SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We have identified "anonymous" in the solutions of problems 2064, 2065, 2067, 2068, 2069, 2071 and 2077 as MARIA MERCEDES SÁNCHEZ BENITO, I.B. Luis Buñuel, Madrid, Spain.

The name of VICTOR OXMAN, University of Haifa, Haifa, Israel, should be added to the list of solvers of problem 2091. The name of ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick, should be added to the list of solvers to problem 2104.

2113. [1996: 35] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Prove the inequality

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \geq \left(\sum_{i=1}^{n}(a_i+b_i)\right)\left(\sum_{i=1}^{n}\frac{a_ib_i}{a_i+b_i}\right)$$

for any positive numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$.

I. *Solution by Vedula N. Murty, Andhra University, Visakhapatnam, India.*

The given inequality follows from the easily verified identity:

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) - \left(\sum_{i=1}^{n}(a_i+b_i)\right)\left(\sum_{i=1}^{n}\frac{a_ib_i}{a_i+b_i}\right) = \sum_{1\leq i<j\leq n}\frac{(a_ib_j-a_jb_i)^2}{(a_i+b_i)(a_j+b_j)}.$$

II. *Solution by Federico Ardila, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA.*

We know from Cauchy's inequality that

$$\left(\sum_{i=1}^{n}(a_i-b_i)\right)^2 \leq \left(\sum_{i=1}^{n}(a_i+b_i)\right)\left(\sum_{i=1}^{n}\frac{(a_i-b_i)^2}{a_i+b_i}\right).$$

Therefore

$$\begin{aligned}
\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) &= \frac{1}{4}\left(\left(\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i\right)^2 - \left(\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i\right)^2\right) \\
&= \frac{1}{4}\left(\left(\sum_{i=1}^{n}(a_i + b_i)\right)^2 - \left(\sum_{i=1}^{n}(a_i - b_i)\right)^2\right) \\
&\geq \frac{1}{4}\left(\left(\sum_{i=1}^{n}(a_i + b_i)\right)^2\right. \\
&\qquad \left. - \left(\sum_{i=1}^{n}(a_i + b_i)\right)\left(\sum_{i=1}^{n}\frac{(a_i - b_i)^2}{a_i + b_i}\right)\right) \\
&= \left(\sum_{i=1}^{n}(a_i + b_i)\right)\left(\sum_{i=1}^{n}\frac{(a_i + b_i)^2 - (a_i - b_i)^2}{4(a_i + b_i)}\right) \\
&= \left(\sum_{i=1}^{n}(a_i + b_i)\right)\left(\sum_{i=1}^{n}\frac{a_i b_i}{a_i + b_i}\right),
\end{aligned}$$

which completes the proof.

III. *Solution by Kee-Wai Lau, Hong Kong.*

For $n = 1$, the result is clear. Suppose that the inequality holds for $n = k$. Then for $n = k + 1$, we have

$$\begin{aligned}
&\left(\sum_{i=1}^{k+1} a_i\right)\left(\sum_{i=1}^{k+1} b_i\right) - \left(\sum_{i=1}^{k+1}(a_i + b_i)\right)\left(\sum_{i=1}^{k+1}\frac{a_i b_i}{a_i + b_i}\right) \\
&= \left(\sum_{i=1}^{k} a_i\right)\left(\sum_{i=1}^{k} b_i\right) - \left(\sum_{i=1}^{k}(a_i + b_i)\right)\left(\sum_{i=1}^{k}\frac{a_i b_i}{a_i + b_i}\right) \\
&\qquad + a_{k+1}\left(\sum_{i=1}^{k} b_i\right) + b_{k+1}\left(\sum_{i=1}^{k} a_i\right) \\
&\qquad - (a_{k+1} + b_{k+1})\sum_{i=1}^{k}\frac{a_i b_i}{a_i + b_i} - \frac{a_{k+1}b_{k+1}}{a_{k+1} + b_{k+1}}\sum_{i=1}^{k}(a_i + b_i) \\
&\geq \sum_{i=1}^{k}\left(a_{k+1}b_i + b_{k+1}a_i - (a_{k+1} + b_{k+1})\frac{a_i b_i}{a_i + b_i}\right. \\
&\qquad \left. - \frac{a_{k+1}b_{k+1}}{a_{k+1} + b_{k+1}}(a_i + b_i)\right) \qquad \text{by the induction hypothesis} \\
&= \frac{1}{a_{k+1} + b_{k+1}}\sum_{i=1}^{k}\frac{(a_{k+1}b_i - b_{k+1}a_i)^2}{a_i + b_i} \;\geq\; 0,
\end{aligned}$$

completing the induction.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; HAN PING DAVIN CHOR, Student, Cambridge, MA, USA; THEODORE CHRONIS, student, Aristotle University of Thessaloniki, Greece; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; DAVID E. MANES, State University of New York, Oneonta, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer. There were also one anonymous solution and one incorrect submission.

Klamkin remarked that this is a known inequality due to E.A. Milne [1], that came up in establishing an integral inequality. It also appeared as problem 67 in [2]. This was also noticed by Janous and Tsaoussoglou. Arslanagić, Herzig, Hess, Manes and Tsaoussoglou all pointed out that equality holds in the proposed inequality if and only if the vectors $(a_1, a_2, \dots, a_n)$ *and* $(b_1, b_2, \dots, b_n)$ *are linearly dependent. This is obvious from proof* I *above.*

References

[1.] E.A. Milne, Note on Rosseland's Integral for the Stellar Absorption Coefficient, *Monthly Notices Royal Astronomical Soc.* **85** (1925) 979–984.

[2.] G.H. Hardy, J.E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, London, 1934, pp. 61–62.

2114. [1996: 75] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a square with incircle Γ. A tangent ℓ to Γ meets the sides AB and AD and the diagonal AC at P, Q and R respectively. Prove that

$$\frac{AP}{PB} + \frac{AR}{RC} + \frac{AQ}{QD} = 1.$$

Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, and Maria Ascensión López Chamorro, I.B. Leopoldo Cano, Valladolid, Spain.

We suppose that the equation of Γ is $x^2 + y^2 = 1$, and the coordinates of A and C are $(1,1)$ and $(-1,-1)$ respectively. Let $T(\cos t, \sin t)$ be the coordinates of the point of tangency of line ℓ with Γ, with $0 < t < \frac{\pi}{2}$.

Then the equation of PQ is

$$y \cdot \sin t + x \cdot \cos t = 1,$$

and the coordinates of the involved points are

$$R\left(\frac{1}{\sin t + \cos t}, \frac{1}{\sin t + \cos t}\right), P\left(\frac{1 - \cos t}{\sin t}\right), Q\left(\frac{1 - \sin t}{\cos t}, 1\right),$$

and so

$$\frac{AP}{PB} = \frac{\sin t + \cos t - 1}{\sin t - \cos t + 1},$$
$$\frac{AR}{RC} = \frac{\sin t + \cos t - 1}{\sin t + \cos t + 1},$$
$$\frac{AQ}{QD} = \frac{\cos t + \sin t - 1}{\cos t - \sin t + 1},$$

from which an easy calculation shows that

$$\frac{AP}{PB} + \frac{AQ}{QD} + \frac{AR}{RC} = 1.$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; CLAUDIO ARCONCHER, Jundiaí, Brazil; FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; SAM BAETHGE, Science Academy, Austin, Texas, USA; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; TIM CROSS, King Edward's School, Birmingham, England; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; DAVID HANKIN, Hunter College Campus Schools, New York, NY, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; PETER HURTHIG, Columbia College, Burnaby, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; P. PENNING, Delft, the Netherlands; WALDEMAR POMPE, student, University of Warsaw, Poland; D.J. SMEENK, Zaltbommel, the Netherlands; MELETIS VASILIOU, Elefsis, Greece; and the proposer.

2115. [1996: 75] *Proposed by Toby Gee, student, the John of Gaunt School, Trowbridge, England.*

Find all polynomials f such that $f(p)$ is a prime for every prime p.

Solution by Luis V. Dieulefait, IMPA, Rio de Janeiro, Brazil.

Let f be a polynomial such that $f(p)$ is prime for every prime value of p. We will prove that f is a constant polynomial (the constant being, of course, a prime number) or f is the identity; that is $f(x) = x$, for every integer x. Suppose that f is not the identity. Then $f(x) = x$ possesses only a finite number of solutions.

Therefore, there exist p, q primes, $p \neq q$ such that $f(p) = q$. Applying Dirichlet's Theorem, we know that the arithmetical progression $p + kq$, $k = 0, 1, 2, \ldots$, contains infinitely many primes. Besides, for every $k \geq 0$,

$f(p+kq) \equiv f(p) \equiv 0 \pmod q$, and so for the infinitely many primes r_i of the form $p+kq$ we must have $f(r_i) = q$. But f being a polynomial that takes the value q infinitely many times means that $f(x) = q$ for all x.

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; MANSUR BOASE, student, St. Paul's School, London, England; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL JOSEPHY, Universidad de Costa Rica, San José, Costa Rica; JOEL SCHLOSBERG, student, Hunter College High School, New York NY, USA; and the proposer. There were 3 incorrect submissions.

2116. [1996: 75] *Proposed by Yang Kechang, Yueyang University, Hunan, China.*

A triangle has sides a, b, c and area F. Prove that

$$a^3b^4c^5 \geq \frac{25\sqrt{5}(2F)^6}{27}.$$

When does equality hold?

Several solvers pointed out that this proposal is just a special case of [1984: 19], proposed by M.S. Klamkin, which asks for the maximum value of $P \equiv \sin^\alpha A.\sin^\beta B.\sin^\gamma C$, where A, B, C are the angles of a triangle and α, β, γ are given positive numbers. The featured solution by Walther Janous [1985: 908] establishes the maximum to be

$$P_{max} = \left\{\frac{\alpha(\alpha+\beta+\gamma)}{(\alpha+\beta)(\alpha+\gamma)}\right\}^{\alpha/2} \cdot \left\{\frac{\beta(\alpha+\beta+\gamma)}{(\beta+\gamma)(\beta+\alpha)}\right\}^{\beta/2} \cdot \left\{\frac{\gamma(\alpha+\beta+\gamma)}{(\gamma+\alpha)(\gamma+\beta)}\right\}^{\gamma/2}.$$

In the humble opinion of the editor, nothing in the submissions added anything substantive to [1985: 908].

Solved by PETER HURTHIG, Columbia College, Burnaby, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; DIGBY SMITH, Mount Royal College, Calgary, Alberta; and PANOS E. TSAOUSSOGLOU, Athens, Greece.

2117. [1996: 76] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with $AB > AC$, and the bisector of $\angle A$ meets BC at D. Let P be an interior point of the side AC. Prove that $\angle BPD < \angle DPC$.

Two solutions by Shailesh Shirali, Rishi Valley School, India.

[*Editor's comment*: Shirali strengthens the result by replacing the condition $AB > AC$ by the less stringent condition $\angle ABC < 90°$. (That seems to be sufficient for the inequality in his first proof: $90° > \angle ABC > \angle PBC$ implies $\sin\angle ABC > \sin\angle PBC$.) Neither he nor the other solvers explicitly stated that they had stronger results.]

Solution I. Let E be the point where the internal bisector of $\angle P$ of $\triangle PBC$ meets side BC. Then

$$\frac{BE}{EC} = \frac{PB}{PC} = \frac{\sin\angle PCB}{\sin\angle PBC} > \frac{\sin\angle ACB}{\sin\angle ABC} = \frac{AB}{AC} = \frac{BD}{DC},$$

so that E lies between D and C. It follows that $\angle BPD < \angle DPC$.

[*Editor's further comment*: The above argument clearly continues to work for some positions of P on the side AC even when $\angle ABC \geq 90°$. A set of conditions on P and $\triangle ABC$ under which $\angle BPD < \angle DPC$ is implicit in Shirali's second argument.]

Solution II. The set of points X in the plane of the triangle for which $\angle BXD = \angle DXC$ is Ω, the *circle of Apollonius through A with respect to B and C*. [Equivalently, Ω is the locus of points X for which $\frac{XB}{XC} = \frac{BD}{DC}$; it is the circle through A and D whose centre lies on BC; inversion in this circle fixes A and D, and interchanges B with C; see, for example, H.S.M. Coxeter, *Introduction to Geometry*, §6.6 pages 88-89.] When $AB > AC$ [so that $BD > DC$], the set of points X for which $\angle BXD < \angle DXC$ is the set of points interior to Ω. In this case the centre of Ω lies on the extension of BC beyond C, so the segment AC lies entirely within Ω. It follows that for all points P on AC, we have $\angle BPD < \angle DPC$. [When $AB < AC$, the centre of Ω lies on the part of BC beyond B, so that now we seek points P outside Ω; when $\angle B \leq 90°$, the entire line segment AC lies outside Ω, which agrees with the first proof. When $\angle B$ is obtuse, then some points of AC lie inside Ω – for those points, the desired inequality no longer holds.]

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JORDI DOU, Barcelona, Spain; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; WALDEMAR POMPE, student, University of Warsaw, Poland; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrino, Greece; MELETIS VASILIOU, Elefsis, Greece; and the proposer.

2118. [1996: 76] *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

The primitive Pythagorean triangle with sides 2547 and 40004 and hypotenuse 40085 has area 50945094, which is an 8–digit number of the form $abcdabcd$. Find another primitive Pythagorean triangle whose area is of this form.

Solution by the proposer.
There are three such primitive Pythagorean triangles:

	m	n	$m^2 - n^2$	$2mn$	$m^2 + n^2$	Area
(i)	146	137	2547	40004	40085	50945094
(ii)	146	9	21235	2628	21397	27902790
(iii)	105	32	10001	6720	12049	33603360

8–digit numbers of the form $abcdabcd$ can be factored as $10001 \cdot N = 73 \cdot 137 \cdot N$ for an integer N in the range $1000 \leq N < 10000$. The sides of a primitive Pythagorean triangle are given by $m^2 - n^2$, $2mn$, $m^2 + n^2$, with relatively prime integers m and n of different parity. The area of the triangle is $\triangle = mn(m-n)(m+n)$. Note that the factors m, n, $m+n$, and $m-n$ are pairwise relatively prime, and exactly one of them is even. These numbers are in the order

(i) $m + n > m > n > m - n$, or

(ii) $m + n > m > m - n > n$.

[Clearly, none of m, n or $m - n$ can be $10001r$. Also $m + n \neq 10001$; otherwise, $mn \geq (10000)(1)$ and $\triangle > 10^8$.] Each of 137 and 73 divides exactly one of the four numbers $m, n, m \pm n$. We list the twelve different types in the table below, in which h and k are positive integers.

Type	$m+n$	m	n	$m-n$	conditions
(1)	$2(137h)+73k$	$137h+73k$	$137h$	$73k$	k odd
(2)	$137h+2(73)k$	$137h+73k$	$73k$	$137h$	h odd
(3)	$137h+73k$	$73k$	$137h$	$73k-137h$	$h+k$ odd
(4)	$-137h+2(73)k$	$73k$	$-137h+73k$	$137h$	h odd
(5)	$73k$	$-137h+73k$	$137h$	$73k-2(137)h$	k odd
(6)	$73k$	$\frac{1}{2}(137h+73k)$	$\frac{1}{2}(-137h+73k)$	$137h$	$h \cdot k$ odd
(7)	$137h+73k$	$137h$	$73k$	$173h-73k$	$h+k$ odd
(8)	$2(137h)-73k$	$137h$	$137h-73k$	$73k$	k odd
(9)	$137h$	$137h-73k$	$73k$	$137h-2(73k)$	h odd
(10)	$137h$	$\frac{1}{2}(137h+73k)$	$\frac{1}{2}(137h-73k)$	$73k$	$h \cdot k$ odd
(11)	$137h$	$73k$	$137h-73k$	$-137h+2(73k)$	h odd
(12)	$73k$	$137h$	$-137h+73k$	$2(137)h-73k$	k odd

In Types (1) and (2), $\triangle > ((137h)(73k))^2 = (hk)^2(10001)^2 > 10^8$.

In each of Types (3), (4), (5), (6), note that $137h > 100$. This cannot be the smallest of the four numbers $m, n, m \pm n$, and must therefore be the *third* in order. The area of the triangle is greater than $(137h)^3 \cdot 1$. We need therefore consider only h satisfying $(137h)^3 < 10^8$; $h < \frac{10^{\frac{8}{3}}}{137} \approx 3.388$; in other words, $h = 1, 2, 3$. In each of these types, the smallest number is a linear function $-p\,137h + q\,73k$, where each of p and q is 1, 2 or $\frac{1}{2}$.

Note that $-p\,137h + q\,73k < \frac{1}{h}73$ since the product of the two smallest measurements cannot exceed 10001.
It follows that $0 < -p\,137h + q\,73k < \frac{1}{h}73$. From this,

$$\frac{p}{q}\cdot\frac{137}{73}\cdot h < k < \frac{p}{q}\cdot\frac{137}{73}\cdot h + \frac{1}{qh}.$$

In each of Types (7), (8), (9), (10), either n or $m - n$ is $73k$, so $\triangle > (73k)^3$. In other words, $k \le 6$. Also, from $0 < p137h - q73k < \frac{1}{k}137$, we have

$$\frac{q}{p}\cdot\frac{73}{137}\cdot k < h < \frac{q}{p}\cdot\frac{73}{137}\cdot k + \frac{1}{pk}.$$

Type	p	q	possible (h,k)	relevant (h,k)	triangle
(3)	1	1	$(1,2)$, $(2,4)$	$(1,2)$	(i)
(4)	1	1	$(1,2)$, $(2,4)$	$(1,2)$	(ii)
(5)	2	1	$(1,4)$, $(2,8)$	none	
(6)	$\frac{1}{2}$	$\frac{1}{2}$	$(1,2)$, $(1,3)$, $(2,4)$, $(3,6)$	$(1,3)$	$m = 178$, $n = 41$
(7)	1	1	$(1,1)$	none	
(8)	1	1	$(1,1)$	$(1,1)$	$m = 137$, $n = 64$
(9)	1	2	$(2,1)$	none	
(10)	$\frac{1}{2}$	$\frac{1}{2}$	$(1,1)$, $(2,1)$, $(2,2)$,	$(1,1)$	(iii)
			$(2,3)$, $(3,5)$	$(3,5)$	$m = 388$, $n = 23$

Type (11). We consider (h,k) in either of the parallelograms

$$137h - 73k = 0, \quad 137h - 73k = 100,$$
$$-137h + 2(73k) = 0, \quad -137h + 2(73k) = 464;$$
$$137h - 73k = 0, \quad 137h - 73k = 464,$$
$$-137h + 2(73k) = 0, \quad -137h + 2(73k) = 100.$$

(Note: $464 = [\sqrt[3]{10^8}]$). These are the points

$$(h,k) = (1,1), (2,3), (t,t) \text{ for } t = 2,\dots,8,$$

and we need only consider $(h,k) = (1,1)$ (since h is odd and gcd $(h,k) = 1$). But this triangle, with $m = 73$, $n = 64$, is too small; it has area 5760576.

Type (12). We consider (h,k) in either of the parallelograms

$$-137h + 73k = 0, \quad -137h + 73k = 100,$$
$$2(137)h - 73k = 0, \quad 2(137)h - 73k = 464;$$

$$-137h + 73k = 0, \quad -137h + 73k = 464,$$

$$2(137)h - 73k = 0, \quad 2(137)h - 73k = 100.$$

These are the points

$$(h, k) = (1, 2), (1, 3), (2, 7), (3, 11), (2, 4), (3, 6),$$

and we need only consider $(h, k) = (1, 3), (2, 7)$ and (3,11). But the corresponding triangles are all too big.

Remark Enlarging the primitive triangle with $m = 73, n = 64$ by the factors 2, 3, 4, we obtain non-primitive Pythagorean triangles of areas 23042304, 51845184, and 92169216 respectively. Enlargements of the primitive triangles (i), (ii), (iii) all have areas exceeding 10^8. The non-primitive triangle from $m = 137$, $n = 73$ has area 13441344.

Both the additional triangles were also found by TIM CROSS, King Edward's School, Birmingham, England; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; and RICHARD I. HESS, Rancho Palos Verdes, California, USA.

The following readers each found one of the triangles: CHARLES ASHBACHER, Cedar Rapids, Iowa, USA; SAM BAETHGE, Science Academy, Austin, Texas, USA; MANSUR BOASE, student, St. Paul's School, London, England; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; DAVID HANKIN, Hunter College Campus Schools, New York, NY, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; DAVID E. MANES, State University of New York, Oneonta, NY, USA; P. PENNING, Delft, the Netherlands; CORY PYE, student, Memorial University of Newfoundland, St. John's, Newfoundland; JOEL SCHLOSBERG, student, Hunter College High School, New York NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DIGBY SMITH, Mount Royal College, Calgary, Alberta; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece. There was one incorrect submission.

2119. [1996: 76] *Proposed by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore.*

(a) Show that for any positive integer $m \geq 3$, there is a permutation of m 1's, m 2's and m 3's such that

   (i) no block of consecutive terms of the permutation (other than the entire permutation) contains equal numbers of 1's, 2's and 3's; and

(ii) there is no block of m consecutive terms of the permutation which are all equal.

(b) For $m = 3$, how many such permutations are there?

Solution by P. Penning, Delft, the Netherlands.

(b) I found two:

$$aabbcbcca \qquad \text{and} \qquad abbcbccaa,$$

where a, b, c is any permutation of $1, 2, 3$. (In fact these are mirror-images of one another.) Thus there are $3! \cdot 2 = 12$ such permutations of 1's, 2's and 3's.

(a) To generate a solution for $m > 3$, start with three blocks of m equal terms, and to make it satisfy condition (ii) let the last term of each block change places with the first term of the next block (and do the same with the first and last block):

$$1\,\overbrace{22\ldots2}^{m-2}\,3\;2\,\overbrace{33\ldots3}^{m-2}\,1\;3\,\overbrace{11\ldots1}^{m-2}\,2.$$

Note that for $m = 3$ this permutation violates condition (i). [It is easy to check that it works whenever $m > 3$.—*Ed.*] Also note that if the permutation is considered as a cycle then there are solutions only if $m > 3$.

Also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; MANSUR BOASE, student, St. Paul's School, London, England; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; DAVID E. MANES, State University of New York, Oneonta, NY, USA; JOEL SCHLOSBERG, student, Hunter College High School, New York, NY, USA; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; and the proposer. Part (b) *only was solved by CHARLES ASHBACHER, Cedar Rapids, Iowa, USA.*

Note that Penning's solution shows that for $m > 3$, condition (ii) *can be strengthened just a teensy bit to:*

(ii)′ *there is no block of $m - 1$ consecutive terms of the permutation which are all equal.*

However, there is an even better result. For any $m \geq 3$ there is a permutation of m 1's, m 2's and m 3's which satisfies condition (i) *and also*

(ii)* *there is no block of three consecutive terms of the permutation which are all equal.*

(*Thanks to expert colleague James Currie for suggesting examples of such permutations, which readers might enjoy finding for themselves!*)

2120. [1996: 76] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Let $A_1A_3A_5$ and $A_2A_4A_6$ be non-degenerate triangles in the plane. For

$i = 1, \ldots, 6$ let ℓ_i be the perpendicular from A_i to line $A_{i-1}A_{i+1}$ (where of course $A_0 = A_6$ and $A_7 = A_1$). If ℓ_1, ℓ_3, ℓ_5 concur, prove that ℓ_2, ℓ_4, ℓ_6 also concur.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

Let the vector position of the vertex A_i relative to the point of concurrency of ℓ_1, ℓ_3, ℓ_5 be $\vec{a_i}$. Then $\vec{a_1}\cdot(\vec{a_2}-\vec{a_6}) = \vec{a_3}\cdot(\vec{a_4}-\vec{a_2}) = \vec{a_5}\cdot(\vec{a_6}-\vec{a_4}) = 0$. In particular, the sum is also zero, so

$$\vec{a_2}\cdot(\vec{a_1}-\vec{a_3}) + \vec{a_4}\cdot(\vec{a_3}-\vec{a_5}) + \vec{a_6}\cdot(\vec{a_5}-\vec{a_1}) = 0. \qquad (1)$$

Now suppose that the perpendicular from A_2 to A_1A_3 meets the perpendicular from A_4 to A_3A_5 at B, with position vector $\vec{b}$. (These perpendiculars cannot be parallel.) Then

$$(\vec{b}-\vec{a_2})\cdot(\vec{a_1}-\vec{a_3}) = 0 \quad \text{and} \quad (\vec{b}-\vec{a_4})\cdot(\vec{a_3}-\vec{a_5}) = 0.$$

Adding gives $\vec{b}\cdot(\vec{a_1}-\vec{a_5}) = \vec{a_2}\cdot(\vec{a_1}-\vec{a_3}) + \vec{a_4}(\vec{a_3}-\vec{a_5})$, which, by (1), is equal to $-\vec{a_6}\cdot(\vec{a_5}-\vec{a_1})$.
Hence $(\vec{b}-\vec{a_6})\cdot(\vec{a_1}-\vec{a_5}) = 0$, which means that $BA_6 \perp A_1A_5$.

Also solved by CLAUDIO ARCONCHER, Jundiaí, Brazil; FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, the Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; MELETIS VASILIOU, Elefsis, Greece; and the proposer.

Problem 2120 can be found in many reference texts. Dan Pedoe provides two proofs in his Geometry, A comprehensive Course (*Dover, 1998*). *Clerk Maxwell's remarkable treatment of the problem, reducing it to an observation about the radical axes of appropriate circle, appears in §28.4, page 115. Earlier, in §6.1, pages 35-37, Pedoe uses barycentric coordinates for a proof of a closely related problem involving* reciprocal triangles. *He comments that whole books have been written on the subject of reciprocal figures.*

Seimiya uses Steiner's theorem in his solutions (Jacob Steiner, *Gesammelte Werke* I, p. 189): A necessary and sufficient condition in order for ℓ_1, ℓ_3, ℓ_5 to be concurrent is

$$A_1A_2^2 + A_3A_4^2 + A_5A_6^2 = A_2A_3^2 + A_4A_5^2 + A_6A_1^2.$$

Since (by permuting the subscripts) this is also a necessary and sufficient condition for ℓ_2, ℓ_4, ℓ_6 to concur, our problem follows immediately. Arconcher refers to this condition as Carnot's Theorem, *but he provides no* *reference. Ardila discovered the theorem for himself and showed that it follows fairly quickly from Pythagoras' Theorem, so maybe we should refer to it as the Pythagoras-Ardila-Steiner-Carnot Theorem. Seimiya added that the problem may also be found in* Aref and Wernick, Problems and Solutions in Euclidean Geometry (*Dover*), *p. 55 problems 2.32, He includes two of his interesting generalizations that have appeared in Japanese, one in 1928 and one in 1967.*

Finally, Heuver found the problem as Ex. 7 of §54 of George Salmon's A Treatise on Conic Sections, *6th ed. His solution exploits Salmon's imaginative use of pencils of lines in Cartesian coordinates.*

2121. [1996: 76] *Proposed by Krzysztof Chelmiński, Technische Hochschule Darmstadt, Germany; and Waldemar Pompe, student, University of Warsaw, Poland.*

Let $k \geq 2$ be an integer. The sequence (x_n) is defined by $x_0 = x_1 = 1$ and

$$x_{n+1} = \frac{x_n^k + 1}{x_{n-1}} \qquad \text{for} \quad n \geq 1.$$

(a) Prove that for each positive integer $k \geq 2$ the sequence (x_n) is a sequence of integers.

(b) If $k = 2$, show that $x_{n+1} = 3x_n - x_{n-1}$ for $n \geq 1$.

(c)* Note that for $k = 2$, part (a) follows immediately from (b). Is there an analogous recurrence relation to the one in (b), not necessarily linear, which would give an immediate proof of (a) for $k \geq 3$?

I *Solution* ((*a*) *and* (*b*) *only*) *by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

(a) We immediately get $x_2 = 2$ and $x_3 = 2^k + 1$. Now we use mathematical induction for the proof. Assume that $x_0, x_1, \ldots, x_n$ are all natural numbers. We must show that $x_{n+1} \in \mathbb{N}$. First we note that since $x_{n-2} \cdot x_n = x_{n-1}^k + 1$ it follows that x_{n-2} and x_{n-1} are relatively prime. Using $x_n = (x_{n-1}^k + 1)/x_{n-2}$ we infer that

$$x_{n+1} = \frac{x_n^k + 1}{x_{n-1}} = \frac{(x_{n-1}^k + 1)^k + x_{n-2}^k}{x_{n-2}^k x_{n-1}}.$$

Thus obviously x_{n-2}^k divides $N = (x_{n-1}^k + 1)^k + x_{n-2}^k$ since x_n is a natural number. Furthermore, modulo x_{n-1} we have:

$$N \equiv 1 + x_{n-2}^k = x_{n-3} \cdot x_{n-1} \equiv 0.$$

That is, x_{n-1} also divides N and we are done.

(b) Now,

$$x_{n+1} = \frac{x_n^2+1}{x_{n-1}} \iff x_{n-1}\cdot x_{n+1} - x_n^2 = 1.$$

That is, the sequence $\{y_n\} = \{x_{n-1}\cdot x_{n+1} - x_n^2\}$ is constant. Setting $y_{n+1} = y_n$ we have

$$\begin{aligned} x_n \cdot x_{n+2} - x_{n+1}^2 &= x_{n-1}\cdot x_{n+1} - x_n^2 \\ \iff \quad x_n(x_n + x_{n+2}) &= x_{n+1}(x_{n-1}+x_{n+1}) \\ \iff \quad \frac{x_n + x_{n+2}}{x_{n+1}} &= \frac{x_{n-1}+x_{n+1}}{x_n}. \end{aligned}$$

That is, the sequence $\{z_n\} = \{(x_{n-1}+x_{n+1})/x_n\}$ is constant. From $z_1 = 3$ we get $(x_{n-1}+x_{n+1})/x_n = 3$; that is, $x_{n+1} = 3x_n - x_{n-1}$ for all $n \geq 1$, as claimed.

II *Solution by Christopher J. Bradley, Clifton College, Bristol, UK.*

We first establish the following lemma:

Lemma. Suppose a and b have highest common factor 1 and $(a^k+b^k+1)/ab$ is an integer (where k is an integer, $k \geq 2$), then
(i) $c = (b^k+1)/a$ is an integer;
(ii) $(b^k+c^k+1)/bc$ is an integer;
(iii) b and c have highest common factor 1.
Proof. Let $a^k+b^k+1 = \lambda(a,b)ab$ where $\lambda(a,b)$ is an integer. Then since a divides $\lambda(a,b)ab - a^k$ it also divides b^k+1; that is, $(b^k+1)/a = c$ is an integer, which proves (i). Also

$$\frac{b^k+c^k+1}{bc} = \frac{ac+c^k}{bc} = \frac{a+c^{k-1}}{b} = \frac{a^k+(b^k+1)^{k-1}}{a^{k-1}b}.$$

Now $b^k+1 = ac$ has a as a factor, so $a^k+(b^k+1)^{k-1}$ is divisible by a^{k-1} and hence the numerator is divisible by a^{k-1}. Also a^k+1 is divisible by b (from part (i)), so multiplying out the numerator by the Binomial Theorem we see that the numerator is divisible by b. But a and b have highest common factor 1, so the numerator is divisible by $a^{k-1}b$. Hence

$$\lambda(b,c) = \frac{b^k+c^k+1}{bc}$$

is an integer, which proves (ii). Since $b^k+c^k+1 = \lambda(b,c)bc$, if b and c have a common factor h, then h divides $b^k+c^k-\lambda(b,c)bc$; that is, h divides 1. Hence b and c have highest common factor 1, which proves (iii), and the lemma is proved.

We claim that the recurrence relation sought is

$$x_{n+1} = \lambda(x_{n-1},x_n)x_n - x_{n-1}^{k-1} \qquad (*)$$

with $\lambda(x_{n-1}, x_n) = (x_n^k + x_{n-1}^k + 1)/(x_n x_{n-1})$, which is but one step away from $x_{n+1} = (x_n^k + 1)/x_{n-1}$. That $\lambda(x_{n-1}, x_n)$ is always an integer follows by induction, using the lemma, with the start of induction satisfied since $x_0 = x_1 = 1$ have highest common factor 1 and $\lambda(x_0, x_1) = 3$ is an integer. Relation $(*)$ then establishes that $\{x_n\}$ is a sequence of integers. This gives us parts (c) and (a). The term $(x_n^2 + x_{n-1}^2 + 1)/(x_n x_{n-1})$ is known from work on alternate terms of the Fibonacci sequence to be equal to 3 for all $n \geq 2$ and is also 3 for $n = 1$, which proves (b).

Parts (a) and (b) together were also solved by FEDERICO ARDILA, student, Massachusetts Institute of Technology, Cambridge, Massachusetts, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; and the proposers. Part (b) alone was solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and TIM CROSS, King Edward's School, Birmingham, England. There was one incorrect solution.

JANOUS comments that this was posed as a problem of the final round of the third Austrian Mathematical Olympiad held in 1972, and refers the interested reader to the book "Österreichische Matematik-Olympiaden" 1970-1989, *G. Baron & E. Windischbacher, Innsbruck 1990, problem 42.*

2122. [1996: 77] *Proposed by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.*

Little Sam is a unique child and his math marks show it. On four tests this year his scores out of 100 were all two-digit numbers made up of eight different non-zero digits. What's more, the average of these scores is the same as the average if each score is reversed (so 94 becomes 49, for example), and this average is an integer none of whose digits is equal to any of the digits in the scores. What is Sam's average?

I. *Solution by Mansur Boase, student, St. Paul's School, London, England.*

Let the four marks be $10a + b$, $10c + d$, $10e + f$ and $10g + h$. Then

$$10a + b + 10c + d + 10e + f + 10g + h = 10b + a + 10d + c + 10f + e + 10h + g,$$

so

$$9(a + c + e + g) = 9(b + d + f + h)$$

and

$$a + c + e + g = b + d + f + h. \qquad (1)$$

The average of the four marks is therefore

$$\frac{10a + b + 10b + a + 10c + d + 10d + c + 10e + f + 10f + e + 10g + h + 10h + g}{4 \cdot 2}$$

$$= \frac{11(a + b + c + d + e + f + g + h)}{8}. \qquad (2)$$

The average must consist only of k's and 0's, the digits *not* equal to a, b, c, d, e, f, g or h. Thus (2) is

$$\frac{11(1+2+\cdots+9-k)}{8} = \frac{11(45-k)}{8},$$

and so $8 \mid (45-k)$. Since $k \leq 9$, the only solution for k is $k = 5$, and therefore Sam's average is $11(40)/8 = 55$. An example of four such marks is $98, 76, 34, 12$.

II. *Solution by Tara McCabe, student, Mount Allison University, Sackville, New Brunswick.*

[*Editor's note*: McCabe first obtained equation (1) as above, using the same notation.] Since $a, \ldots, h$ are all different and non-zero, their total must lie between $1+\cdots+8 = 36$ and $2+\cdots+9 = 44$ inclusively. From (1),

$$\frac{36}{2} = 18 \leq a+c+e+g \leq 22 = \frac{44}{2}. \tag{3}$$

Letting the average be the two digit number xy,

$$\frac{10a+b+10c+d+10e+f+10g+h}{4} = 10x+y,$$

which by (1) means

$$11(a+c+e+g) = 4(10x+y).$$

Consequently, $a+c+e+g$ must be divisible by 4, and from (3), $a+c+e+g = 20$. Therefore $11(20) = 4(10x+y)$, and so $55 = 10x+y$. Little Sam has an average of 55.

A natural addition to this problem is to try to find Little Sam's four test scores ab, cd, ef and gh. We know that $a+c+e+g = 20$, and the problem becomes: how many sets of test scores are possible?

First, choose four digits from $\{1,2,3,4,6,7,8,9\}$ to be a, c, e and g (0 and 5 are not allowed). Notice that for a sum of 20 to be possible, two digits must be chosen from $\{1,2,3,4\}$ and two from $\{6,7,8,9\}$. There are $\binom{4}{2} = 6$ ways to choose two digits from $\{1,2,3,4\}$. The sum S_1 of these two digits is such that $3 \leq S_1 \leq 7$, and the only sum that can occur twice is 5. Similarly, there are 6 ways to choose two digits from $\{6,7,8,9\}$, the sum S_2 of these two digits is such that $13 \leq S_2 \leq 17$, and the only sum that can occur twice is 15. There are $2 \times 2 = 4$ combinations when $S_1 = 5$ (and $S_2 = 15$) and one combination for each other sum S_1. Therefore, there are eight ways to choose four digits from $\{1,2,3,4,6,7,8,9\}$ such that the sum is 20.

Once a, c, e and g are chosen, b, d, f and h are simply the four remaining digits. There are $4! = 24$ different ways to assign these four digits to be b, d, f and h. Therefore there are $8 \times 24 = 192$ different sets of four test scores possible for Little Sam. He's not such a unique child after all!

Also solved by SAM BAETHGE, Nordheim, Texas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; TIM CROSS, King Edward's School, Birmingham, UK; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; NOEL EVANS and CHARLES DIMINNIE, Angelo State University, San Angelo, Texas, USA; FLORIAN HERZIG, student, Perchtoldsdorf, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KATHLEEN E. LEWIS, SUNY Oswego, Oswego, New York, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; JOHN GRANT MCLOUGHLIN, Okanagan University College, Kelowna, B.C.; P. PENNING, Delft, the Netherlands; CORY PYE, student, Memorial University of Newfoundland, St. John's, Newfoundland; JOEL SCHLOSBERG, student, Hunter College High School, New York, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer. Three other solvers did not prove that 55 *is the only possible answer. There was also one incorrect answer sent in.*

At the end of his solution, the proposer asks for the number of sets of test scores satisfying the problem, but only McCabe was able to read his mind and answer this too!

2123. [1996: 77] *Proposed by Sydney Bulman–Fleming and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

It is known (e.g., exercise 23, page 78 of Kenneth H. Rosen's *Elementary Number Theory and its Applications*, Third Edition) that every natural number greater than 6 is the sum of two relatively prime integers, each greater than 1. Find all natural numbers which can be expressed as the sum of *three* pairwise relatively prime integers, each greater than 1.

Solution by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.

Since every number greater than 6 is the sum of two relatively prime integers, then each even number greater than 6 can be expressed as the sum of of two relatively prime **odd** numbers. If we add 2 to each of these expressions we see that every even number greater than 8 can be expressed as the sum of three pairwise relatively prime integers (two relatively prime odd integers and 2).

All odd numbers can be expressed in the form $18N + k$, where N is a non-negative integer and k is an odd number less than 18. Note that:

$$\begin{aligned}
18N + 1 &= (6N - 3) + (6N - 1) + (6N + 5) \quad \text{for } N \geq 1 \\
18N + 3 &= (6N - 1) + (6N + 1) + (6N + 3) \quad \text{for } N \geq 1 \\
18N + 5 &= (6N - 1) + (6N + 1) + (6N + 5) \quad \text{for } N \geq 1
\end{aligned}$$

$$\begin{aligned}
18N+7 &= (6N-1)+(6N+3)+(6N+5) && \text{for } N \geq 1\\
18N+9 &= (6N+1)+(6N+3)+(6N+5) && \text{for } N \geq 1\\
18N+11 &= (6N+1)+(6N+3)+(6N+7) && \text{for } N \geq 1\\
18N+13 &= (6N+1)+(6N+5)+(6N+7) && \text{for } N \geq 1\\
18N+15 &= (6N+3)+(6N+5)+(6N+7) && \text{for } N \geq 0\\
18N+17 &= (6N+1)+(6N+7)+(6N+9) && \text{for } N \geq 1
\end{aligned}$$

Clearly each of the pairs of terms in each expression is relatively prime, since if there is a number which divides each term in a pair it must divide the difference. The restriction on N ensures that each term is greater than 1. Putting all this together shows that the numbers which can be so written are 10, 12, 14, 15, 16, and all the natural numbers greater than 17.

[Ed: it is easily verified that all other integers do not have the desired property.]

Also solved by RICHARD I. HESS, Rancho Palos Verdes, California, USA; PETER HURTHIG, Columbia College, Burnaby, BC; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL JOSEPHY, Universidad de Costa Rica, San José, Costa Rica; KEE-WAI LAU, Hong Kong; DAVID E. MANES, State University of New York, Oneonta, NY, USA; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, Newfoundland; P. PENNING, Delft, the Netherlands; GOTTFRIED PERZ, Pestalozzi-gymnasium, Graz, Austria; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; and the proposers. There was 1 incorrect submission.

Most solvers used **mod** 18 *or* **mod** 12 *arithmetic to handle the odd values of* n. *MANES pointed out that both this problem and the problem in Rosen's book can be found in Sierpiński's book,* "250 Problems in Elementary Number Theory", *American Elsevier, New York, 1970, (problems 47 and 48) where they are both solved.*