# PROBLEMS

*Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk* ($\star$) *after a number indicates that a problem was submitted without a solution.*

*In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.*

*To facilitate their consideration, please send your proposals and solutions on signed and separate standard* $8\frac{1}{2}''\times 11''$ *or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 November 1997**. *They may also be sent by email to* **crux-editors@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX). *Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication.*

**2226**. *Proposed by K.R.S. Sastry, Dodballapur, India.*

An old man willed that, upon his death, his three sons would receive the $u$'th, $v$'th, $w$'th parts of his herd of camels respectively. He had $uvw-1$ camels in the herd when he died. Obviously, their sophisticated calculator could not divide $uvw-1$ exactly into $u$, $v$ or $w$ parts. They approached a distinguished ***CRUX*** problem solver for help, who rode over on his camel, which he added to the herd and then fulfilled the old man's wishes, and took the one camel that remained, which was, of course, his own.

Dear ***CRUX*** reader, how many camels were there in the herd?

**2227**. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Evaluate

$$\prod_p \left[\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2p)^{2k}}\right].$$

where the product is extended over all prime numbers.

**2228**. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Let $A$ be the set of all real numbers from the interval $(0, 1)$ whose decimal representation consists only of $1$'s and $7$'s; that is, let

$$A = \left\{ \sum_{k=1}^{\infty} \frac{a_k}{10^k} : a_k \in \{1, 7\} \right\}.$$

Let $B$ be the set of all reals that cannot be expressed as finite sums of members of $A$. Find $\sup B$.

**2229**. *Proposed by Kenneth Kam Chiu Ko, Mississauga, Ontario.*

(a) Let $m$ be any positive integer greater than $2$, such that $x^2 \equiv 1 \pmod{m}$ whenever $(x, m) = 1$.

Let $n$ be a positive integer. If $m | n+1$, prove that the sum of all divisors of $n$ is divisible by $m$.

(b)★ Find all possible values of $m$

**2230**. *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Triangles $BCD$ and $ACE$ are constructed outwardly on sides $BC$ and $CA$ of triangle $ABC$ such that $AE = BD$ and $\angle BDC + \angle AEC = 180^\circ$. The point $F$ is chosen to lie on the segment $AB$ so that

$$\frac{AF}{FB} = \frac{DC}{CE}.$$

Prove that

$$\frac{DE}{CD + CE} = \frac{EF}{BC} = \frac{FD}{AC}.$$

**2231**. *Proposed by Herbert Gülicher, Westfalische Wilhelms-Universität, Münster, Germany.*

In quadrilateral $P_1P_2P_3P_4$, suppose that the diagonals intersect at the point $M \neq P_i$ $(i = 1, 2, 3, 4)$. Let $\angle MP_1P_4 = \alpha_1$, $\angle MP_3P_4 = \alpha_2$, $\angle MP_1P_2 = \beta_1$ and $\angle MP_3P_2 = \beta_2$.

Prove that

$$\lambda_{13} := \frac{|P_1M|}{|MP_3|} = \frac{\cot\alpha_1 \pm \cot\beta_1}{\cot\alpha_2 \pm \cot\beta_2},$$

where the $+(-)$ sign holds if the line segment $P_1P_3$ is located inside (outside) the quadrilateral.

**2232**. *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Find all solutions of the inequality:

$$n^2+n-5<\left\lfloor\frac{n}{3}\right\rfloor+\left\lfloor\frac{n+1}{3}\right\rfloor+\left\lfloor\frac{n+2}{3}\right\rfloor<n^2+2n-2,\ (n\in\mathbb{N}).$$

(Note: If $x$ is a real number, then $\lfloor x\rfloor$ is the largest integer not exceeding $x$.)

**2233**. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $x$, $y$, $z$ be non-negative real numbers such that $x+y+z=1$, and let $p$ be a positive real number.

(a) If $0<p\leq 1$, prove that

$$x^p+y^p+z^p\ \geq\ C_p\left((xy)^p+(yz)^p+(zx)^p\right),$$

where

$$C_p\ =\ \begin{cases} 3^p & \text{if}\quad p\leq\frac{\log 2}{\log 3-\log 2},\\ 2^{p+1} & \text{if}\quad p\geq\frac{\log 2}{\log 3-\log 2}.\end{cases}$$

(b)* Prove the same inequality for $p>1$.

Show that the constant $C_p$ is best possible in all cases.

**2234**. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Given triangle $ABC$, its centroid $G$ and its incentre $I$, construct, using only an unmarked straightedge, its orthocentre $H$.

**2235**. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Triangle $ABC$ has angle $\angle CAB=90^\circ$. Let $\Gamma_1(O,R)$ be the circumcircle and $\Gamma_2(T,r)$ be the incircle. The tangent to $\Gamma_1$ at $A$ and the polar line of $A$ with respect to $\Gamma_2$ intersect at $S$. The distances from $S$ to $AC$ and $AB$ are denoted by $d_1$ and $d_2$ respectively.

Show that

(a) $ST\|BC$,

(b) $|d_1-d_2|=r$.

[For the benefit of readers who are not familiar with the term "polar line", we give the following definition as in, for example, *Modern Geometries*, 4th Edition, by James R. Smart, Brooks/Cole, 1994:

*The line through an inverse point and perpendicular to the line joining the original point to the centre of the circle of inversion is called the polar of the original point, whereas the point itself is called the pole of the line.*]

**2236**. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Let $ABC$ be an arbitrary triangle and let $P$ be an arbitrary point in the interior of the circumcircle of $\triangle ABC$. Let $K$, $L$, $M$, denote the feet of the perpendiculars from $P$ to the lines $AB$, $BC$, $CA$, respectively.

Prove that $[KLM] \leq \frac{[ABC]}{4}$.

Note: $[XYZ]$ denotes the area of $\triangle XYZ$.

**2237**. *Proposed by Meletis D. Vasiliou, Elefsis, Greece.*

$ABCD$ is a square with incircle $\Gamma$. Let $\ell$ be a tangent to $\Gamma$. Let $A'$, $B'$, $C'$, $D'$ be points on $\ell$ such that $AA'$, $BB'$, $CC'$, $DD'$ are all perpendicular to $\ell$.

Prove that $AA' \cdot CC' = BB' \cdot DD'$.

## Correction

**2173**. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $n \geq 2$ and $x_1, \ldots, x_n > 0$ with $x_1 + \ldots + x_n = 1$.

Consider the terms

$$l_n = \sum_{k=1}^{n} (1 + x_k) \sqrt{\frac{1 - x_k}{x_k}}$$

and

$$r_n = C_n \prod_{k=1}^{n} \frac{1 + x_k}{\sqrt{1 - x_k}}$$

where

$$C_n = (\sqrt{n-1})^{n+1} (\sqrt{n})^n / (n+1)^{n-1}.$$

[Ed: there is no $x$ in the line above.]

1. Show $l_2 \leq r_2$.
2. Prove or disprove: $l_n \geq r_n$ for $n \geq 3$.