(x \Leftrightarrow y)(3x + 3y + 1) = y^2 \quad (1)$$
$$(x \Leftrightarrow y)(4x + 4y + 1) = x^2 \quad (2)$$

(1) $\times$ (2) yields $(x \Leftrightarrow y)^2(3x + 3y + 1)(4x + 4y + 1) = (xy)^2$ which implies that $(3x + 3y + 1)(4x + 4y + 1)$ is a perfect square. But clearly $\gcd(3x+3y+1, 4x+4y+1) = 1$ since $4(3x+3y+1) \Leftrightarrow 3(4x+4y+1) = 1$. Therefore, $3x + 3y + 1$ and $4x + 4y + 1$ are both squares, which, together with (1) (or (2)), implies that $x \Leftrightarrow y$ is also a square.

Comment: This would be a much better problem had it asked to show only that $x \Leftrightarrow y$ is a square. This is an example of a case when asking to prove too many things actually gives the solution away, in some sense.

Shan and Wang also proposed the following problem inspired by this one.

The Diophantine equation $3x^2 + x = 4y^2 + y$ is satisfied when $x = 30$ and $y = 26$.

(a) Find another solution in positive integers.

(b) Are there infinitely many solutions in positive integers? Is so, describe all of them.

2nd SELECTION ROUND

1. It is given that $0 \le x_i \le 1$, $i = 1, 2, \ldots, n$. Find the maximum of the expression

$$\frac{x_1}{x_2x_3\ldots x_n + 1} + \frac{x_2}{x_1x_3x_4\ldots x_n + 1} + \cdots + \frac{x_n}{x_1x_2\ldots x_{n-1} + 1}.$$

Solutions by Murray S. Klamkin, University of Alberta, Edmonton, Alberta; and by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We first give the solution of Shan and Wang.

Clearly $n \ge 2$ for the question to make sense. Let S_n denote the given sum. We prove that $S_n \le n \Leftrightarrow 1$. For $n = 2$, equality holds if and only if $x_1 = 0$, $x_2 = 1$ or $x_1 = 1$, $x_2 = 0$ or $x_1 = x_2 = 1$. For $n \ge 3$, equality holds if and only if one of the x_i's is 0 and the others are all equal to 1. We first establish a lemma:

Lemma. Suppose that the x_i's are reals such that $0 \leq x_i \leq 1$ for all $i = 1, 2, \ldots, n$, where $n \geq 2$. Then $x_1 + x_2 + \cdots + x_n \leq n \Leftrightarrow 1 + x_1 x_2 \cdots x_n$ with equality holding if and only if $x_i \neq 1$ for at most one i, $i = 1, 2, \ldots, n$.

Proof. For $n = 2$, $x_1 + x_2 \leq x_1 x_2 + 1 \Leftrightarrow (1 \Leftrightarrow x_1)(1 \Leftrightarrow x_2) \geq 0$, which is clearly true. Equality holds if and only if $x_1 = 1$ or $x_2 = 1$. Suppose the assertion holds for some $n \geq 2$. Then

$$\begin{aligned} x_1 + x_2 + \cdots + x_n + x_{n+1} &\leq n \Leftrightarrow 1 + x_1 x_2 \cdots x_n + x_{n+1} \\ &\leq n \Leftrightarrow 1 + x_1 x_2 \cdots x_n x_{n+1} + 1 \\ &= n + x_1 x_2 \cdots x_{n+1}. \end{aligned}$$

(The 2^{nd} inequality is by the $n = 2$ case.)

If equality holds, then it must hold in both inequalities above. By the induction hypotheses, we then have

(i) at most one of $x_1, x_2, \ldots, x_n$ is different from 1 and

(ii) either $x_{n+1} = 1$ or $x_1 x_2 \cdots x_n = 1$.

Since $x_1 x_2 \cdots x_n = 1$ clearly implies $x_1 = x_2 = \cdots = x_n = 1$, our assertion about the equality follows.

Now we proceed to prove the claim about S_n. For $n = 2$,

$$S_2 = \frac{x_1}{x_2 + 1} + \frac{x_2}{x_1 + 1} \leq \frac{x_1}{x_2 + x_1} + \frac{x_2}{x_1 + x_2} = 1.$$

It is readily seen that equality holds if and only if $x_1 = 0$, $x_2 = 1$ or $x_1 = 1$, $x_2 = 0$, or $x_1 = x_2 = 1$. For $n \geq 3$ we apply the lemma above and obtain

$$S_n \leq \frac{x_1 + x_2 + \cdots + x_n}{x_1 x_2 \cdots x_n + 1} \leq \frac{(n \Leftrightarrow 1) + x_1 x_2 \cdots x_n}{x_1 x_2 \cdots x_n + 1} \leq n \Leftrightarrow 1.$$

If equality holds, then the 2^{nd} inequality implies $x_j \neq 1$ for at most one j and the 3^{rd} inequality implies $(n \Leftrightarrow 1)x_1 x_2 \cdots x_n = x_1 x_2 \cdots x_n$, which implies that $x_i = 0$ for at least one i. Hence $x_i = 0$ for some i and $x_j = 1$ for all $j \neq i$. It is obvious that this condition is also sufficient. This completes the proof.

And here is Klamkin's somewhat more abbreviated solution with generalization.

Since the expression is convex in each of the variables, the maximum value is achieved when each variable takes on 0 or 1. Clearly this occurs when one variable is 0 and the rest are 1 giving the maximum value of $n \Leftrightarrow 1$. The same maximum occurs if any of the numerators x_i are replaced by $x_i^{\alpha_i}$ where $\alpha_i \geq 1$.

A similar result, using convexity, that

$$\sum \frac{x_i^u}{1 + s \Leftrightarrow x_i} + \prod (1 \Leftrightarrow x_i)^v \leq 1,$$

where $0 \leq x_i \leq 1$, $u, v \geq 1$, $s = \sum x_i$ and the sum and product are over $i = 1, 2, \ldots, n$, is given in [1].

Reference:

[1] M.S. Klamkin, *USA Mathematical Olympiads* 1972–1986, M.A.A., Washington D.C., 1988, p. 82.

3. A triangle ABC is given. From the vertex B, n rays are constructed intersecting the side AC. For each of the $n+1$ triangles obtained, an incircle with radius r_i and excircle (which touches the side AC) with radius R_i is constructed. Prove that the expression

$$\frac{r_1 r_2 \ldots r_{n+1}}{R_1 R_2 \ldots R_{n+1}}$$

depends on neither n nor on which rays are constructed.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

If A, B, C are the angles of a triangle, then

$$r = s\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} \qquad \text{and} \qquad r_a = s\tan\frac{A}{2},$$

where r, r_a are the inradius and the radius of the excircle opposite A, and s is the semiperimeter.

It follows that

$$\frac{r}{r_a} = \tan\frac{B}{2}\tan\frac{C}{2}.$$

Next we apply this result to each of $n+1$ triangles obtained (see figure at the top of the next page). This yields

$$\begin{aligned}
\tfrac{r_1}{R_1} &= \tan\tfrac{A}{2}\tan\tfrac{\alpha_1}{2} \\
\tfrac{r_2}{R_2} &= \tan\tfrac{180^\circ-\alpha_1}{2}\tan\tfrac{\alpha_2}{2} \\
&\quad \cdots\cdots\cdots \\
&\quad \cdots\cdots\cdots \\
\tfrac{r_n}{R_n} &= \tan\tfrac{180^\circ-\alpha_{n-1}}{2}\tan\tfrac{\alpha_n}{2} \\
\tfrac{r_{n+1}}{R_{n+1}} &= \tan\tfrac{180^\circ-\alpha_n}{2}\tan\tfrac{C}{2}.
\end{aligned}$$

Multiplying these equalities, we observe that the product of all the right hand members is

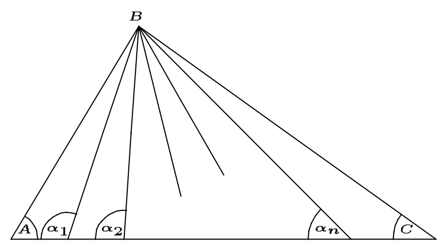
$$\tan\frac{A}{2}\cdot\left(\tan\frac{\alpha_1}{2}\cot\frac{\alpha_1}{2}\right)\cdot\left(\tan\frac{\alpha_2}{2}\cot\frac{\alpha_2}{2}\right)\cdots\left(\tan\frac{\alpha_n}{2}\cot\frac{\alpha_n}{2}\right)\cdot\tan\frac{C}{2}$$

$$= \tan\frac{A}{2}\cdot\tan\frac{C}{2},$$

and we get

$$\frac{r_1 r_2 \cdots r_{n-1}}{R_1 R_2 \cdots R_{n+1}} = \tan\frac{A}{2}\cdot\tan\frac{C}{2}$$

which depends on neither n nor on which rays are constructed.

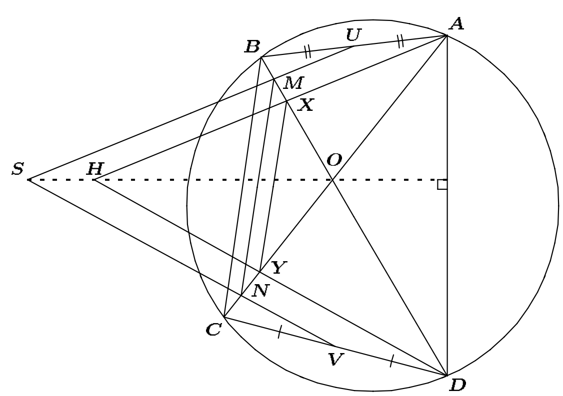


3^{rd} SELECTION ROUND

3. Let $ABCD$ be an inscribed quadrilateral. Its diagonals intersect at O. Let the midpoints of AB and CD be U and V. Prove that the lines through O, U and V, perpendicular to AD, BD and AC respectively, are concurrent.

Solution by Toshio Seimiya, Kawasaki, Japan.

Case 1. Neither is AC orthogonal to BD, nor is AD a diameter.



Let M, N be the feet of the perpendiculars from U, V to BD, AC respectively, and let S be the intersection of UM and VN. Let X, Y be the feet of the perpendiculars from A, D to BD, AC respectively, and let H be the intersection of AX and DY. Since U is the midpoint of AB, and $UM \| AX$, M is the midpoint of BX. Similarly N is the midpoint of CY.

Since $\angle AXD = \angle AYD = 90°$, A, X, Y, D are concyclic. Therefore $\angle YXD = \angle YAD = \angle CAD = \angle CBD$. Thus we have $XY \| BC$.

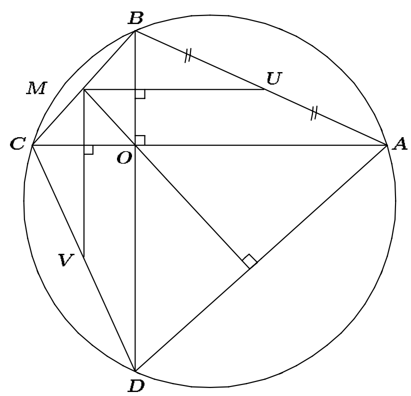
Since M, N are the midpoints of BX, CY respectively, we have $MN \| XY$.

Since $SM \| HX$, $XN \| HY$, and $MN \| XY$, MX, NY and SH are concurrent at O. Therefore S, H, O are collinear.

Since $AH \perp OD$ and $DH \perp OA$, H is the orthocentre of ΔOAD, so that $HO \perp AD$. Thus we have $SO \perp AD$.

Thus the lines through O, U and V, perpendicular to AD, BD and AC respectively, are concurrent at S.

Case 2. AC is orthogonal to BD.

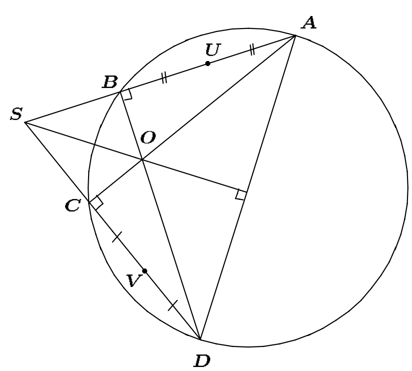


Let M be the midpoint of BC. Since U is the midpoint of AB, we get $UM \perp AC$, so that $UM \perp BD$. Similarly we have $VM \perp AC$.

Since $AC \perp BD$ and M is the midpoint of BC, by Brahmagupta's Theorem we have $MO \perp AD$. (See: H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*, p. 59).

Thus the lines through O, U and V, perpendicular to AD, BD and AC respectively, are concurrent at M.

Case 3. AD is a diameter.



Let S be the intersection of AB and CD. Since $AB \perp BD$ and $CD \perp AC$, S is the orthocentre of ΔOAD. Thus $SO \perp AD$.

Hence the lines through O, U and V, perpendicular to AD, BD and AC respectively, are concurrent at S.

That completes this number of the Olympiad Corner. Send me your nice solutions and suggestions for future issues.