THE OLYMPIAD CORNER

No. 180

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We begin this number of the Corner with the problems of the 3rd Mathematical Olympiad of the Republic of China (Taiwan). The contest was written April 14 and 15, 1994. Many thanks go to Richard Nowakowski, Canadian Team Leader to the IMO in Hong Kong for collecting these problems and many others.

3rd MATHEMATICAL OLYMPIAD OF THE REPUBLIC OF CHINA (Taiwan)

First Day — April 14, 1994

1. Let $ABCD$ be a quadrilateral with $\overline{AD} = \overline{BC}$ and $\angle A + \angle B = 120°$. Three equilateral triangles $\triangle ACP$, $\triangle DCQ$ and $\triangle DBR$ are drawn on $\overline{AC}$, $\overline{DC}$ and $\overline{DB}$ away from $\overline{AB}$. Prove that the three new vertices P, Q and R are collinear.

2. Let a, b, c be positive real numbers, α be a real number. Suppose that

$$f(\alpha) = abc(a^\alpha + b^\alpha + c^\alpha),$$

$$g(\alpha) = a^{\alpha+2}(b+c-a) + b^{\alpha+2}(a-b+c) + c^{\alpha+2}(a+b-c).$$

Determine $|f(\alpha) - g(\alpha)|$.

3. Let a be a positive integer such that $(5^{1994} - 1) \mid a$. Show that the expression of the number a in the base 5 contains at least 1994 digits different from zero.

Second Day — April 15, 1994

4. Prove that there are infinitely many positive integers n having the following property: for every arithmetic progression $a_1, a_2, \ldots, a_n$ of integers with n terms, both the mean and standard deviation of the set $\{a_1, a_2, \ldots, a_n\}$ are integers. (Note: For any set $\{x_1, x_2, \ldots, x_n\}$ of real numbers, the mean of the set is defined to be the number

$$\overline{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

and the standard deviation of the set is defined to be the number

$$\sqrt{\frac{\sum(x_i - \overline{x})^2}{n}}.$$

5. Let $X = \{0, a, b, c\}$ and $M(X) = \{f \mid f : X \to X\}$ be the set of all functions from X into itself. Here, an addition table of X is given as follows:

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

(1) If $S = \{f \in M(X) \mid f(x \oplus y \oplus x) = f(x) \oplus f(y) \oplus f(x),\ \forall x, y \in X\}$, determine the number of elements of S.
(2) If $I = \{f \in M(X) \mid f(x \oplus x) = f(x) \oplus f(x),\ \forall x \in X\}$, determine the number of elements of I.

6. For $-1 \le x \le 1$ define

$$T_n(x) = \frac{1}{2^n}\left[\left(x + \sqrt{1-x^2}\right)^n + \left(x - \sqrt{1-x^2}\right)^n\right]$$

(1) Prove that, for $-1 \le x \le 1$, $T_n(x)$ is a monic polynomial of degree n in the x-variable and the maximum value of $T_n(x)$ is $\frac{1}{2^{n-1}}$.
(2) Suppose that $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a monic polynomial with real coefficients such that for all x in $-1 \le x \le 1$, $p(x) > -\frac{1}{2^{n-1}}$. Prove that there exists x_* in $-1 \le x \le 1$ such that $p(x_*) \ge \frac{1}{2^{n-1}}$.

Last issue we gave Five Klamkin Quickies. Here we give his "Quick" solutions plus another 5 problems. Many thanks to Murray S. Klamkin, the University of Alberta.

ANOTHER FIVE KLAMKIN QUICKIES
October 21, 1996

6. Determine the four roots of the equation $x^4 + 16x - 12 = 0$.

7. Prove that the smallest regular n-gon which can be inscribed in a given regular n-gon is one whose vertices are the midpoints of the sides of the given regular n-gon.

8. If 31^{1995} divides $a^2 + b^2$, prove that 31^{1996} divides ab.

9. Determine the minimum value of

$$S = \sqrt{(a+1)^2 + 2(b-2)^2 + (c+3)^2} + \sqrt{(b+1)^2 + 2(c-2)^2 + (d+3)^2} + \sqrt{(c+1)^2 + 2(d-2)^2 + (a+3)^2} + \sqrt{(d+1)^2 + 2(a-2)^2 + (b+3)^2}$$

where a, b, c, d are any real numbers.

10. A set of 500 real numbers is such that any number in the set is greater than one-fifth the sum of all the other numbers in the set. Determine the least number of negative numbers in the set.

We will give the solutions to these in the next issue so you can have some fun looking for the answers. Next we give his solutions to the five Quickies we gave last issue.

FIVE KLAMKIN QUICKIES
October 21, 1996

1. For $x, y, z > 0$, prove that

(i) $1 + \dfrac{1}{(x+1)} \geq \left\{1 + \dfrac{1}{x(x+2)}\right\}^x$,

(ii) $[(x+y)(x+z)]^x[(y+z)(y+x)]^y[(z+x)(z+y)]^z \geq [4xy]^x[4yz]^y[4zx]^z$.

Solution. Both inequalities will follow by a judicious application of the weighted arithmetic-geometric mean inequality (W–A.M.–G.M.) which for three weights is

$$u^a v^b w^c \leq \left[\frac{au + bv + cw}{a+b+c}\right]^{a+b+c},$$

where $a, b, c, u, v, w \geq 0$.

(i) The inequality can be rewritten in the more attractive form

$$\left[1 + \frac{1}{x}\right]^x \leq \left[1 + \frac{1}{x+1}\right]^{x+1},$$

and which now follows by the W–A.M.–G.M.

$$\left[1 + \frac{1}{x}\right]^x \leq \left\{\frac{1 + x\left(1 + \frac{1}{x}\right)}{1+x}\right\}^{x+1} = \left[1 + \frac{1}{x+1}\right]^{x+1}.$$

(ii) Also, the inequality here can be rewritten in the more attractive form

$$\left[\frac{2x}{z+x}\right]^{z+x}\left[\frac{2y}{x+y}\right]^{x+y}\left[\frac{2z}{y+z}\right]^{y+z} \leq 1.$$

But this follows by applying the W–A.M.–G.M. to

$$1 = \sum[z+x]\left[\frac{2x}{z+x}\right] \div \sum[z+x].$$

2. If $ABCD$ is a quadrilateral inscribed in a circle, prove that the four lines joining each vertex to the nine point centre of the triangle formed by the other three vertices are concurrent.

Solution. The given result still holds if we replace the nine point centres by either the orthocentres or the centroids.

A vector representation is particularly à propos here, since (with the circumcentre O as an origin and F denoting the vector from O to any point F) the orthocentre H_a, the nine point centre N_a, the centroid G_a of $\triangle BCD$ are given simply by $H_a = B + C + D$, $N_a = (B + C + D)/2$, $G_a = (B + C + D)/3$, respectively, and similarly for the other three triangles. Since the proofs for each of the three cases are practically identical, we just give the one for the orthocentres. The vector equation of the line L_a joining A to H_a is given by $L_a = A + \lambda_a[B + C + D - A]$ where λ_a is a real parameter. By letting $\lambda_a = 1/2$, one point on the line is $[A + B + C + D]/2$ and similarly this point is on the other three lines. For the nine point centres, the point of concurrency will be $2[A + B + C + D]/3$, while for the centroids, the point of concurrency will be $3[A + B + C + D]/4$.

3. How many six digit perfect squares are there each having the property that if each digit is increased by one, the resulting number is also a perfect square?

Solution. If the six digit square is given by

$$m^2 = a \cdot 10^5 + b \cdot 10^4 + c \cdot 10^3 + d \cdot 10^2 + e \cdot 10 + f,$$

then

$$n^2 = (a+1)\cdot 10^5 + (b+1)\cdot 10^4 + (c+1)\cdot 10^3 + (d+1)\cdot 10^2 + (e+1)\cdot 10 + (d+1),$$

so that

$$n^2 - m^2 = 111,111 = (111)(1,001) = (3 \cdot 37)(7 \cdot 11 \cdot 13).$$

Hence,

$$n + m = d_i \quad \text{and} \quad n - m = 111,111/d_i$$

where d_i is one of the divisors of $111,111$. Since $111,111$ is a product of five primes it has 32 different divisors. But since we must have $d_i > 111,111/d_i$, there are at most 16 solutions given by the form $lm = \frac{1}{2}(d_i - 111,111/d_i)$. Then since m^2 is a six digit number, we must have

$$632.46 \approx 200\sqrt{10} < 2m < 2,000.$$

On checking the various divisors, there are four solutions. One of them corresponds to $d_i = 3 \cdot 13 \cdot 37 = 1,443$ so that $m = \frac{1}{2}(1,443 - 7 \cdot 11) = 683$ and $m^2 = 466,489$. Then, $466,489 + 111,111 = 577,600 = 760^2$. The others are given by the table

d_i	m	m^2	n^2	n
$3 \cdot 7 \cdot 37 = 777$	317	$100,489$	$211,600$	460
$3 \cdot 11 \cdot 37 = 1,221$	565	$319,225$	$430,336$	656
$7 \cdot 11 \cdot 13 = 1,001$	445	$198,025$	$309,136$	556

4. Let v_iw_i, $i = 1, 2, 3, 4$, denote four cevians of a tetrahedron $v_1v_2v_3v_4$ which are concurrent at an interior point P of the tetrahedron. Prove that

$$pw_1 + pw_2 + pw_3 + pw_4 \le \max v_iw_i \le \text{longest edge.}$$

Solution. We choose an origin, o, outside of the space of the tetrahedron and use the set of 4 linearly independent vectors $V_i = ov_i$ as a basis. Also the vector from o to any point q will be denoted by Q. The interior point p is then given by $P = x_1V_1 + x_2V_2 + x_3V_3 + x_4V_4$ where $x_i > 0$ and $\sum_i x_i = 1$. It now follows that $W_i = \frac{P - x_iV_i}{1-x_i}$ (for other properties of concurrent cevians via vectors, see [1987: 274–275]) and then that

$$pw_i = \left|\frac{P - x_iV_i}{1 - x_i} - P\right| = \left|\frac{x_i(P - V_i)}{1 - x_i}\right| = \left|x_i \sum_j x_j \frac{V_j - V_i}{1 - x_i}\right|,$$

$$v_iw_i = \left|\frac{P - x_iV_i}{1 - x_i} - V_i\right| = \left|\frac{P - V_i}{1 - x_i}\right| = \left|\sum_j x_j \frac{V_j - V_i}{1 - x_i}\right|.$$

Summing

$$\sum_i pw_i = \sum_i \left|x_i \sum_j x_j \frac{V_j - V_i}{1 - x_i}\right| = \sum_i x_i(v_iw_i) \le \max_i v_iw_i,$$

and with equality only if v_iw_i is constant. Also,

$$v_iw_i \le \sum_{j \ne i} \left[\frac{x_j}{1 - x_i}\right] \max_r |V_r - V_i| = \max_r |V_r - V_i|.$$

Finally,

$$\sum pw_i \le \max_i v_iw_i \le \max_{r,s} |V_r - V_s|.$$

Comment: In a similar fashion, it can be shown that the result generalizes to n-dimensional simplexes. The results for triangles are due to Paul Erdős, Amer. Math. Monthly, Problem 3746, 1937, p. 400; Problem 3848, 1940, p. 575.

5. Determine the radius r of a circle inscribed in a given quadrilateral if the lengths of successive tangents from the vertices of the quadrilateral to the circle are a, a, b, b, c, c, d, d, respectively.

Solution. Let $2A$, $2B$, $2C$, $2D$ denote the angles between successive pairs of radii vectors to the points of tangency and let r be the inradius. Then

$$r = \frac{a}{\tan A} = \frac{b}{\tan B} = \frac{c}{\tan C} = \frac{d}{\tan D}.$$

Also, since $A + B + C + D = \pi$, we have $\tan(A + B) = \tan(C + D) = 0$, or

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} + \frac{\tan C + \tan D}{1 - \tan C \tan D} = 0,$$

so that

$$\frac{r(a+b)}{r^2 - ab} + \frac{r(c+d)}{r^2 - cd} = 0.$$

Finally,

$$r^2 = \frac{abc + bcd + cda + dab}{a + b + c + d}.$$

Now we turn our attention to solutions by our readers to problems given in the September 1995 number of the Corner where we gave the 16th Austrian Polish Mathematics Competition [1995: 221–222].

16TH AUSTRIAN POLISH MATHEMATICS COMPETITION

First Day — June 30, 1993

Time: 4.5 hours (individual competition)

1. Determine all natural numbers $x, y \geq 1$ such that $2^x - 3^y = 7$.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Amengual's solution.

First of all we note that both numbers x and y must be even. Suppose to the contrary that one of the numbers is odd.

If x is odd, the number $2^x + 1$ (which we have from the factorization $A^x + 1 = (A+1)(A^{x-1} - A^{x-2} + \cdots - A + 1)$ that $2^x + 1$) is a multiple of 3. Consequently $2^x - 3^y \equiv -1 \bmod 3$, while $7 \equiv 1 \bmod 3$, and the given equation is invalid $\bmod 3$.

If y is odd, we will use the modular technique as in the previous case, but this time modulo 8. We have $3^2 \equiv 1 \bmod 8$. It follows that $3^{2k+1} \equiv 3 \bmod 8$ for $k = 0, 1, 2, \ldots$.

Consequently $3^y + 7 \equiv 0 \bmod 8$ and since $2^x \equiv 3^y + 7$ we must have $x \leq 2$. If $x = 1$, we have $2 - 3^y = 7$, which is impossible. If $x = 2$, we have $2^2 - 3^y = 7$, which is also impossible.

Suppose that the numbers x and y are even. So $x = 2l$, $y = 2m$, with l and m natural numbers. The given equation can then be written in the form $(2^l + 3^m)(2^l - 3^m) = 7$, where $2^l + 3^m$ and $2^l - 3^m$ are natural numbers, which implies that $2^l + 3^m = 7$ and $2^l - 3^m = 1$. These two equations determine the values of l, m, namely $l = 2$, $m = 1$ for which we have $x = 4$, $y = 2$.

The only two natural numbers $x, y \geq 1$ such that $2^x - 3^y = 7$ are therefore $x = 4$ and $y = 2$.

5. Determine all real solutions x, y, z of the system of equations:

$$\begin{aligned} x^3 + y &= 3x + 4, \\ 2y^3 + z &= 6y + 6, \\ 3z^3 + x &= 9z + 8. \end{aligned}$$

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and by Panos E. Tsaoussoglou, Athens, Greece. We give the solution of Tsaoussoglou.

From $x^3 + y = 3x + 4$, we have

$$x^3 - 1 - 1 - 3x = 2 - y,$$

or

$$(x-2)(x+1)^2 = 2 - y. \tag{1}$$

From $2y^3 - 2 - 2 - 6y = 2 - z$, we have

$$2(y-2)(y+1)^2 = (2 - z), \tag{2}$$

and $3z^3 - 3 - 3 - 9z = 2 - x$ gives

$$3(z-2)(z+1)^2 = (2 - x), \tag{3}$$

so that

$$\begin{aligned} (x-2)(x+1)^2 &= -(y-2), \\ 2(y-2)(y+1)^2 &= -(z-2), \\ 3(z-2)(z+1)^2 &= -(x-2), \end{aligned}$$

and

$$(x-2)(y-2)(z-2)\left((x+1)^2(y+1)^2(z+1)^2 + \frac{1}{6}\right) = 0.$$

As the last factor is always positive for real x, y, z, we have

$$(x-2)(y-2)(z-2) = 0.$$

This gives at least one of $x = 2$, $y = 2$, $z = 2$. In conjunction with (1), (2) and (3) this gives the unique solution $x = y = z = 2$.

6. Show: For all real numbers $a, b \geq 0$ the following chain of inequalities is valid

$$\begin{aligned} \left(\frac{\sqrt{a} + \sqrt{b}}{2}\right)^2 &\leq \frac{a + \sqrt[3]{a^2 b} + \sqrt[3]{a b^2} + b}{4} \\ &\leq \frac{a + \sqrt{ab} + b}{3} \leq \sqrt{\left(\frac{\sqrt[3]{a^2} + \sqrt[3]{b^2}}{2}\right)^3}. \end{aligned}$$

Also, for all three inequalities determine the cases of equality.

Solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and by Panos E. Tsaoussoglou, Athens, Greece. We give Tsaossoglou's solution.

1. $\left(\dfrac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \le \dfrac{\sqrt[3]{a^2}(\sqrt[3]{a}+\sqrt[3]{b})+\sqrt[3]{b^2}(\sqrt[3]{a}+\sqrt[3]{b})}{4}$

is equivalent to

$$(\sqrt{a}+\sqrt{b})^2 \le (\sqrt[3]{a^2}+\sqrt[3]{b^2})(\sqrt[3]{a}+\sqrt[3]{b}),$$

which holds by the Cauchy inequality.

Let $\sqrt[6]{a}=A$, $\sqrt[6]{b}=B$, $(A^3+B^3)^2 \le (A^4+B^4)(A^2+B^2)$.

2. $3(a+b)+3\sqrt[3]{ab}(\sqrt[3]{a}+\sqrt[3]{b}) \le 4(a+\sqrt{ab}+b)$,
or equivalently

$$a+3\sqrt[3]{a^2b}+3\sqrt[3]{ab^2}+b \le 2(a+2\sqrt{ab}+b),$$

or

$$(\sqrt[3]{a}+\sqrt[3]{b})^3 \le 2(\sqrt{a}+\sqrt{b})^2,$$

or

$$\left(\frac{A^2+B^2}{2}\right)^3 \le \left(\frac{A^3+B^3}{2}\right)^2,$$

(with A, B as above), a known inequality.

3. $\dfrac{a+\sqrt{ab}+b}{3} \le \sqrt{\left(\dfrac{\sqrt[3]{a^2}+\sqrt[3]{b^2}}{2}\right)^3}$.

With A and B as above this is equivalent to

$$\left(\frac{A^6+A^3B^3+B^6}{3}\right)^2 \le \left(\frac{A^4+B^4}{2}\right)^3.$$

For this it is enough to prove that

$$\left(\frac{A^4+B^4}{2}\right)^3 - \left(\frac{A^6+A^3B^3+B^6}{3}\right)^2 \ge 0,$$

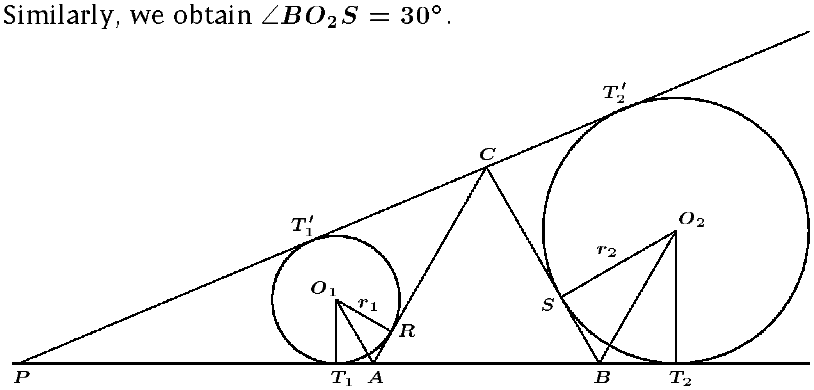
or

$$\begin{aligned}
&9(A^4+B^4)^3-8(A^6+A^3B^3+B^6)^2\\
&= A^{12}-16A^9B^3+27A^8B^4-24A^6B^6+27A^4B^8-16A^3B^9+B^{12}\\
&= (A-B)^4\left[A^8+4A^7B+10A^6B^2+4A^5B^3-2A^4B^4\right.\\
&\qquad\left.+4A^3B^5+10A^2B^6+4AB^7+B^8\right]\\
&\ge (A-B)^4(A^3B^3(A-B)^2) \ge 0.
\end{aligned}$$

9. Let $\triangle ABC$ be equilateral. On side AB produced, we choose a point P such that A lies between P and B. We now denote a as the length of sides of $\triangle ABC$; r_1 as the radius of incircle of $\triangle PAC$; and r_2 as the exradius of $\triangle PBC$ with respect to side BC. Determine the sum $r_1 + r_2$ as a function of a alone.

Solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; and by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina. We give Amengual's solution and comment.

Looking at the figure, we see that $\angle T_1O_1R = 60°$ since it is the supplement of $\angle T_1AR = 120°$ (as an exterior angle for $\triangle ABC$). Hence, $\angle AO_1R = 30°$. Similarly, we obtain $\angle BO_2S = 30°$.



Since tangents drawn to a circle from an external point are equal, we have

$$\begin{aligned} T_1T_2 &= T_1A + AB + BT_2 \;=\; RA + AB + SB \\ &= r_1 \tan 30° + a + r_2 \tan 30° \;=\; \frac{r_1 + r_2}{\sqrt{3}} + a, \end{aligned}$$

and

$$T_1'T_2' = T_1'C + CT_2' = CR + CS = (a - RA) + (a - SB) = 2a - \frac{r_1 + r_2}{\sqrt{3}}.$$

Since common external tangents to two circles are equal, $T_1T_2 = T_1'T_2'$. Hence,

$$\frac{r_1 + r_2}{\sqrt{3}} + a = 2a - \frac{r_1 + r_2}{\sqrt{3}},$$

whence we find that

$$r_1 + r_2 = \frac{a\sqrt{3}}{2}.$$

Comment. This problem is identical to problem 2.1.11, page 25, of H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems*, (The Charles Babbage Research Centre, 1989).

We next give the solution to one problem of the VII Nordic Mathematical Contest.

2. [1995: 223] *VII Nordic Mathematical Contest*

A hexagon is inscribed in a circle with radius r. Two of its sides have length 1, two have length 2 and the last two have length 3. Prove that r is a root of the equation

$$2r^3 - 7r - 3 = 0.$$

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

Equal chords subtend equal angles at the centre of a circle; if each of sides of length i subtends an angle α_i ($i = 1, 2, 3$) at the centre of the given circle, then

$$2\alpha_1 + 2\alpha_2 + 2\alpha_3 = 360^\circ,$$

whence

$$\frac{\alpha_1}{2} + \frac{\alpha_2}{2} = 90^\circ - \frac{\alpha_3}{2},$$

and

$$\cos\left(\frac{\alpha_1}{2} + \frac{\alpha_2}{2}\right) = \cos\left(90^\circ - \frac{\alpha_3}{2}\right) = \sin\frac{\alpha_3}{2}.$$

Next we apply the addition formula for the cosine:

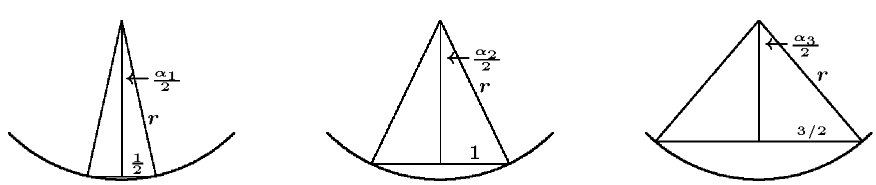
$$\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2} - \sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2} = \sin\frac{\alpha_3}{2}, \qquad (1)$$

where (see figures)

$$\sin\frac{\alpha_1}{2} = \frac{1/2}{r}, \quad \cos\frac{\alpha_1}{2} = \frac{\sqrt{4r^2-1}}{2r};$$

$$\sin\frac{\alpha_2}{2} = \frac{1}{r}, \quad \cos\frac{\alpha_2}{2} = \frac{\sqrt{r^2-1}}{r};$$

$$\sin\frac{\alpha_3}{2} = \frac{3/2}{r}.$$



A B C

We substitute these expressions into (1) and obtain, after multiplying both sides by $2r^2$,

$$\sqrt{4r^2-1}\cdot\sqrt{r^2-1} - 1 = 3r.$$

Now write it in the form

$$\sqrt{(4r^2-1)(r^2-1)} = 3r+1,$$

and square, obtaining

$$(4r^2-1)(r^2-1) = 9r^2+6r+1,$$

which is equivalent to

$$r(2r^3-7r-3) = 0.$$

Since $r \neq 0$, we have

$$2r^3-7r-3 = 0,$$

which was to be shown.

To finish this number of the Corner we turn to two problems from the 32nd Ukrainian Mathematical Olympiad given in the October 1995 number of the Corner [1995 : 266].

32nd UKRAINIAN MATHEMATICAL OLYMPIAD

March 1992 — Selected Problems

2. (8) There are real numbers a, b, c, such that $a \geq b \geq c > 0$. Prove that

$$\frac{a^2-b^2}{c} + \frac{c^2-b^2}{a} + \frac{a^2-c^2}{b} \geq 3a-4b+c.$$

Solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Panos E. Tsaoussoglou, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give the solution of Arslanagić.

From $a \geq b \geq c > 0$ we have

$$\frac{a+b}{c} \geq 2, \quad 0 < \frac{b+c}{a} \leq 2 \quad \text{and} \quad \frac{a+c}{b} \geq 1.$$

Now we get

$$\frac{a^2-b^2}{c} \geq 2(a-b), \quad \text{because } a \geq b;$$

$$\frac{c^2-b^2}{a} \geq 2(c-b), \quad \text{because } c \leq b;$$

and

$$\frac{a^2-c^2}{b} \geq a-c, \quad \text{because } a \geq c.$$

After addition of these inequalities, we have

$$\frac{a^2-b^2}{c} + \frac{c^2-b^2}{a} + \frac{a^2-c^2}{b} \geq 2(a-b)+2(c-b)+(a-c),$$

that is,

$$\frac{a^2-b^2}{c}+\frac{c^2-b^2}{a}+\frac{a^2-c^2}{b}\geq 3a-4b+c.$$

The equality holds if and only if $a = b = c > 0$.

5. (10) Prove that there are no real numbers x, y, z, such that

$$x^2 + 4yz + 2z = 0,$$

$$x + 2xy + 2z^2 = 0,$$

$$2xz + y^2 + y + 1 = 0.$$

Solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; by D.J. Smeenk, Zaltbommel, the Netherlands; by Panos E. Tsaoussoglou, Athens, Greece; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario. We give Wang's solution.

Label the three given equations (1), (2) and (3), (in that order), respectively. If $x = 0$ or $z = 0$, then from (3), $y^2 + y + 1 = 0$, which has no real solutions. Hence we may assume that $xz \neq 0$. From (1) and (2), we get $x^2 = -2z(2y+1)$ and $2z^2 = -x(2y+1)$, which, when multiplied gives $2x^2z^2 = 2xz(2y+1)^2$ or $xz = (2y+1)^2$. Substituting into (3) we get

$$2(2y+1)^2 + y^2 + y + 1 = 0$$

or

$$3y^2 + 3y + 1 = 0,$$

which has no solution.

That completes the Corner for this issue. Olympiad season is upon us. Send me your Olympiads, as well as your nice solutions.