# MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

**All material intended for inclusion in this section should be sent to the Mayhem Editor, Naoki Sato, Department of Mathematics, University of Toronto, Toronto, ON Canada M5S 1A1**. The electronic address is

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The Assistant Mayhem Editor is Cyrus Hsia (University of Toronto). The rest of the staff consists of Richard Hoshino (University of Waterloo), Wai Ling Yee (University of Waterloo), and Adrian Chan (Upper Canada College).

## A Journey to the Pole — Part II

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In this second (and last, for your relief) article, we look at some advanced topics like inversion or the applications of calculus to the theory of curves.

### Inversion

Inversion is a transformation determined by a point called the *centre of inversion* $O$ and an *inversion ratio* $\pm k^2$. The image of a point $P$ is a point $P'$ such that $P'$ is on line $OP$ and $|OP'| = \pm k^2|OP|$. It is evident that a curve $r = f(\theta)$ can be inverted by letting $r = \pm\dfrac{k^2}{f(\theta)}$. Inversion is a *conformal* transformation, meaning that the angles between intersecting curves are preserved. We will prove this in a later section.

**Exercise 1.** Prove that the inverse curve of a straight line is itself if it passes through the origin or a circle through the origin if it does not.

**Example 1.** By inspection of the equations of the conic $r = \dfrac{de}{1 - e\cos(\theta - \phi)}$ and the limaçon $r = b + a\cos\theta$, it is evident that the inverse of a conic about its focus is a limaçon. I imagine that trying to prove this theory with synthetic geometry would result in a severe headache.

This last result provides a different definition of conics as loci if we invert the definition of the limaçon given above. Consider a circle or straight line and a point $O$ not on it. Draw a circle through $O$ tangent to the circle or line. The diameter through $O$ intersects the circle at $P$. The locus of all $P$'s is a conic with $O$ at one focus.

**Tangent Lines**

We leave the realm of elementary geometry to enter calculus, where we will teach the same old dog new tricks. The first is how to find the tangent line to a polar curve.

Our starting point will be the equation of the straight line $d = r\sin(\phi - \theta)$. The tangent line at $\theta_0$ is a first-order approximation to the curve involving $r(\theta_0)$ and $\left.\dfrac{dr}{d\theta}\right|_{\theta_0}$. Differentiating the equation of the straight line with respect to $\theta$ at $\theta_0$, we get $0 = \left.\dfrac{dr}{d\theta}\right|_{\theta_0} \sin(\phi - \theta_0) - r(\theta_0)\cos(\phi - \theta_0)$, which implies that $\tan(\phi - \theta_0) = \left.\frac{r}{(dr/d\theta)}\right|_{\theta_0}$. This quantity can sometimes be useful in itself, as $\phi - \theta_0$ represents the angle between the radius vector and the tangent line. We will make use of it in the next example.

The orientation $\phi$ is determined from its tangent, and

$$\tan(\phi - \theta_0 + \theta_0) = \frac{\tan(\phi - \theta_0) + \tan\theta_0}{1 - \tan(\phi - \theta_0)\tan\theta_0}$$

implies that

$$\tan\phi = \frac{r(\theta_0) + \frac{dr}{d\theta}|_{\theta_0}\tan\theta_0}{\frac{dr}{d\theta}|_{\theta_0} - r(\theta_0)\tan\theta_0}.$$

The parameter $d$ in the equation of the tangent line is given (after some trigonometric manipulations) by

$$d = \frac{r(\theta_0)\tan(\phi - \theta_0)}{\sqrt{1 + \tan^2(\phi - \theta_0)}},$$

which gives

$$d = \frac{r^2(\theta_0)}{\sqrt{r^2(\theta_0) + (dr/d\theta)^2|_{\theta_0}}}.$$

**Example 2.** Proof that inversion preserves angles. Let $r = f(\theta)$ and $r = g(\theta)$ be two curves that intersect at $\theta_0$. Their directions at $\theta_0$ are $\phi_1$ and $\phi_2$, and the angle between them satisfies

$$\tan(\phi_2 - \phi_1) = \tan(\phi_2 - \theta_0 + \theta_0 - \phi_1) = \frac{\tan(\phi_2 - \theta_0) - \tan(\phi_1 - \theta_0)}{1 + \tan(\phi_2 - \theta_0)\tan(\phi_1 - \theta_0)}.$$

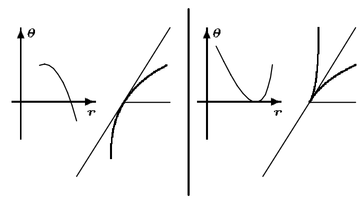
Now, the inverted curves, $r' = 1/f(\theta)$ and $r' = 1/g(\theta)$, also intersect at $\theta_0$ and their directions $\phi_1'$ and $\phi_2'$ satisfy

$$\tan(\phi_1' - \theta_0) = \frac{1/f}{\frac{-1}{f^2}\left(\frac{df}{d\theta}\right)} = \frac{-f}{(df/d\theta)} = \tan(\theta_0 - \phi_1)$$

(and similarly for $\phi_2'$). Hence, $\tan(\phi_2 - \phi_1) = \tan(\phi_1' - \phi_2')$ and we are done.

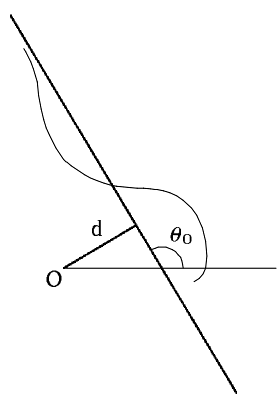
**Tangent lines through the origin**

An interesting special case is when $r(\theta_0) = 0$. In that case, $\tan\phi = \tan -\theta_0$ or $\phi = \theta_0$. In words, if the curve crosses the origin for a given $\theta_0$, the equation of the tangent line at the origin is $\theta = \theta_0$.




When sketching curves, a useful result is that if $r(\theta)$ has an odd-order root at $\theta_0$, then the curve is smooth at the origin, but if the root is even-order, then there is a cusp (see the figure).

**Asymptotes**




In certain cases $r$ tends to infinity for some finite value of $\theta$, signalling the possibility of an asymptote. In handling asymptotes, it is convenient to consider $s(\theta) = 1/r(\theta)$, in which case $\tan(\phi - \theta_0) = \frac{-s}{(ds/d\theta)}|_{\theta_0}$ and $\tan\phi = \frac{\frac{ds}{d\theta}|_{\theta_0}\tan\theta_0 - s(\theta_0)}{\frac{ds}{d\theta}|_{\theta_0} + s(\theta_0)\tan\theta_0}$. There is a possible asymptote if $s(\theta_0) = 0$ and its equation is: $s(\theta) = \frac{\sin(\theta - \theta_0)}{d}$, or $\theta = \theta_0$ if $d = 0$.

This is because, as in the case of tangent lines through the origin, the slope of the tangent line is $\tan\phi = \tan\theta_0$. The parameter $d$ is given by

$$d = \lim_{\theta\to\theta_0} \frac{1}{\sqrt{s^2(\theta) + (ds/d\theta)^2}}.$$

If this does not diverge, there is an asymptote.

When sketching curves, it is useful to know from which side of the asymptote the curve approaches infinity. This is achieved by studying the sign of $s(\theta) - \frac{1}{d}\sin(\theta - \theta_0)$, which tends to $0$ at $\theta = \theta_0$. If it tends to $0+$, the curve is closer to the origin than the asymptote (see the figure on the last page), and if it tends to $0-$, the curve is farther from the origin than the asymptote.

**Exercise 2.** Sketch the curve $r = \ln\theta$, its asymptote and the tangent line at the origin.

**Example 3.** The parabola $r = \dfrac{1}{1-\cos\theta}$ satisfies $\lim_{\theta\to 0} r(\theta) = \infty$, but it has no asymptotes since $\dfrac{1}{\sqrt{(1-\cos\theta_0)^2 + (\sin\theta_0)^2}} = \dfrac{1}{\sqrt{2(1-\cos\theta_0)}}$, which diverges at $\theta_0$.

**Arc Length**

Another application of calculus is the computation of curve lengths. Usually one would take the expression for the line element in cartesian coordinates, $dl^2 = dx^2 + dy^2$ and transform it to polar coordinates. To use only polar coordinates, one could apply Pythagoras' Theorem to $(dr)$ and $(rd\theta)$. Although this gives the right answer, it is not rigorous. A rigorous argument that does not rely on rectangular coordinates follows.

Applying the cosine rule to side $PP'$ of $POP'$ (see figure) we have

$$dl^2 = r^2(\theta + d\theta) + r^2(\theta) - 2r(\theta)r(\theta+d\theta)\cos(d\theta).$$

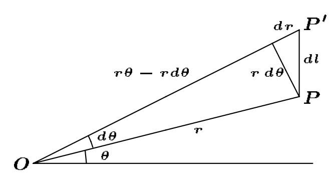
Expanding each term in a Taylor series up to the second order in $d\theta$, we get

$$\begin{aligned} dl^2 &= r^2 + 2r\frac{dr}{d\theta} + \left[\left(\frac{dr}{d\theta}\right)^2 + r\frac{d^2r}{d\theta^2}\right]d\theta^2 \\ &\qquad + r^2 - 2r\left(r + \frac{dr}{d\theta} + \frac{1}{2}\frac{d^2r}{d\theta^2}\right)\left(1 - \frac{1}{2}d\theta^2\right). \end{aligned}$$

Keeping terms up to second order in $d\theta$ we have

$$dl^2 = \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]d\theta^2.$$

We can thus write the expression for the length of a curve in polar coordinates as follows: $l = \int_{\theta_0}^{\theta}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_{\theta_0}^{\theta}\frac{1}{s^2}\sqrt{s^2 + \left(\frac{ds}{d\theta}\right)^2}\,d\theta$, where $s = \frac{1}{r}$.




**Exercise 3.** Derive the polar expression for arc length from the cartesian expression $dl^2 = dx^2 + dy^2$.

**Curvature**

It seems tautological to say that curvature is an important feature of curves, but the fact is that a planar curve is uniquely determined (up to translations and rotations) if its curvature is known as a function of arc length. This is generally of little practical importance, since the resulting differential equations can only be solved if you know the solution! We will give the formula for curvature in terms of $s(\theta) = l/r(\theta)$ and some applications.

Curvature, $\kappa$, can be defined as the rate of change of the direction of the tangent line per unit arc length. We have

$$\kappa = \frac{d\phi}{ds} = \frac{d\theta}{ds} \times \frac{d\phi}{d\theta} = \frac{s^2}{\sqrt{s^2 + \left(\frac{ds}{d\theta}\right)^2}} \times \left[1 + \frac{d(\phi - \theta)}{d\theta}\right].$$

Now, $\tan(\phi - \theta) = \dfrac{-s}{(ds/d\theta)}$, so that

$$\begin{aligned}
\frac{d(\phi-\theta)}{d\theta} &= \frac{d\left(\arctan \frac{-s}{(ds/d\theta)}\right)}{d\theta} \\
&= \frac{1}{1 + \frac{s^2}{(ds/d\theta)}} \times \frac{-\left(\frac{ds}{d\theta}\right)^2 + s\left(\frac{d^2 s}{d\theta^2}\right)}{(ds/d\theta)^2}
\end{aligned}$$

giving

$$\begin{aligned}
\kappa &= \frac{s^2}{\sqrt{s^2 + \left(\frac{ds}{d\theta}\right)^2}} \times \left[1 + \frac{-\left(\frac{ds}{d\theta}\right)^2 + s\left(\frac{d^2 s}{d\theta^2}\right)}{s^2 + \left(\frac{ds}{d\theta}\right)^2}\right] \\
&= \frac{s^2}{\sqrt{s^2 + \left(\frac{ds}{d\theta}\right)^2}} \times \left[\frac{s + \left(\frac{d^2 s}{d\theta^2}\right)^2}{s^2 + \left(\frac{ds}{d\theta}\right)^2}\right] \\
&= \frac{s + \left(\frac{d^2 s}{d\theta^2}\right)}{\left[1 + \left(\frac{1}{2}\frac{ds}{d\theta}\right)^2\right]^{3/2}}.
\end{aligned}$$

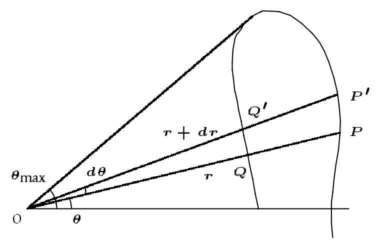
In terms or $r$, we have

$$\kappa = \frac{r^2 + 2(dr/d\theta)^2 - r(d^2 r/d\theta^2)}{[r^2 + (dr/d\theta)^2]^{3/2}}.$$

**Example 4.** If curvature is zero we obtain the differential equation $s + s'' = 0$, with general solution $s = d\cos(\theta - \theta_0)$, that is, the equation of a straight line.

**Exercise 4.** Check that the curvature at each point of a lemniscate is proportional to the distance to the origin. (Hint: to simplify the algebra, divide numerator and denominator by $r^3$ in the curvature formula above.)

**Area Enclosed by Polar Curves**




The last application of calculus is the calculation of areas. The natural surface element is the 'triangle' defined by a segment of curve and the radius vectors at the endpoints ($OPP'$ in the figure). The area of this triangle is, to first order in $d\theta$, $dA = \frac{1}{2}r^2 d\theta$.

If the origin does not lie inside the curve the equation $r = r(\theta)$ will have more than one branch, as shown, and the sign of the 'enclosed area' $dA$ depends on the orientation given to the curve. In the figure the curve is traversed counterclockwise, and so the outer branch ($PP'$) has positive sign (the positive sense of $d\theta$ coincides with the direction of the curve) and the inner branch ($QQ'$) has negative sign (the positive sense of $d\theta$ as opposed to the direction of the curve). The same applies when calculating the area enclosed by two intersecting curves.

**Example 5.** As our last example, we will evaluate the area enclosed by the circle $(r - R\cos\theta)^2 = \rho^2 - R^2\sin^2\theta$. The area is given by

$$A = \int_{\sin\theta=-\rho/R}^{\sin\theta=\rho/R} \frac{1}{2}(r_+^2 - r_-^2)d\theta,$$

where $r_\pm = R\cos\theta \pm \sqrt{\rho^2 - R^2\sin^2\theta}$. We have

$$A = \int \frac{1}{2}(r_+ + r_-)(r_+ - r_-)d\theta = \int 2(R\cos\theta)\sqrt{\rho^2 - R^2\sin^2\theta}d\theta.$$

Letting $R\sin\theta = \rho\sin\phi$ and $R\cos\theta d\theta = \rho\cos\phi d\phi$, we have

$$A = \int_{\phi=-\pi/2}^{\pi/2} 2\rho^2\cos^2\phi d\phi = \pi\rho^2$$

as expected.

# A Pattern in Permutations

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Let $p_n(k)$ be the number of permutations on $n$ elements (say, $n$ large anteaters) with exactly $k$ fixed points (i.e. a permutation which takes $k$ elements to themselves), so for example, $p_3(0) = 2$, $p_3(1) = 3$, $p_3(2) = 0$, and $p_3(3) = 1$. It should be clear that $\sum_{k=0}^{n} p_n(k) = n!$, but may be not so obvious that $\sum_{k=0}^{n} kp_n(k) = n!$ (this was problem 1 on the 1987 IMO). It may be even more surprising to learn that for $n \geq 2$, $\sum_{k=0}^{n} k^2 p_n(k) = 2n!$. What kind of pattern ensues? As my old analysis prof would no doubt say, "this is good exercise," and it is kind of fun to follow a trail like this and see where it leads.

Based on the above results, for $n \geq 1$ and $t \geq 0$, let

$$Q(n,t) = \frac{1}{n!} \sum_{k=0}^{n} k^t p_n(k).$$

Then $Q(n,0) = Q(n,1) = 1$ for all $n \geq 1$ and $Q(n,2) = 2$ for all $n \geq 2$ (not $n \geq 1$, and we will see why soon). Our first conjecture would probably be then that indeed each $Q(n,t)$ is an integer. But we will have to see a little more before we can prove anything.

Take $n = 5$. Then we can make the following table:

| $k$ | $p_5(k)$ | $kp_5(k)$ | $k^2p_5(k)$ | $k^3p_5(k)$ | $k^4p_5(k)$ | $k^5p_5(k)$ | $k^6p_5(k)$ |
|---|---|---|---|---|---|---|---|
| 0 | 44 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 45 | 45 | 45 | 45 | 45 | 45 | 45 |
| 2 | 20 | 40 | 80 | 160 | 320 | 640 | 1280 |
| 3 | 10 | 30 | 90 | 270 | 810 | 2430 | 7290 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 1 | 5 | 25 | 125 | 625 | 3125 | 15625 |
| $\sum$ | 120 | 120 | 240 | 600 | 1800 | 6240 | 24240 |
| $\sum/5!$ | 1 | 1 | 2 | 5 | 15 | 52 | 202 |

Table 1.

Thus, $Q(5,0) = 1$, $Q(5,1) = 1$, $Q(5,2) = 2$, $Q(5,3) = 5$, and so on. So far so good. We can make a second table with the actual $Q(n,t)$ values:

| $n\backslash t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 2 | 4 | 8 | 16 | 32 |
| 3 | 1 | 1 | 2 | 5 | 14 | 41 | 122 |
| 4 | 1 | 1 | 2 | 5 | 15 | 51 | 187 |
| 5 | 1 | 1 | 2 | 5 | 15 | 52 | 202 |
| 6 | 1 | 1 | 2 | 5 | 15 | 52 | 203 |

Table 2.

Now we are getting somewhere. Notice how the rows seem to converge to a single sequence of integers, with one new term kicking in with each row. This sequence begins $1, 1, 2, 5, 15, 52, 203, \ldots$. I could not find this sequence in any of my references, so I did what any enterprising student would do. I sent an e-mail to `sequences@research.att.com`, with "`lookup 1 1 2 5 15 52 203`" in the body. For those not familiar, it is an on-line sequence server that tries to solve or match any sequence you might send it; I should also add that they ask that you send at most one request per hour. Soon enough, I had a response, which indicated that this was a sequence known as the Bell numbers, which satisfy

$$B(0) = 1, \quad B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k).$$

This recursion is striking, because if one looks at the second row of table 2, it may remind one of the identity $2^n = \sum_{k=0}^{n} \binom{n}{k}$. Could it be? Yes, in fact the same recursion that generates the Bell numbers is what generates successive rows of table 2. Now we really have something.

**Claim.** For all $n \geq 1$ and $t \geq 0$, $Q(n+1, t+1) = \sum_{i=0}^{t} \binom{t}{i} Q(n,i)$.

**Proof.** First, note that $p_n(k) = \binom{n}{k} p_{n-k}(0)$ [why?]. Then

$$\begin{aligned} Q(n+1,t+1) &= \frac{1}{(n+1)!} \sum_{k=0}^{n+1} k^{t+1} p_{n+1}(k) \\ &= \frac{1}{(n+1)!} \sum_{k=0}^{n+1} k^{t+1} \binom{n+1}{k} p_{n+1-k}(0) \end{aligned}$$

$$
\begin{aligned}
&= \frac{n+1}{(n+1)!}\sum_{k=1}^{n+1}\frac{n!}{(k-1)!(n+1-k)!}\frac{k^{t+1}}{k}p_{n+1-k}(0) \\
&= \frac{1}{n!}\sum_{k=1}^{n+1}k^t\binom{n}{k-1}p_{n+1-k}(0) \\
&= \frac{1}{n!}\sum_{k=1}^{n+1}k^t p_n(k-1) \\
&= \frac{1}{n!}\sum_{k=0}^{n}(k+1)^t p_n(k) \\
&= \frac{1}{n!}\sum_{k=1}^{n+1}\sum_{i=0}^{t}\binom{t}{i}k^i p_n(k) \\
&= \sum_{i=0}^{t}\binom{t}{i}\left(\frac{1}{n!}\sum_{k=0}^{n}k^i p_n(k)\right) \\
&= \sum_{i=0}^{t}\binom{t}{i}Q(n,i).
\end{aligned}
$$

So we have proven quite a bit actually, including:

1. Each $Q(n,t)$ is an integer (i.e., $\sum_{k=0}^{n} k^t p_n(k)$ is divisible by $n!$), and
2. For fixed $t$, $Q(n,t)$ eventually becomes $B(t)$ for sufficiently high $n$.

The claim looks complicated, but we know what we want to prove, and it turns out to be just a little algebraic manipulation. So in the end, we have a nice result from a simple observation.

# IMO CORRESPONDENCE PROGRAM

Canadian students wishing to participate in this program should first contact Professor Edward J. Barbeau, Department of Mathematics, University of Toronto, Toronto, Ontario. Please note that there is a fee for participation in the program: \$12. Please make the cheque payable to Edward J. Barbeau.

## PROBLEM SET 1

### *Algebra*

1. Solve the system of equations

$$x^2 + 2yz = x,$$
$$y^2 + 2xz = z,$$
$$z^2 + 2xy = y.$$

2. Let $m$ be a real number. Solve, for $x$, the equation

$$|x^2 - 1| + |x^2 - 4| = mx.$$

3. Let $\{x_1, x_2, \ldots, x_n, \ldots\}$ be a sequence of nonzero real numbers. Show that the sequence is an arithmetic progression if and only if, for each integer $n \geq 2$,

$$\frac{1}{x_1x_2} + \frac{1}{x_2x_3} + \cdots + \frac{1}{x_{n-1}x_n} = \frac{n-1}{x_1x_n}.$$

4. Suppose that $x$ and $y$ are two unequal positive real numbers. Let

$$r = \left(\frac{x^2 + y^2}{2}\right)^{1/2} \qquad g = (xy)^{1/2}$$
$$a = \frac{x+y}{2} \qquad h = \frac{2xy}{x+y}\ .$$

Which of the numbers $r - a$, $a - g$, $g - h$ is largest and which is smallest?

5. Simplify

$$\frac{x^3 - 3x + (x^2 - 1)\sqrt{x^2 - 4} - 2}{x^3 - 3x + (x^2 - 1)\sqrt{x^2 - 4} + 2}$$

to a fraction whose numerator and denominator are of the form $u\sqrt{v}$ with $u$ and $v$ each linear polynomials. For which values of $x$ is the equation valid?

6. Prove or disprove: if $x$ and $y$ are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

7. $X$ is a collection of objects upon which the operation of addition, subtraction and multiplication are defined so as to satisfy the following axioms:

   (1) if $x, y$ belong to $X$, then $x+y$ and $xy$ both belong to $X$;

   (2) for all $x, y$ in $X$, $x+y = y+x$;

   (3) for all $x, y, z$ in $X$, $x+(y+z) = (x+y)+z$ and $x(yz) = (xy)z$;

   (4) for all $x, y, z$ in $x$, $x(y+z) = xy + xz$;

   (5) there is an element $0$ such that $0+x = x+0 = x$ and for each $x$ in $X$, there exists a unique element denoted by $-x$ for which $x + (-x) = 0$;

   (6) $x - y = x + (-y)$ for each pair $x, y$ of elements of $X$;

   (7) $x^3 - x = x + x + x = 0$ for $x$ in $X$.

   Note that these axioms do not rule out the possibility that the product of two non-zero elements of $X$ may be zero, and so it may not be valid to cancel terms.

   On $X$, we define a relation $\leq$ by the following condition:

   $$x \leq y \text{ if and only if } x^2y - xy^2 - xy + x^2 = 0.$$

   Prove that the following properties obtain:

   (i) $x \leq x$ for each element $x$ of $X$;

   (ii) if $x \leq y$ and $y \leq x$, then $x = y$;

   (iii) if $x \leq y$ and $y \leq z$, then $x \leq z$.

8. Let $n$ be a positive integer and suppose that $u$ and $v$ are positive real numbers. Determine necessary and sufficient conditions on $u$ and $v$ such that there exist real numbers $a_1, a_2, \ldots, a_n$ satisfying

   $$a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$$

   $$u = a_1 + a_2 + \cdots + a_n$$

   $$v = a_1^2 + a_2^2 + \cdots + a_n^2.$$

   When such a representation is possible, determine the maximum and minimum values of $a_1$.

9. Suppose that $x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x} = t$, where $x, y, z$ are not all equal. Determine $xyz$.

10. Let $a \geq 0$. The polynomial $x^3 - ax + 1$ has three distinct real roots. For which values of $a$ does the root $u$ of least absolute value satisfy $\frac{1}{a} < u < \frac{2}{a}$?

11. Determine the range of values of $cd$ subject to the constraints $ab = 1$, $ac + bd = 2$, where $a$, $b$, $c$, $d$ are real.

12. Find polynomials $p(x)$ and $q(x)$ with integer coefficients such that

$$\frac{p(\sqrt{2}+\sqrt{3}+\sqrt{5})}{q(\sqrt{2}+\sqrt{3}+\sqrt{5})} = \sqrt{2}+\sqrt{3}.$$

# Mayhem Problems

The Mayhem Problems editors are:

| | |
|---|---|
| ***Cyrus Hsia*** | *Mayhem Advanced Problems Editor,* |
| ***Richard Hoshino*** | *Mayhem High School Problems Editor,* |
| ***Ravi Vakil*** | *Mayhem Challenge Board Problems Editor.* |

Note that all correspondence should be sent to the appropriate editor — see the relevant section. In this issue, you will find only problems — the next issue will feature only solutions.

We warmly welcome proposals for problems and solutions. With the new schedule of eight issues per year, we request that solutions to the new problems in this issue be submitted by 1 August 1997, for publication in the issue 5 months ahead; that is, issue 8. We also request that **only students** submit solutions (see editorial [1997: 30]), but we will consider particularly elegant or insightful solutions from others. Since this rule is only being implemented now, you will see solutions from many people in the next few months, as we clear out the old problems from Mayhem.

# High School Problems

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There is a correction for **H220**; the expression $2n \times \frac{T}{S}$ should be $2^n \times \frac{T}{S}$.

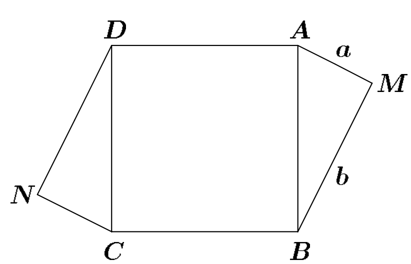
**H221**. Let $P = 19^5 + 660^5 + 1316^5$. It is known that $25$ is one of the forty-eight positive divisors of $P$. Determine the largest divisor of $P$ that is less than $10,000$.

**H222**. McGregor becomes very bored one day and decides to write down a three digit number $ABC$, and the six permutations of its digits. To his surprise, he finds that $ABC$ is divisible by $2$, $ACB$ is divisible by $3$, $BAC$ is divisible by $4$, $BCA$ is divisible by $5$, $CAB$ is divisible by $6$, and $CBA$ is a divisor of $1995$. Determine $ABC$.

**H223**. There are $n$ black marbles and two red marbles in a jar. One by one, marbles are drawn at random out of the jar. Jeanette wins as soon as two black marbles are drawn, and Fraserette wins as soon as two red marbles are drawn. The game continues until one of the two wins. Let $J(n)$ and $F(n)$ be the two probabilities that Jeanette and Fraserette win, respectively.

1. Determine the value of $F(1) + F(2) + \cdots + F(3992)$.
2. As $n$ approaches infinity, what does $J(2) \times J(3) \times J(4) \times \cdots \times J(n)$ approach?

**H224**. Consider square $ABCD$ with side length $1$. Select a point $M$ exterior to the square so that $\angle AMB$ is $90°$. Let $a = AM$ and $b = BM$. Now, determine the point $N$ exterior to the square so that $CN = a$ and $DN = b$. Find, as a function of $a$ and $b$, the length of line segment $MN$.



# Advanced Problems

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**A197**. Calculate

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin(2N+1)\theta}{\sin\theta}\,d\theta,$$

where $N$ is a non-negative integer.

**A198**. Given positive real numbers $a$, $b$, and $c$ such that $a+b+c=1$, show that $a^ab^bc^c + a^bb^cc^a + a^cb^ac^b \leq 1$.

**A199**. Let $P$ be a point inside triangle $ABC$. Let $A'$, $B'$, and $C'$ be the reflections of $P$ through the sides $BC$, $AC$, and $AB$ respectively. For what points $P$ are the six points $A$, $B$, $C$, $A'$, $B'$, and $C'$ concyclic?

**A200**. Given positive integers $n$ and $k$, for $0 \leq i \leq k-1$, let

$$S_{n,k,i} = \sum_{j \equiv i \pmod{k}} \binom{n}{j}.$$

Do there exist positive integers $n$, $k > 2$, such that $S_{n,k,0}$, $S_{n,k,1}$, ..., $S_{n,k,k-1}$ are all equal?

# Challenge Board Problems

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There are no new Challenge Board Problems this month — we reprint those from issue 1 this year [1997: 44]'

**C70**. Prove that the group of automorphisms of the dodecahedron is $S_5$, the symmetric group on five letters, and that the rotation group of the dodecahedron (the subgroup of automorphisms preserving orientation) is $A_5$.

**C71**. Let $L_1$, $L_2$, $L_3$, $L_4$ be four general lines in the plane. Let $p_{ij}$ be the intersection of lines $L_i$ and $L_j$. Prove that the circumcircles of the four triangles $p_{12}p_{23}p_{31}$, $p_{23}p_{34}p_{42}$, $p_{34}p_{41}p_{13}$, $p_{41}p_{12}p_{24}$ are concurrent.

**C72**. A finite group $G$ acts on a finite set $X$ transitively. (In other words, for any $x, y \in X$, there is a $g \in G$ with $g \cdot x = y$.) Prove that there is an element of $G$ whose action on $X$ has no fixed points.