

# THE OLYMPIAD CORNER

No. 181

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We begin this number with two contests. Thanks go to Richard Nowakowski, Canadian Team Leader to the 35th IMO in Hong Kong, for collecting them and forwarding them to us.

## SELECTED PROBLEMS FROM THE ISRAEL MATHEMATICAL OLYMPIADS, 1994

**1.**  $p$  and  $q$  are positive integers.  $f$  is a function defined for positive numbers and attains only positive values, such that  $f(xf(y)) = x^p y^q$ . Prove that  $q = p^2$ .

**2.** The sides of a polygon with 1994 sides are  $a_i = \sqrt{4+i^2}$ ,  $i = 1, 2, \dots, 1994$ . Prove that its vertices are not all on integer mesh points.

**3.** A “standard triangle” in the plane is a (filled) isosceles right triangle whose sides are parallel to the  $x$  and  $y$  axes. A finite family of standard triangles, containing at least three, is given. Every three of this family have a common point. Prove that there is a point common to all triangles in that family.

**4.** A shape  $c'$  is called “a copy of the planar shape  $c$ ” if the following conditions hold:

(i) There are two planes  $\sigma$  and  $\sigma'$  and a point  $P$  that does not belong to either of them.

(ii)  $c \in \sigma$  and  $c' \in \sigma'$ .

(iii) A point  $X'$  satisfies  $X' \in c'$  iff  $X'$  is the intersection of  $\sigma'$  with the line passing through  $X$  and  $P$ .

Given a planar trapezoid, prove that there is a square which is a copy of this trapezoid.

**5.** Find all polynomials  $p(x)$ , with real coefficients, satisfying

$$(x-1)^2 p(x) = (x-3)^2 p(x+2)$$

for all  $x$ .



## PROBLEMS FROM THE BI-NATIONAL ISRAEL-HUNGARY COMPETITION, 1994

**1.**  $a_1, \dots, a_k, a_{k+1}, \dots, a_n$  are positive numbers ( $k < n$ ). Suppose that the values of  $a_{k+1}, \dots, a_n$  are fixed. How should one choose the values of  $a_1, \dots, a_k$  in order to minimize  $\sum_{i,j,i \neq j} \frac{a_i}{a_j}$ ?

**2.** Three given circles pass through a common point  $P$  and have the same radius. Their other points of pairwise intersections are  $A, B, C$ . The 3 circles are contained in the triangle  $A'B'C'$  in such a way that each side of  $\triangle A'B'C'$  is tangent to two of the circles. Prove that the area of  $\triangle A'B'C'$  is at least 9 times the area of  $\triangle ABC$ .

**3.**  $m, n$  are two different natural numbers. Show that there exists a real number  $x$ , such that  $\frac{1}{3} \leq \{xn\} \leq \frac{2}{3}$  and  $\frac{1}{3} \leq \{xm\} \leq \frac{2}{3}$ , where  $\{a\}$  is the fractional part of  $a$ .

**4.** An “ $n$ - $m$  society” is a group of  $n$  girls and  $m$  boys. Show that there exist numbers  $n_0$  and  $m_0$  such that every  $n_0$ - $m_0$  society contains a subgroup of five boys and five girls in which all of the boys know all of the girls or none of the boys knows none of the girls.

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Last issue we gave five more Klamkin Quickies. Next we give his “Quickie” solutions to these problems. Many thanks to Murray S. Klamkin, the University of Alberta, for creating the problems and solutions.

### ANOTHER FIVE KLAMKIN QUICKIES

October 21, 1996

**6.** Determine the four roots of the equation  $x^4 + 16x - 12 = 0$ .

*Solution.* Since

$$x^4 + 16x - 12 = (x^2 + 2)^2 - 4(x - 2)^2 = (x^2 + 2x - 2)(x^2 - 2x + 6) = 0,$$

the four roots are  $-1 \pm \sqrt{3}$  and  $1 \pm i\sqrt{5}$ .

**7.** Prove that the smallest regular  $n$ -gon which can be inscribed in a given regular  $n$ -gon is one whose vertices are the midpoints of the sides of the given regular  $n$ -gon.

*Solution.* The circumcircle of the inscribed regular  $n$ -gon must intersect each side of the given regular  $n$ -gon. The smallest that such a circle can be is the inscribed circle of the given  $n$ -gon, and it touches each of its sides at its midpoints.

**8.** If  $31^{1995}$  divides  $a^2 + b^2$ , prove that  $31^{1996}$  divides  $ab$ .

*Solution.* If one calculates  $1^2, 2^2, \dots, 30^2 \pmod{31}$  one finds that the sum of no two of these equals  $0 \pmod{31}$ . Hence,  $a = 31a_1$  and  $b = 31b_1$  so that  $31^{1993}$  divides  $a_1^2 + b_1^2$ . Then,  $a_1 = 31a_2$  and  $b_1 = 31b_2$ . Continuing in this fashion (with  $p = 31$ ), we must have  $a = p^{998}m$  and  $b = p^{998}n$  so that  $ab$  is divisible by  $p^{1996}$ .

More generally, if a prime  $p = 4k + 3$  divides  $a^2 + b^2$ , then both  $a$  and  $b$  must be divisible by  $p$ . This follows from the result that “a natural  $n$  is the sum of squares of two relatively prime natural numbers if and only if  $n$  is divisible neither by 4 nor by a natural number of the form  $4k + 3$ ” (see J.W. Sierpiński, *Elementary Theory of Numbers*, Hafner, NY, 1964, p. 170).

**9.** Determine the minimum value of

$$S = \sqrt{(a+1)^2 + 2(b-2)^2 + (c+3)^2} + \sqrt{(b+1)^2 + 2(c-2)^2 + (d+3)^2} \\ + \sqrt{(c+1)^2 + 2(d-2)^2 + (a+3)^2} + \sqrt{(d+1)^2 + 2(a-2)^2 + (b+3)^2}$$

where  $a, b, c, d$  are any real numbers.

*Solution.* Applying Minkowski's inequality,

$$S \geq \sqrt{(4+s)^2 + 2(s-8)^2 + (s+12)^2} = \sqrt{4s^2 + 288}$$

where  $s = a + b + c + d$ . Consequently,  $\min S = 12\sqrt{2}$  and is taken on for  $a = b = c = d = 0$ .

**10.** A set of 500 real numbers is such that any number in the set is greater than one-fifth the sum of all the other numbers in the set. Determine the least number of negative numbers in the set.

*Solution.* Letting  $a_1, a_2, a_3, \dots$  denote the numbers of the set and  $S$  the sum of all the numbers in the set, we have

$$a_1 > \frac{S - a_1}{5}, \quad a_2 > \frac{S - a_2}{5}, \quad \dots, \quad a_6 > \frac{S - a_6}{5}.$$

Adding, we get  $0 > S - a_1 - a_2 - \dots - a_6$  so that if there were six or less negative numbers in the set, the right hand side of the inequality could be positive. Hence, there must be at least seven negative numbers.

**Comment.** This problem where the “5” is replaced by “1” is due to Mark Kantrowitz, Carnegie–Mellon University.



First a solution to one of the 36th IMO problems:

**2.** [1995: 269] 36th IMO

Let  $a$ ,  $b$ , and  $c$  be positive real numbers such that  $abc = 1$ . Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

*Solution by Panos E. Tsaoussoglou, Athens, Greece.*

By the Cauchy-Schwartz inequality

$$\begin{aligned} [a(b+c) + b(c+a) + c(a+b)] \left[ \frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \right] \\ \geq \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)^2, \end{aligned}$$

or

$$\begin{aligned} 2(ab+ac+bc) \left[ \frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \right] \\ \geq \frac{(ab+ac+bc)^2}{(abc)^2}, \end{aligned}$$

or

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{ab+ac+bc}{2},$$

because  $abc = 1$ .

Also

$$\frac{ab+ac+bc}{3} \geq \sqrt[3]{a^2b^2c^2} = 1.$$

Therefore

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

holds.

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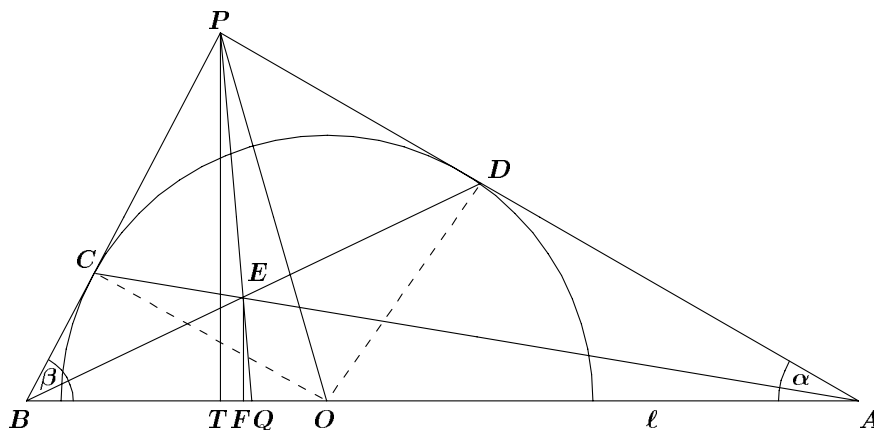
Now we turn to some of the readers' solutions to problems proposed to the jury but not used at the 35th IMO in Hong Kong [1995: 299–300].

## PROBLEMS PROPOSED BUT NOT USED AT THE 35th IMO IN HONG KONG

### Selected Problems

**3.** A semicircle  $\Gamma$  is drawn on one side of a straight line  $\ell$ .  $C$  and  $D$  are points on  $\Gamma$ . The tangents to  $\Gamma$  at  $C$  and  $D$  meet  $\ell$  at  $B$  and  $A$  respectively, with the center of the semicircle between them. Let  $E$  be the point of intersection of  $AC$  and  $BD$ , and  $F$  be the point on  $\ell$  such that  $EF$  is perpendicular to  $\ell$ . Prove that  $EF$  bisects  $\angle CFD$ .

*Solutions by Toshio Seimiya, Kawasaki, Japan; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give Seimiya's write-up.*



Let  $P$  be the intersection of  $AD$  and  $BC$ . Then  $\angle PCO = \angle PDO = 90^\circ$ ,  $\angle CPO = \angle DPO$  and  $PC = PD$ . Let  $Q$  be the intersection of  $PE$  with  $AB$ . Then by Ceva's Theorem, we get

$$\frac{BQ}{QA} \cdot \frac{AD}{DP} \cdot \frac{PC}{CB} = \frac{BQ}{QA} \cdot \frac{AD}{CB} = 1.$$

Thus we get

$$\frac{BQ}{QA} = \frac{BC}{AD}. \quad (1)$$

Since  $\angle BPO = \angle APO$  we get

$$\frac{PB}{PA} = \frac{BO}{AO}. \quad (2)$$

We put  $\angle PAB = \alpha$ ,  $\angle PBA = \beta$ .

Let  $T$  be the foot of the perpendicular from  $P$  to  $AB$ . Then from (1) and (2) we have

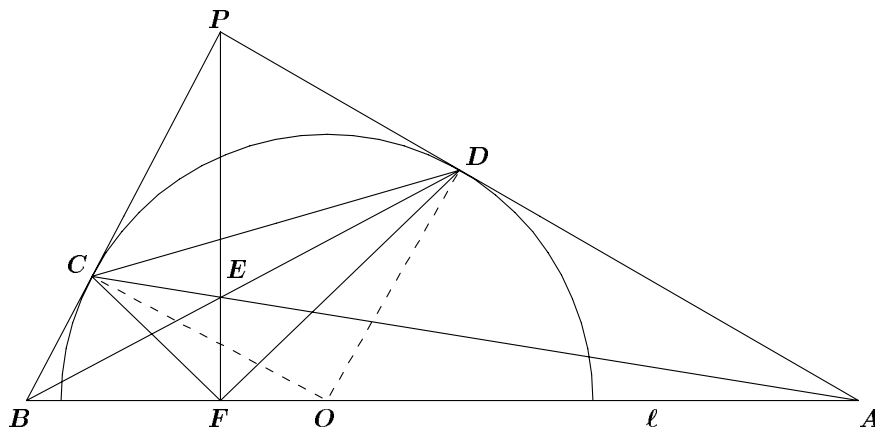
$$\frac{BC}{AD} = \frac{BO \cos \beta}{AO \cos \alpha} = \frac{PB \cos \beta}{PA \cos \alpha} = \frac{PT}{TA}. \quad (3)$$

From (1) and (3) we have

$$\frac{BQ}{QA} = \frac{PT}{TA}.$$

Hence  $Q$  coincides with  $T$  so that  $P, E, F$  are collinear. [See page 136.]

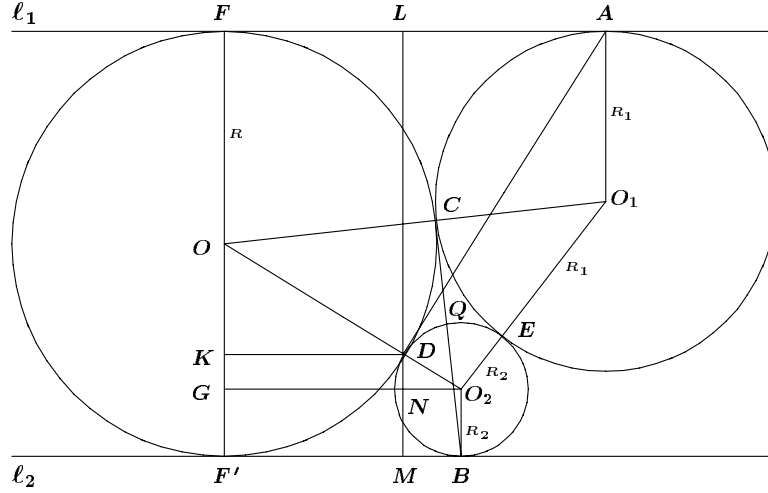
Because  $\angle PCO = \angle PDO = \angle PFO = 90^\circ$ ,  $P, C, F, O, D$  are concyclic. Hence  $\angle CFE = \angle CFP = \angle CDP = \angle DCP = \angle DFP = \angle DFE$ . Thus  $EF$  bisects  $\angle CFD$ .



**4.** A circle  $\omega$  is tangent to two parallel lines  $\ell_1$  and  $\ell_2$ . A second circle  $\omega_1$  is tangent to  $\ell_1$  at  $A$  and to  $\omega$  externally at  $C$ . A third circle  $\omega_2$  is tangent to  $\ell_2$  at  $B$ , to  $\omega$  externally at  $D$  and to  $\omega_1$  externally at  $E$ .  $AD$  intersects  $BC$  at  $Q$ . Prove that  $Q$  is the circumcentre of triangle  $CDE$ .

*Solutions by Toshio Seimiya, Kawasaki, Japan; and by D.J. Smeenk, Zaltbommel, the Netherlands. We give Smeenk's solution.*

We denote the three circles as  $\omega(O, R)$ ,  $\omega_1(O_1, R_1)$ ,  $\omega_2(O_2, R_2)$ . Now let  $\omega$  touch  $\ell_1$  at  $F$  and  $\ell_2$  at  $F'$ . Let the line through  $O_2$  parallel to  $\ell_1$  intersect  $FF'$  at  $G$  and the production of  $AO_1$  at  $H$ .



Let the line through  $D$  parallel to  $\ell_1$  intersect  $FF'$  at  $K$ .

Let the line through  $D$  parallel to  $FF'$  intersect  $\ell_1$  at  $L$ ,  $\ell_2$  at  $M$  and  $GO_2$  at  $N$ . Now  $AF$  is a common tangent of  $\omega$  and  $\omega_1$ , so

$$AF = 2\sqrt{RR_1} \quad (1)$$

and

$$BF' = 2\sqrt{RR_2} = GO_2. \quad (2)$$

It follows that

$$HO_2 = |2\sqrt{RR_2} - 2\sqrt{RR_1}|;$$

$$HO_1 = 2R - R_1 - R_2.$$

In right triangle  $O_1HO_2$ ,

$$(2\sqrt{RR_2} - 2\sqrt{RR_1})^2 + (2R - R_1 - R_2)^2 = (R_1 + R_2)^2.$$

After some reduction  $R = 2\sqrt{R_1R_2}$ .

Next consider triangle  $GOO_2$ .

$$GO = R - R_2, \quad GO_2 = 2\sqrt{RR_2}, \quad DO_2 = R_2, \quad DO = R, \quad KD \parallel GO_2.$$

$$\text{We find that } GN = FL = \frac{R}{R + R_2} \cdot GO_2 = \frac{2R\sqrt{RR_2}}{R + R_2}.$$

With (1) we have

$$AL = 2\sqrt{RR_1} - \frac{2R\sqrt{RR_2}}{R + R_2}. \quad (3)$$

Furthermore  $DN = \frac{R_2}{R + R_2} \cdot GO = \frac{R_2(R - R_2)}{R + R_2}$  and

$$DL = 2R - R_2 - \frac{R_2(R - R_2)}{R + R_2} = \frac{2R^2}{R + R_2}. \quad (4)$$

Now  $AD^2 = AL^2 + DL^2$ . With (3) and (4),

$$\left(2\sqrt{RR_1} - \frac{2R\sqrt{RR_2}}{R_1 + R_2}\right)^2 + \left(\frac{2R^2}{R_1 + R_2}\right)^2 = 4RR_1 = AE^2.$$

So  $AD = AE$ .

That means that  $AD$  touches  $\omega$  at  $D$  and  $AD$  is a common tangent and the radical axis of  $\omega$  and  $\omega_2$ .

In the same way  $BC$  is the radical axis of  $\omega$  and  $\omega_1$  and  $Q$  is the radical point of  $\omega$ ,  $\omega_1$  and  $\omega_2$ .

So  $QC = QD = QE$ , as required.

**5.** A line  $\ell$  does not meet a circle  $\omega$  with center  $O$ .  $E$  is the point on  $\ell$  such that  $OE$  is perpendicular to  $\ell$ .  $M$  is any point on  $\ell$  other than  $E$ . The tangents from  $M$  to  $\omega$  touch it at  $A$  and  $B$ .  $C$  is the point on  $MA$  such that  $EC$  is perpendicular to  $MA$ .  $D$  is the point on  $MB$  such that  $ED$  is perpendicular to  $MB$ . The line  $CD$  cuts  $OE$  at  $F$ . Prove that the location of  $F$  is independent of that of  $M$ .

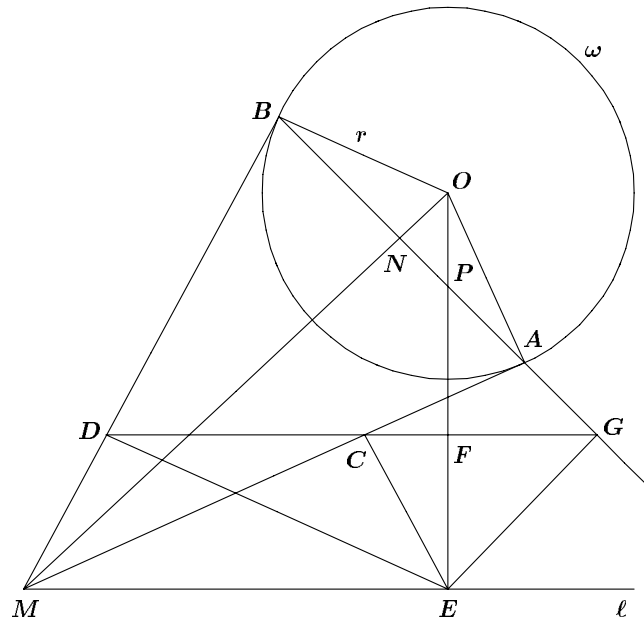
*Solution by Toshio Seimiya, Kawasaki, Japan.*

As  $MA$ ,  $MB$  are tangent to  $\omega$  at  $A$ ,  $B$  respectively, we get  $\angle OAM = \angle OBM = 90^\circ$  and  $OM \perp AB$ . Let  $N$ ,  $P$  be the intersections of  $AB$  with  $OM$  and  $OE$  respectively.

Since  $M$ ,  $E$ ,  $P$ ,  $N$  lie on the circle with diameter  $MP$  we get  $ON \cdot OM = OB^2 = r^2$  where  $r$  is the radius of  $\omega$ . Hence  $P$  is a fixed point. ( $P$  is the pole of  $\ell$ .)

Let  $G$  be the foot of the perpendicular from  $E$  to  $AB$ . As  $\angle OBM = \angle OAM = \angle OEM = 90^\circ$ ,  $O$ ,  $B$ ,  $M$ ,  $E$ ,  $A$  are concyclic, so that by Simson's Theorem  $C$ ,  $D$ ,  $G$  are collinear.





Since  $A, C, E, G$  lie on the circle with diameter  $AE$  we get

$$\angle EGF = \angle EGC = \angle EAC = \angle EAM. \quad (1)$$

As  $O, M, E, A$  are concyclic and  $OM$  is parallel to  $EG$  we have

$$\angle EAM = \angle EDM = \angle DEG = \angle FEG. \quad (2)$$

From (1) and (2) we get

$$\angle EGF = \angle FEG. \quad (3)$$

Since  $\angle EGP = 90^\circ$  we get

$$\angle FGP = \angle FPG. \quad (4)$$

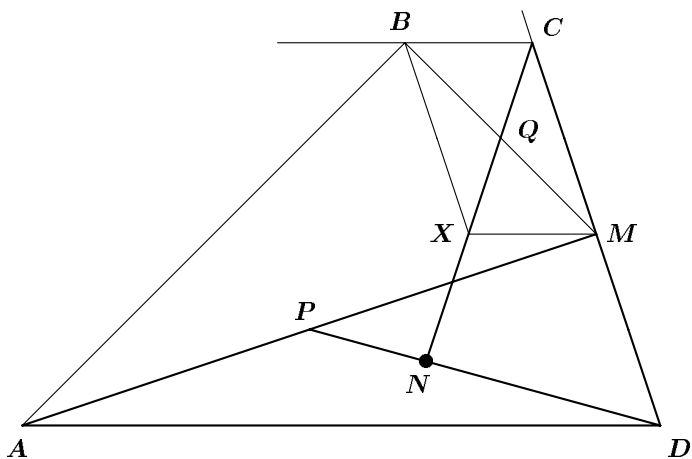
From (3) and (4) we have  $EF = FG = FP$ . Thus  $F$  is the midpoint of  $EP$ . Hence  $F$  is a fixed point.

Next, we give a counterexample to the first problem of the set of problems proposed to the jury, but not used at the 35th IMO in Hong Kong given in the December 1995 number of the corner.

**1.** [1995: 334] *Problems proposed but not used at the 35th IMO in Hong Kong.*

$ABCD$  is a quadrilateral with  $BC$  parallel to  $AD$ .  $M$  is the midpoint of  $CD$ ,  $P$  that of  $MA$  and  $Q$  that of  $MB$ . The lines  $DP$  and  $CQ$  meet at  $N$ . Prove that  $N$  is not outside triangle  $ABM$ .

*Counterexamples by Joanna Jaszńska, student, Warsaw, Poland; and by Toshio Seimiya, Kawasaki, Japan. We give Jaszńska's example.*



We draw a triangle  $ADM$  and denote the midpoint of  $MA$  by  $P$ . Let  $C$  be a point on the half-line  $DM$  such that  $M$  is the midpoint of  $CD$ .

Let  $N$  be any point of the segment  $PD$ , *inside* triangle  $ADM$ .

We construct a parallelogram  $MCBX$  such that  $MX$  and  $BC$  are parallel to  $AD$  and  $X$  lies on the segment  $CN$ .

Let us denote the point where the diagonal  $MB$  of this quadrilateral meets  $CN$  by  $Q$ .  $Q$  is then the midpoint of  $MB$ .

Connect points  $A$  and  $B$ . We have thus constructed a quadrilateral  $ABCD$  with  $BC$  parallel to  $AD$ ,  $M$  is the midpoint of  $CD$ ,  $P$  that of  $MA$  and  $Q$  that of  $MB$ . Lines  $DP$  and  $CQ$  meet at  $N$ .

$N$  is inside triangle  $ADM$ ; hence it is outside triangle  $ABM$ .



Next we look back to some further solutions to problems of the Sixth Irish Mathematical Olympiad given in [1995: 151–152] and for which some solutions were given in [1997: 9–13]. An envelope from Michael Selby arrived which I misfiled. It contains solutions to problems 1, 2, and 4 of Day 1, and problems 1, 2, 3 and 4 of Day 2.

**1.** [1995: 152] *Second Paper, Sixth Irish Mathematical Olympiad.*

Given five points  $P_1, P_2, P_3, P_4, P_5$  in the plane having integer coordinates, prove that there is at least one pair  $(P_i, P_j)$  with  $i \neq j$  such that the line  $P_iP_j$  contains a point  $Q$  having integer coordinates and lying strictly between  $P_i$  and  $P_j$ .

*Solution by Michael Selby, University of Windsor, Windsor, Ontario.*

The points can be characterized according to the parity of their  $x$  and  $y$  coordinates. There are only four such classes: (even, even), (even, odd), (odd, even), (odd, odd).

Since we are given five such points, at least two must have the same parity of coordinates by the Pigeonhole Principle. Suppose they are  $P_i$  and  $P_j$ ,  $P_i = (x_i, y_i)$ ,  $P_j = (x_j, y_j)$ . Then  $x_i + x_j$  is even and  $y_i + y_j$  is even, since the  $x_i, x_j$  have the same parity and  $y_i, y_j$  have the same parity. Hence the midpoint

$$Q = \left( \frac{x_i + x_j}{2}, \frac{y_i + y_j}{2} \right)$$

has integral coordinates.

**2.** [1995: 152] *Second Paper, Sixth Irish Mathematical Olympiad.*

Let  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$  be  $2n$  real numbers, where  $a_1, a_2, \dots, a_n$  are distinct, and suppose that there exists a real number  $\alpha$  such that the product

$$(a_i + b_1)(a_i + b_2) \dots (a_i + b_n)$$

has the value  $\alpha$  for all  $i$  ( $i = 1, 2, \dots, n$ ). Prove that there exists a real number  $\beta$  such that the product

$$(a_1 + b_j)(a_2 + b_j) \dots (a_n + b_j)$$

has the value  $\beta$  for all  $j$  ( $j = 1, 2, \dots, n$ ).

*Solution by Michael Selby, University of Windsor, Windsor, Ontario.*

Define

$$P_n(x) = (x + b_1)(x + b_2) \dots (x + b_n) - \alpha. \quad (1)$$

Then  $P_n(a_i) = 0$  for  $i = 1, 2, \dots, n$ .

Therefore  $P_n(x) = (x - a_1)(x - a_2) \dots (x - a_n)$  by the Factor Theorem.

Now  $(-1)^n P_n(-x) = (x + a_1)(x + a_2) \cdots (x + a_n)$ . So

$$\begin{aligned} (-1)^n P_n(-b_i) &= (b_i + a_1)(b_i + a_2) \cdots (b_i + a_n) \\ &= (-1)^{n+1} \alpha \quad \text{by (1)}. \end{aligned}$$

Hence  $(b_i + a_1)(b_i + a_2) \cdots (b_i + a_n) = (-1)^{n+1} \alpha$  for  $i = 1, 2, \dots, n$ .

Thus, the result is true with  $\beta = (-1)^{n+1} \alpha$ .

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That completes the Corner for this number. We are in high Olympiad season. Send me your nice solutions and contests.

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## Do you believe what occurs in print?

The last sentence of the quoted passage, taken from *The Daughters of Cain* by Colin Dexter (Macmillan, 1994), contains two factual errors. What are they?

*‘Have you heard of “Pythagorean Triplets”?’*

*‘We did Pythagoras Theorem at school.’*

*‘Exactly. The most famous of all the triplets, that is —*

*“3, 4, 5”  $3^2 + 4^2 = 5^2$ . Agreed?’*

*‘Agreed.’*

*‘But there are more spectacular examples than that.*

*The Egyptians, for example, knew all about “5961, 6480, 8161”.’*

Contributed by J.A. McCallum, Medicine Hat, Alberta.