

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk () after a number indicates that a problem was submitted without a solution.*

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

*To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}'' \times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than **1 October 1997**. They may also be sent by email to crux-editors@cms.math.ca. (It would be appreciated if email proposals and solutions were written in \LaTeX). Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication.*

2214. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $n \geq 2$ be a natural number. Show that there exists a constant $C = C(n)$ such that for all real $x_1, \dots, x_n \geq 0$ we have

$$\sum_{k=1}^n \sqrt{x_k} \leq \sqrt{\prod_{k=1}^n (x_k + C)}.$$

Determine the minimum $C(n)$ for some values of n .

[For example, $C(2) = 1$.]

2215* *Proposed by Theodore Chronis, student, Aristotle University of Thessaloniki, Greece.*

Let P be a point inside a triangle ABC . It is known how to determine P such that $PA + PB + PC$ is a minimum (known as Fermat's Problem for Torricelli).

Determine P such that $PA + PB + PC$ is a maximum.

2216. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that $\lambda \geq 1$ is a natural number.

1. Determine the set of all λ 's such that the diophantine equation $x^\lambda + y^2 = z^2$ has infinitely many solutions.
- 2.* For any such λ , determine all solutions of this equation.

2217. *Proposed by Bill Sands, University of Calgary, Calgary, Alberta.*

- (a) Prove that for every sufficiently large positive integer n , there are arithmetic progressions a_1, a_2, a_3 and b_1, b_2, b_3 of positive integers such that $n = a_1b_1 + a_2b_2 + a_3b_3$.
- (b) What happens if we require $a_1 = b_1 = 1$?

(This is a variation of problem 3 of the 1995/96 Alberta High School Mathematics Competition, Part II, which will appear in a future Skoliad Corner.)

2218. *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

Suppose that a, b, c are positive real numbers and that

$$abc = (a + b - c)(b + c - a)(c + a - b).$$

Clearly $a = b = c$ is a solution. Determine all others.

2219. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Show that there are an infinite number of solutions of the simultaneous equations:

$$\begin{aligned}x^2 - 1 &= (u + 1)(v - 1) \\y^2 - 1 &= (u - 1)(v + 1)\end{aligned}$$

with x, y, u, v positive integers and $x \neq y$.

2220. *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

Let V be the set of an icosahedron's twelve vertices, which can be partitioned into four classes of three vertices, each one in such a way that the three selected vertices of each class belong to the same face.

How many ways can this be done?

2221. Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.

Find all members of the sequence $a_n = 3^{2n-1} + 2^{n-1}$, ($n \in \mathbb{N}$) which are the squares of any positive integer.

2222. Proposed by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.

Find the value of the continued root:

$$\sqrt{4 + 27\sqrt{4 + 29\sqrt{4 + 31\sqrt{4 + 33\sqrt{\dots}}}}}$$

NOTE: This was inspired by the problems in chapter 26 “*Ramanujan, Infinity and the Majesty of the Quattuordecillion*”, pp 193–195, in “*Keys to Infinity*” by Clifford A. Pickover, John Wiley and Sons, 1995.

2223. Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.

We are given a bag with n identical bolts and n identical nuts, which are to be used to secure the n holes of a gadget.

The $2n$ pieces are drawn from the bag at random one by one. Throughout the draw, bolts and nuts are screwed together in the holes, but if the number of bolts exceeds the number of available nuts, the bolt is put into a hole until one obtains a nut, whereas if the number of nuts exceed the number of bolts, the nuts are piled up, one on top of the other, until one obtains a bolt.

Let L denote the discrete random variable which measures the height of the pile of nuts.

Find $E[L] + E[L]^2$.

2224. Proposed by Waldemar Pompe, student, University of Warsaw, Poland.

Point P lies inside triangle ABC . Triangle BCD is erected outwardly on side BC such that $\angle BCD = \angle ACP$ and $\angle CBD = \angle ABC$. Prove that if the area of quadrilateral $PBDC$ is equal to the area of triangle ABC , then triangles ACP and BCD are similar.

2225. Proposed by Kenneth Kam Chiu Ko, Mississauga, Ontario.

(a) For any positive integer n , prove that there exists a unique n -digit number

N such that:

- (i) N is formed with only digits 1 and 2; and
- (ii) N is divisible by 2^n .

(b) Can digits “1” and “2” in (a) be replaced by any other digits?