MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

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Matrix Exponentials: An Introduction

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Let us start with the well-known power series for the exponential function,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

which works for all $x \in \mathbb{C}$. Likewise, we will define the exponential function for matrices (with complex entries) as

$$e^M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \frac{M^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{M^k}{k!}$$

for all $n \times n$ matrices M.

Matrix Exponentials Really *Do* Exist

First of all, we must show that this sum actually converges. Otherwise, the answer we get would be meaningless, and that would be bad. To start us off, we will define the **norm** of an $n \times n$ matrix, which we will denote $|A|$, as follows:

$$|A| = \max_{1 \le i \le n} \left(\sum_{j=1}^{n} |a_{ij}| \right).$$

That is, for each row, take the sum of the absolute values of the components in that row, and the norm will be the largest of these row sums.

Lemma 1 For $n \times n$ matrices A and B,

$$|AB| \leq |A||B|.$$

Proof We label the entries of A and B in the standard way, a_{ij} and b_{ij}, respectively.

For any row in A, define the row sum as $\sum_{j=1}^{n} |a_{ij}|$. Since $(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$, the row sum of row i in AB is

$$\begin{aligned}
\sum_{j=1}^{n} \left| \sum_{k=1}^{n} a_{ik}b_{kj} \right| &\leq \sum_{j=1}^{n}\sum_{k=1}^{n} |a_{ik}b_{kj}| = \sum_{k=1}^{n}\sum_{j=1}^{n} |a_{ik}|\,|b_{kj}| \\
&= \sum_{k=1}^{n} \left(|a_{ik}| \sum_{j=1}^{n} |b_{kj}| \right) \leq \sum_{k=1}^{n} (|a_{ik}|\,|B|) \\
&= |B| \sum_{k=1}^{n} |a_{ik}| \leq |A||B|.
\end{aligned}$$

Now, $|AB|$, the maximum row sum, is still a row sum, thus $|AB| \leq |A||B|$. ■

Lemma 2 For $n \times n$ matrices A and B,

$$|A + B| \leq |A| + |B|.$$

The proof is left as an exercise to the reader.

Theorem 1 The sum e^M converges for all $n \times n$ matrices M.

Proof Using the two lemmas, we get

$$e^M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \frac{M^4}{4!} + \cdots,$$

so that

$$\begin{aligned}
\left|e^M\right| &= \left|I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \frac{M^4}{4!} + \cdots\right| \\
&\leq 1 + |M| + \frac{|M|^2}{2!} + \frac{|M|^3}{3!} + \frac{|M|^4}{4!} + \cdots \\
&= e^{|M|}.
\end{aligned}$$

Since $|M|$ is a real number, $e^{|M|}$ is finite, and $|e^M|$ is bounded. Thus, e^M converges. ■

Exercises

1. Prove Lemma 2.

Some Linear Algebra Lingo

Recall that two $n \times n$ matrices A and B are **similar** if

$$B = C^{-1}AC$$

for some invertible $n \times n$ matrix C. (It is usually easier, when verifying that two matrices are similar, to show that $CB = AC$ instead.) Recall also that a **diagonal** matrix is a matrix with all its non-zero entries along the main diagonal. (But entries along the main diagonal are not necessarily non-zero). For example:

$$\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

If a matrix A is similar to a diagonal matrix, we say that A is **diagonalizable**.

Calculating Matrix Exponentials

Suppose D is a diagonal matrix, with diagonal elements λ_1, λ_2, $\ldots$, λ_n. We have

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}, \qquad D^k = \begin{pmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{pmatrix},$$

and

$$\begin{aligned} e^D &= I + D + \frac{D^2}{2!} + \cdots \\ &= \begin{pmatrix} 1 + \lambda_1 + \frac{\lambda_1^2}{2!} + \cdots & 0 & \cdots & 0 \\ 0 & 1 + \lambda_2 + \frac{\lambda_2^2}{2!} + \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 + \lambda_n + \frac{\lambda_n^2}{2!} + \cdots \end{pmatrix} \\ &= \begin{pmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \end{pmatrix}. \end{aligned}$$

Jordan Matrices

A **Jordan block matrix** is an $n \times n$ matrix of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

with λ's all down the main diagonal and 1's directly above them (except for the first column), and 0's everywhere else. Examples include

$$(5), \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}, \text{ and } \begin{pmatrix} -3 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

A **Jordan matrix** is an $n \times n$ matrix with Jordan blocks down the diagonal. An example containing our three examples above:

$$\begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}.$$

Notice that all diagonal matrices are also Jordan matrices.

Exercises

1. Verify that

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}^k$$

$$= \begin{pmatrix} \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \cdots & \binom{k}{n-1}\lambda^{k-n+1} & \binom{k}{n}\lambda^{k-n} \\ 0 & \lambda^k & \binom{k}{1}\lambda^{k-1} & \cdots & \binom{k}{n-2}\lambda^{k-n+2} & \binom{k}{n-1}\lambda^{k-n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda^k & \binom{k}{1}\lambda^{k-1} \\ 0 & 0 & 0 & \cdots & 0 & \lambda^k \end{pmatrix}.$$

2. Compute e^B when B is a Jordan block matrix.

3. Compute e^J when J is a Jordan matrix.

So why did we just go through all that?

Here's the reason:

Theorem 2 For any $n \times n$ matrix A, there exists a Jordan matrix J which is similar, that is, $C^{-1}AC = J$ for some C.

J is known as a **Jordan canonical form** of A. Some matrices have more than one Jordan canonical form.

The proof is beyond the scope of this article, but can be found in any good advanced linear algebra textbook. But we *will* prove the following:

Theorem 3 If J is a Jordan canonical form of $n \times n$ matrix A, with $A = CJC^{-1}$, then $e^A = C(e^J)C^{-1}$.

Proof

$$\begin{aligned}
A^k &= (CJC^{-1})^k \\
&= \underbrace{(CJC^{-1})(CJC^{-1})\cdots(CJC^{-1})}_{k} \\
&= CJ(C^{-1}C)J(C^{-1}C)J(C^{-1}C)\cdots(C^{-1}C)JC^{-1} \\
&= CJ^kC^{-1}. \\
\text{Now, } e^A &= I + A + \frac{A^2}{2!} + \cdots \\
&= CIC^{-1} + CJC^{-1} + C\frac{J^2}{2!}C^{-1} + \cdots \\
&= C\left(I + J + \frac{J^2}{2!} + \cdots\right)C^{-1} \\
&= C(e^J)C^{-1}.
\end{aligned}$$

Thus, $e^A = C(e^J)C^{-1}$. ∎

The process of finding a Jordan canonical form J of a matrix A is also beyond the scope of this article. Once again, I refer you to a good textbook. However, we have shown that it *is* possible to calculate e^A for any $n \times n$ matrix A.

And now for something less theoretical...

Here, we discuss an interesting application of matrix exponentials: first-order systems of linear differential equations. By that, we mean systems of the form:

$$\begin{aligned}
x_1'(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t), \\
x_2'(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t), \\
&\vdots \\
x_n'(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t).
\end{aligned}$$

Letting

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix},$$

we define the derivative $\mathbf{x}(t)$ as

$$\mathbf{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}.$$

We will also let A be the matrix of coefficients for the system. Now we can rewrite the system of equations as $\mathbf{x}'(t) = A\mathbf{x}(t)$.

Exercises

1. Verify that $\mathbf{x}(t) = e^{At}\mathbf{c}$ is a solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ for every constant vector $\mathbf{c}$.

Further Exploration

This is where I become too lazy to show you all the neat stuff you can do with matrix exponentials, and where I encourage you, the reader, to explore on your own.

Exercises

1. We don't have to stop at matrix exponentials. We can define matrix equivalents for functions like $\sin(x)$ and $\cos(x)$ in a very similar fashion. What other types of functions can we extend to the matrices.
2. Prove that $\sin(A)\sin(A) + \cos(A)\cos(A) = I$ for any arbitrary $n \times n$ matrix A.
3. When is $e^{A+B} = e^A e^B$? The fact that matrix multiplication isn't commutative causes some problems.
4. As a corollary to the above problem, show that $e^{2A} = e^A e^A$.
5. Prove or disprove: there exist a 2×2 matrix A with real entries such that
$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}.$$
[1996 Putnam, B4]

Open problems

1. As we have defined e^A, it is very easy to define x^A for $x \in \mathbb{R}$. Can we define B^A for $n \times n$ matrices A and B? What would their properties be? Would $A^{BC} = \left(A^B\right)^C$?
2. Can we define a $\log$ function? Would it have the similar properties to the real $\log$ function?
3. Can we solve other types of vector differential equations with matrix exponentials or (more generally) matrix functions like $\sin$ and $\cos$?

Mayhem Problems

A new year brings new changes and new problem editors. *Cyrus Hsia* now takes over the helm as *Mayhem Advanced Problems Editor*, with *Richard Hoshino* filling his spot as the *Mayhem High School Problems Editor*, and veteran *Ravi Vakil* maintains his post as *Mayhem Challenge Board Problems Editor*. Note that all correspondence should be sent to the appropriate editor — see the relevant section.

In this issue, you will find only solutions — the next issue will feature only problems. We intend to have problems and solutions in alternate issues.

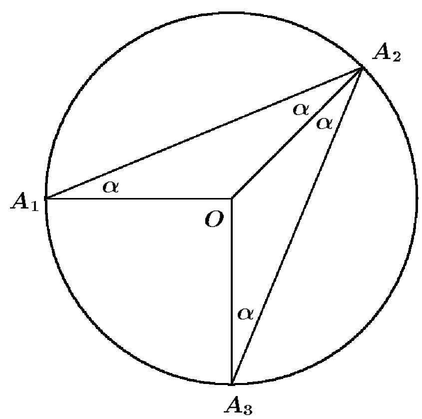
We warmly welcome proposals for problems and solutions. With the new schedule of eight issues per year, we request that solutions from the previous issue be submitted by 1 June 1997, for publication in the issue 5 months ahead; that is, issue 6. We also request that **only students** submit solutions (see editorial [1997: 30]), but we will consider particularly elegant or insightful solutions for others. Since this rule is only being implemented now, you will see solutions from many people in the next few months, as we clear out the old problems from Mayhem.

High School Problems — Solutions

Editor: Richard Hoshino, 17 Norman Ross Drive, Markham, Ontario, Canada. L3S 3E8 **rhoshino@undergrad.math.uwaterloo.ca**

H205. A circular billiard table is given with a cue ball at the circumference. It is shot at an angle of α to the line from the ball to the centre of the table. For what angles α will the ball come back to this point, assuming the ball keeps going indefinitely?

Solution by Samuel Wong, Mary Ward Catholic Secondary School, Toronto.



We let the ball start at A_1 as shown. Then $\angle OA_1A_2 = \angle A_1A_2O = \alpha$ (equal radii) and $\angle A_1OA_2 = 180° - 2\alpha$ (angles sum to 180°). Since the angle of incidence equals the angle of reflection, we have $\angle A_3A_2O = \angle A_1A_2O = \alpha = \angle A_2A_3O$ by equal radii so $\angle A_3OA_2 = \angle A_2OA_1 = 180° - 2\alpha$. Thus, the angles at the centre are always equal. The cue ball returns to A_1 iff

$$k(180° - 2\alpha) = 360°c \implies k(90° - \alpha) = 180°c,$$

for some positive integers c and k. Since α must be real, $90° - \alpha$ must be either a rational or irrational number.

Case I: If $90° - \alpha$ is rational, it can be expressed in the form $\frac{m}{n}$, where m and n are positive, relatively prime integers. Thus $k \cdot \frac{m}{n} = 180c$. We may take $k = 180n$ and then $m = c$, so for all α rational, the cue ball will return to A_1.

Case II: If $90° - \alpha$ is irrational, let $x = 90° - \alpha$. Then $kx = 180°c \implies x = \frac{180°c}{k}$. But x is irrational, so we have a contradiction.

Also solved by MIGUEL CARRIÓN ÁLVAREZ, Universidad Complutense de Madrid, Spain.

H206. For what values of n is an n-digit natural number uniquely determined from the sum and product of its digits?

Solution by Miguel Carrión Álvarez, Universidad Complutense de Madrid, Spain.

The sum and product of the digits are two conditions, and so only two digits can be determined in general. This means $n = 1$ or 2, but when $n = 2$, the order cannot be determined (for example, consider 12 and 21), so the answer is $n = 1$.

H207. Is there a natural number n, such that $\phi(n) = p$, where p is an odd prime number?

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI.

We shall show that there is NO positive integer n such that $\phi(n)$ is an odd prime number. This is an immediate consequence of the fact that $\phi(1) = 1 = \phi(2)$ and the result established below.

Theorem: If n is a positive integer, $n \geq 3$, then 2 divides $\phi(n)$.

Proof: Let n be a positive integer, $n \geq 3$.

Case I: $n = 2^a$, where a is an integer, $a \geq 2$. Then $\phi(n) = 2^{a-1}$, where $a - 1$ is a positive integer, so 2 divides $\phi(n)$.

Case II: n has an odd prime factor p. If the prime factorization of n is $p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where p_k is odd, then

$$\phi(n) = p_1^{a_1-1} p_2^{a_2-1} \cdots p_k^{a_k-1}(p_1 - 1)(p_2 - 1) \cdots (p_k - 1).$$

Because p_k is odd, $p_k - 1$ is even so, 2 divides $\phi(n)$

Also solved by MIGUEL CARRIÓN ÁLVAREZ, Universidad Complutense de Madrid, Spain, SAMUEL WONG, Mary Ward Catholic Secondary School, Toronto.

Advanced Problems — Solutions

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A185. Let b_n be the highest power of 3 dividing $\binom{3^k}{n}$, $0 \leq n \leq 3^k$. Calculate $\displaystyle\sum_{n=0}^{3^k} \frac{1}{b_n}$.

Solution by Wai Ling Yee, student, University of Waterloo, Waterloo, Ontario.

The exponent of the highest power of 3 dividing $t!$ is

$$\left\lfloor \frac{t}{3} \right\rfloor + \left\lfloor \frac{t}{3^2} \right\rfloor + \left\lfloor \frac{t}{3^3} \right\rfloor + \cdots .$$

Thus, the exponent of the highest power of 3 dividing $\binom{3^k}{n} = \dfrac{3^k!}{n!(3^k-n)!}$ is

$$\sum_{j=1}^{k}\left(\left\lfloor\frac{3^k}{3^j}\right\rfloor - \left\lfloor\frac{n}{3^j}\right\rfloor - \left\lfloor\frac{3^k-n}{3^j}\right\rfloor\right).$$

Since $\dfrac{n}{3^j} + \dfrac{3^k-n}{3^j} = 3^{k-j}$, an integer,

$$\left\lfloor\frac{3^k}{3^j}\right\rfloor - \left\lfloor\frac{n}{3^j}\right\rfloor - \left\lfloor\frac{3^k-n}{3^j}\right\rfloor$$

can only take two possible values: 0 and 1.
When $3^j \mid n$, the value is 0. Otherwise, it is 1. Therefore, the value of b_n is 3^k divided by the highest power of 3 dividing n.

Consider the numbers divisible by $3^j, 0 \le j \le k-1$. Ignoring 0, we see that 3^{k-j} of the numbers are divisible by 3^j. These numbers can be written as $3^j \cdot (3m), 3^j \cdot (3m+1)$, or $3^j \cdot (3m+2)$. Therefore, there are $2 \cdot 3^{k-j-1}$ numbers in $\{1, 2, \ldots, 3^k\}$ for which 3^j is the highest power of 3 dividing them.

Each of the $2 \cdot 3^{k-j-1}$ numbers contributes $\dfrac{1}{3^{k-j}}$ to the sum $\displaystyle\sum_{n=0}^{3^k}\frac{1}{b_n}$ for a total of $\frac{2}{3}$. Therefore,

$$\sum_{n=0}^{3^k}\frac{1}{b_n} = \sum_{n=1}^{3^k-1}\frac{1}{b_n} + 2 = \frac{2}{3}k + 2.$$

Also solved by Edward Wang, Wilfred Laurier University, Waterloo, Ontario.

A186. Let a_n be the sequence defined by $a_1 = 1$, and

$$a_{n+1} = \frac{1 + a_1^2 + \cdots + a_n^2}{n}, \quad n \ge 1.$$

Prove that every a_n is an integer.

Solution

An April Fool's joke. It turns out that $a_1, a_2, \ldots, a_{43}$ are all integers, but a_{44} is not an integer.

[Ed: can you prove this?]

A187. Let $S_k(n)$ be the polynomial in n, such that $S_k(n) = \sum_{i=1}^{n} i^k$ for all positive integers n, e.g. $S_0(n) = n$, $S_1(n) = (n^2+n)/2$.
Prove that for $k \geq 1$, $n(n+1) \mid S_k(n)$.
Furthermore, prove that for k odd, $k \geq 3$, $n^2(n+1)^2 \mid S_k(n)$.

Solution by Wai Ling Yee, student, University of Waterloo, Waterloo, Ontario.

Note that $n(n+1) \mid S_1(n) = \dfrac{n(n+1)}{2}$. Assume that $n(n+1)$ divides all $S_k(n)$ up to $k = t$. Consider:

$$
\begin{aligned}
(n+1)^{t+2} &= n^{t+2} + \binom{t+2}{1} n^{t+1} + \binom{t+2}{2} n^t + \cdots + \binom{t+2}{t+1} n + 1, \\
(n-1+1)^{t+2} &= (n-1)^{t+2} + \binom{t+2}{1}(n-1)^{t+1} + \binom{t+2}{2}(n-1)^t \\
&\quad + \cdots + \binom{t+2}{t+1}(n-1) + 1, \\
(n-2+1)^{t+2} &= (n-2)^{t+2} + \binom{t+2}{1}(n-2)^{t+1} + \binom{t+2}{2}(n-2)^t + \\
&\quad \cdots + \binom{t+2}{t+1}(n-2) + 1, \\
&\ \ \vdots \\
(2+1)^{t+2} &= 2^{t+2} + \binom{t+2}{1} 2^{t+1} + \binom{t+2}{2} 2^t + \cdots + \binom{t+2}{t+1} 2 + 1, \\
(1+1)^{t+2} &= 1^{t+2} + \binom{t+2}{1} 1^{t+1} + \binom{t+2}{2} 1^t + \cdots + \binom{t+2}{t+1} 1 + 1.
\end{aligned}
$$

Adding all of the equations together, we have:

$$\sum_{i=2}^{n+1} i^{t+2} = \sum_{i=1}^{n} i^{t+2} + \sum_{i=0}^{t+1} \sum_{j=1}^{n} \binom{t+2}{t+2-i} j^i$$

if and only if

$$(n+1)^{t+2} - n - 1 = \binom{t+2}{1} S_{t+1}(n) + \sum_{i=1}^{t} \binom{t+2}{t+2-i} S_i(n).$$

By the induction hypothesis, we need only to prove that $n(n+1)$ divides $(n+1)^{t+2} - n - 1 = n(n+1)^{t+1}$, which is true.

By induction, $n(n+1) \mid S_k(n)$ for $k \geq 1$.

Now, $S_3(n) = \frac{n^2(n+1)^2}{4}$; therefore $n^2(n+1)^2$ divides $S_3(n)$. Assume that $n^2(n+1)^2$ divides $S_k(n)$ for all odd k up to $k = t-1$, t even. Consider:

$$\begin{aligned}
(n-1)^{t+2} &= n^{t+2} - \binom{t+2}{1} n^{t+1} + \binom{t+2}{2} n^t - \cdots - \binom{t+2}{t+1} n + 1, \\
(n-1-1)^{t+2} &= (n-1)^{t+2} - \binom{t+2}{1}(n-1)^{t+1} + \binom{t+2}{2}(n-1)^t \\
&\quad - \cdots - \binom{t+2}{t+1}(n-1) + 1,
\end{aligned}$$

$$\begin{aligned}
(n-2-1)^{t+2} &= (n-2)^{t+2} - \binom{t+2}{1}(n-2)^{t+1} + \binom{t+2}{2}(n-2)^t \\
&\quad - \cdots - \binom{t+2}{t+1}(n-2) + 1, \\
&\ \ \vdots \\
(2-1)^{t+2} &= 2^{t+2} - \binom{t+2}{1} 2^{t+1} + \binom{t+2}{2} 2^t - \cdots - \binom{t+2}{t+1} 2 + 1, \\
(1-1)^{t+2} &= 1^{t+2} - \binom{t+2}{1} 1^{t+1} + \binom{t+2}{2} 1^t - \cdots - \binom{t+2}{t+1} 1 + 1.
\end{aligned}$$

Subtracting the sum of all the equations in the above group from the sum of all the equations in the first group, we have:

$$\begin{aligned}
(n+1)^{t+2} + n^{t+2} - 1 &= 2\left[\binom{t+2}{1} S_{t+1}(n) + \binom{t+2}{3} S_{t-1}(n) \right. \\
&\quad \left. + \cdots + \binom{t+2}{t-1} S_3(n) + \binom{t+2}{t+1} S_1(n)\right].
\end{aligned}$$

By the induction hypothesis, we have to prove only that $n^2(n+1)^2$ divides

$$(n+1)^{t+2} + n^{t+2} - \binom{t+2}{t+1}(n^2+n) - 1.$$

$$\begin{aligned}
n^{t+2} - \binom{t+2}{t+1}(n^2+n) - 1 &= (n+1-1)^{t+2} - \binom{t+2}{t+1}(n^2+n) - 1 \\
&= (n+1)^{t+2} - \cdots + \binom{t+2}{t}(n+1)^2 - \binom{t+2}{t+1}(n+1)^1 + 1 \\
&\quad - \binom{t+2}{t+1}(n^2+n) - 1 \\
&= (n+1)^{t+2} - \cdots + \binom{t+2}{t}(n+1)^2 - \binom{t+2}{t+1}(n^2+2n+1) \\
&= (n+1)^{t+2} - \cdots + \binom{t+2}{t}(n+1)^2 - \binom{t+2}{t+1}(n+1)^2.
\end{aligned}$$

Therefore $(n+1)^2$ divides $(n+1)^{t+2} + n^{t+2} - \binom{t+2}{t+1}(n^2+n) - 1$.

$$\begin{aligned}
&(n+1)^{t+2} - \binom{t+2}{t+1}n - 1 \\
&= n^{t+2} + \cdots + \binom{t+2}{t}n^2 + \binom{t+2}{t+1}n + 1 - \binom{t+2}{t+1}n - 1 \\
&= n^{t+2} + \cdots + \binom{t+2}{t}n^2.
\end{aligned}$$

Therefore n^2 divides $(n+1)^{t+2} + n^{t+2} - \binom{t+2}{t+1}(n^2+n) - 1$.

Therefore $n^2(n+1)^2$ divides $S_{t+1}(n)$.

By induction, $n^2(n+1)^2$ divides $S_k(n)$ for odd $k \geq 3$.

Challenge Board Problems — Solutions

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We begin by dredging up an old favourite of ours. Well, maybe not so old – a solution appeared in [Mayhem 8: 4, 25: 1996]. We have since received a new solution that is worth printing.

C64. The numbers $x_1, x_2, \ldots, x_n$ are such that $x_1 + x_2 + \cdots + x_n = 0$ and $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Prove that there are two numbers among them whose product is no greater than $-1/n$.

(1991 Tournament of Towns)

Solution by Naoki Sato, student, University of Toronto, Toronto, Ontario.

Choose i such that $x_1 \leq x_2 \leq \cdots \leq x_i \leq 0 \leq x_{i+1} \leq \cdots \leq x_n$. Then

$$\begin{aligned} x_1^2 + x_2^2 + \ldots + x_i^2 &\leq x_1 \cdot x_1 + x_1 \cdot x_2 + \cdots + x_1 \cdot x_i \\ &= x_1(x_1 + x_2 + \cdots + x_i) \\ &= -x_1(x_{i+1} + \cdots + x_n) \\ &\leq -(n-i)x_1 \cdot x_n. \end{aligned}$$

Similarly,

$$\begin{aligned} x_{i+1}^2 + \cdots + x_n^2 &\leq x_n(x_{i+1} + \cdots + x_n) \\ &= -x_n(x_1 + x_2 + \cdots + x_i) \leq -ix_1 \cdot x_n. \end{aligned}$$

Thus $1 = x_1^2 + \cdots + x_n^2 \leq -nx_1 \cdot x_n$.

(As always, please send us new solutions to old problems.)

C68. *Proposed by Vin de Silva, student, Oxford University, Oxford, England.*

Let M be an $n \times n$ orthogonal matrix. (In other words, the rows are n vectors in n-space that are of length 1 and mutually perpendicular.) Let A be the $k \times k$ matrix in the upper-left corner of M, and let B be the $(n-k) \times (n-k)$ matrix in the lower-right corner of M. Prove that $\det A = \det B$.

Solution by Eric Wepsic, D.E.Shaw and Co., New York, NY, USA.

Let c_i be the i^{th} column of the orthogonal matrix Q, and let r_i be the i^{th} row. Let x_i be the i^{th} basis vector. Let M_1 be the top $k \times k$ submatrix of Q, and let M_2 be the bottom $(n-k) \times (n-k)$ submatrix. Then

$$\begin{aligned} \det M_1 &= c_1 \wedge c_2 \wedge \cdots \wedge c_k \wedge x_{k+1} \wedge \cdots \wedge x_n \\ &= Qx_1 \wedge Qx_2 \wedge \cdots \wedge Qx_k \wedge x_{k+1} \wedge \cdots \wedge x_n. \end{aligned}$$

As Q^t has determinant 1, this is equal to:

$$\begin{aligned} &Q^tQx_1 \wedge Q^tQx_2 \wedge \cdots \wedge Q^tQx_k \wedge Q^tx_{k+1} \wedge \cdots \wedge Q^tx_n \\ &= x_1 \wedge x_2 \wedge \cdots \wedge x_k \wedge r_{k+1} \wedge \cdots \wedge r_n \\ &= \det M_2. \end{aligned}$$

Solution by Sam Vandervelde, University of Chicago, Illinois, USA.

Let Q_1 be the matrix whose first k columns are the first k columns of Q, and whose remaining $n-k$ columns are the last $n-k$ columns of the identity matrix. Let Q_2 be the same sort of thing with the first k columns identical to I, and the last $n-k$ columns those of Q^t. Then an easy calculation (using $Q^tQ = I$, $Q^tI = Q^t$) shows that $Q^tQ_1 = Q_2$. Taking determinants of both sides yields the desired result.

The two solutions presented were in some sense identical, although the perspectives were different.

C69. Let Q be a polyhedron in $\mathbb{R}^3$. Let $\vec{n}_1, \ldots, \vec{n}_k$ and $A_1, \ldots, A_k$ be the normal vectors and areas of the faces of Q respectively. Prove that

$$\sum_{i=1}^{k} A_i \vec{n}_i = 0.$$

Solution by Miguel Carrión Álvarez, student, Universidad Complutense de Madrid, Spain.

We have $\sum_{i=1}^{k} A_i \vec{n}_i = \oint d\vec{s}$, where $d\vec{s}$ is the element of area (pointing outwards) and the integral is over the faces of the polyhedron. Then

$$\oint d\vec{s} = \left(\oint ds_x, \oint ds_y, \oint ds_z\right) = \left(\oint \vec{u}_x \cdot d\vec{s}, \oint \vec{u}_y \cdot d\vec{s}, \oint \vec{u}_z \cdot d\vec{s}\right),$$

for constant fields $\vec{u}_x$, $\vec{u}_y$, $\vec{u}_z$. We recall the divergence theorem: $\oint \vec{A} \cdot d\vec{s} = \int_V (\vec{\nabla} \cdot \vec{A}) dV$. But $\vec{u}_x$, $\vec{u}_y$, $\vec{u}_z$ are constant fields, so $\vec{\nabla} \cdot \vec{u}_i = 0$. Thus $\oint d\vec{s} = (0, 0, 0)$.

Comments.

1. *If ρ is the pressure field in a fluid, the total force on a (three-dimensional) object is*

   $$F = -\oint \rho \, d\vec{s} = -\int (\vec{\nabla}\rho) dV.$$

   (Prove this!) In this case, we have constant pressure. What this result says is that a polyhedron sitting in a fluid with constant pressure from all sides (and no other forces) will not move.

2. *This is true in all dimensions.*

3. *Many other interesting results follow from this one, including the following variant of the Pythagorean theorem. Consider a tetrahedron $ABCD$ with AB, AC, AD mutually perpendicular. Then*

   $$(ABC)^2 + (ACD)^2 + (ADB)^2 = (BCD)^2$$

   where the brackets denote the area of the triangle. This is also true in other dimensions. If you can think of any other interesting consequences (including other variants of Pythagoras), please send them in!