THE ACADEMY CORNER

No. 10

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This month, we present a university entrance scholarship examination paper from the 1940's. Thanks to Georg Gunther, Sir Wilfred Grenfell College, Corner Brook, Newfoundland, for providing this. We challenge today's university students to solve these problems — send me your nice solutions!

1. Find all the square roots of

$$1 - x + \sqrt{22x - 15 - 8x^2}.$$

2. Find all the solutions of the system of equations:

$$\begin{aligned} x + y + z &= 2, \\ x^2 + y^2 + z^2 &= 14, \\ xyz &= -6. \end{aligned}$$

3. Suppose that n is a positive integer and that C_k is the coefficient of x^k in the expansion of $(1+x)^n$. Show that

$$\sum_{k=0}^{n} (k+1) C_k^2 = \frac{(n+2)\,(2n-1)!}{n!\,(n-1)!}.$$

4. (a) Suppose that $a \neq 0$ and $c \neq 0$, and that $ax^3 + bx + c$ has a factor of the form $x^2 + px + 1$. Show that $a^2 - c^2 = ab$.

   (b) In this case, prove that $ax^3 + bx + c$ and $cx^3 + bx^2 + a$ have a common quadratic factor.

5. Prove that all the circles in the family defined by the equation

$$x^2 + y^2 - a(t^2 + 2)x - 2aty - 3a^2 = 0$$

(a fixed, t variable) touch a fixed straight line.

6. Find the equation of the locus of a point P which moves so that the tangents from P to the circle $x^2 + y^2 = r^2$ cut off a line segment of length $2r$ on the line $x = r$.

7. If the tangents at A, B and C to the circumcircle of triangle $\triangle ABC$ meet the opposite sides at D, E and F, respectively, prove that D, E and F are collinear.

8. Find the locus of P which moves so that the polars of P, with respect to three non-intersecting circles, are concurrent.

9. Suppose that P is a point within the tetrahedron $OABC$. Prove that $\angle AOB + \angle BOC + \angle COA$ is less than $\angle APB + \angle BPC + \angle CPA$.

10. Two unequal circles of radii R and r touch externally, and P and Q are the points of contact of a common tangent to the circles, respectively. Find the volume of the frustum of a cone generated by rotating PQ about the line joining the centres of the circle.

11. Prove that

$$\sin^2(\theta+\alpha)+\sin^2(\theta+\beta)-2\cos(\alpha-\beta)\sin(\theta+\alpha)\sin(\theta+\beta) = \sin^2(\alpha-\beta).$$

12. Three points A, B and C are on level ground. B is east of A, C is N. 49° E. of A, and C is N. 11°30′ W. of B.

    Find the direction of C as seen from the mid-point of AB.

13. With each corner of a square of side r as a centre, four circles of radius r are drawn.

    Show that the area of the central curvilinear quadrilateral formed inside the square by the intersection of the four circles is

$$r^2\left(1-\sqrt{3}+\frac{\pi}{3}\right).$$

14. An observer on a boat is vertically beneath the centre of a bridge, which crosses a straight canal at right angles. Looking upwards, the observer sees that the angle subtended by the length of the bridge is 2α. The observer then rows a distance δ along the middle of the canal, and then finds that the length of the bridge now subtends an angle of 2β.

    Show that the length of the bridge is

$$\frac{2\delta}{\sqrt{\cot^2\beta-\cot^2\alpha}}.$$