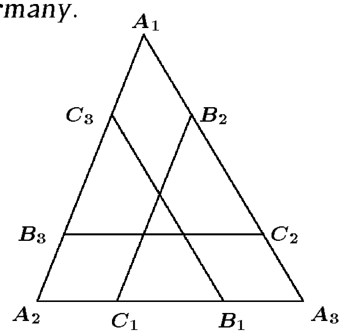
SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

1987. [1994: 250; 1995: 283-285] *Proposed by Herbert Gülicher, Westfalische Wilhelms-Universität, Münster, Germany.*

In the figure, $B_2C_1 \| A_1A_2$, $B_3C_2 \| A_2A_3$ and $B_1C_3 \| A_3A_1$. Prove that B_2C_1, B_3C_2 and B_1C_3 are concurrent if and only if

$$\frac{A_1C_3}{C_3B_3} \cdot \frac{A_2C_1}{C_1B_1} \cdot \frac{A_3C_2}{C_2B_2} = 1.$$



II. *Solution by anonymous* (spotted on the internet by Waldemar Pompe, student, University of Warsaw, Poland).

Just apply Ceva's theorem to $\triangle B_1B_2B_3$.

Editor's comments by Chris Fisher.

1. Note that if we permit B_i to lie on the side of the triangle extended beyond A_{i-1} or A_{i+1} (as Bradley did in solution I) then Ceva's theorem fails to apply exactly when B_1, B_2, B_3 are collinear. Thus the alternative condition in the middle of page 284 (namely $u = pq/(p+q-1)$ in Bradley's notation) applies if and only if $B_1 \in B_2B_3$. As a consequence, Bradley's extended version of the problem can be restated as

$$\frac{A_1A_3}{C_3B_3} \cdot \frac{A_2C_1}{C_1B_1} \cdot \frac{A_3C_2}{C_2B_2} = 1 \text{ if and only if either}$$

$$B_2C_1, B_3C_2, \text{ and } B_1C_3 \text{ are concurrent or } B_1 \in B_2B_3.$$

2. It makes an amusing exercise to prove directly that $B_1 \in B_2B_3$ *if and only if* $C_1 \in C_2C_3$ (in the notation of Gülicher's problem).

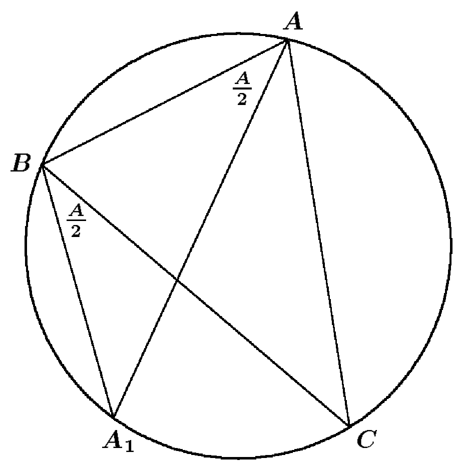
2067. [1995: 235] *Proposed by Moshe Stupel and Victor Oxman, Pedagogical Religious Girls' College "Shanan", Haifa, Israel.*

Triangle ABC is inscribed in a circle Γ. Let AA_1, BB_1 and CC_1 be the bisectors of angles A, B and C, with A_1, B_1 and C_1 on Γ. Prove that the perimeter of the triangle is equal to

$$AA_1 \cos\left(\tfrac{A}{2}\right) + BB_1 \cos\left(\tfrac{B}{2}\right) + CC_1 \cos\left(\tfrac{C}{2}\right) \ .$$

Solution by Miguel Angel Cabezón Ochoa, Logroño, Spain.

Let the internal bisector of $\angle BAC$ meet the circumcircle of $\triangle ABC$ again in A_1. Let R be the radius of the circumcircle.



From $\triangle BAA_1$, we have $\dfrac{AA_1}{\sin\left(B + \frac{A}{2}\right)} = 2R$, so that

$$AA_1 = 2R\,\sin\left(B + \tfrac{A}{2}\right).$$

Thus

$$\begin{aligned} AA_1\,\cos\left(\tfrac{A}{2}\right) &= 2R\,\sin\left(B + \tfrac{A}{2}\right)\cos\left(\tfrac{A}{2}\right) \\ &= R\left(\sin(B + A) + \sin B\right) \\ &= R\sin C + R\sin B \ = \ \frac{b+c}{2}. \end{aligned}$$

Similarly

$$BB_1\,\cos\left(\tfrac{B}{2}\right) \ = \ \frac{c+a}{2}; \qquad CC_1\,\cos\left(\tfrac{C}{2}\right) \ = \ \frac{a+b}{2}.$$

Therefore

$$AA_1\,\cos\left(\tfrac{A}{2}\right) + BB_1\,\cos\left(\tfrac{B}{2}\right) + CC_1\,\cos\left(\tfrac{C}{2}\right) \ = \ \frac{b+c}{2} + \frac{c+a}{2} + \frac{a+b}{2}.$$

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, Berlin, Germany; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SABIN CAUTIS, student, Earl Haig Secondary School, North York, Ontario; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; HIDETOSI FUKAGAWA, Gifu, Japan; TOBY GEE, student, the John of Gaunt School, Trowbridge, England; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; MITKO CHRISTOV KUNCHEV, Rousse, Bulgaria; VEDULA N. MURTY, Andhra University, Visakhapatnam, India; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; WALDEMAR POMPE, student, University of Warsaw, Poland; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; ASHSIH KR. SINGH, student, Kanpur, India; PANOS E. TSAOUSSOGLOU, Athens, Greece; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA; and the proposers. There was one anonymous solution.

Amengual Covas points out that this problem has already appeared in print, in **A Treatise on Plane Trigonometry** *by E.W. Hobson, Cambridge University Press, 2nd Edition, 1897, Example 2 on page 194.*

2069. [1995: 235] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

M is a variable point of side BC of triangle ABC. A line through M intersects the lines AB in K and AC in L so that M is the mid-point of segment KL. Point K' is such that $ALKK'$ is a parallelogram. Determine the locus of K' as M moves on segment BC.

I. *Essentially the same solution by Gottfried Perz, Pestalozzigymnasium, Graz, Austria; Waldemar Pompe, student, University of Warsaw, Poland; and Toshio Seimiya, Kawasaki, Japan.*

Suppose that the dilatation with centre A and ratio two transforms the points B, C and M to B', C' and N respectively. Since M is the mid-point of both KL and AN, we have that $AKNL$ is a parallelogram. By definition, $ALKK'$ is a parallelogram as well, so that K is the mid-point of $K'N$; also, $K'N$ and AC' are parallel. Define C'' to be the point such that A is the mid-point of $C'C''$. We conclude that, as M varies on BC, N varies on $B'C'$, so that the position of K' must vary on the segment $B'C''$.

Editor's comments:

1. Note that this solution provides an explicit construction of the points K and L.

2. Smeenk points out that K' helps in finding the position of M for which the corresponding segment KL has minimum length. ($KL = AK;$, and AK' has its minimum length when K' is the foot of the perpendicular from A to the line $B'C''$.)

II. *Essentially the same solution by Tim Cross, King Edward's School, Birmingham, England; Hidetosi Fukagawa, Gifu, Japan; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Mitko Christov Kunchev, Rousse, Bulgaria; Ashsih Kr. Singh, student, Kanpur, India.*

Take A to be the origin and set the vectors $\overrightarrow{AB} = \overrightarrow{B}$, etc. Then $\overrightarrow{M} = t\overrightarrow{B} + (1-t)\overrightarrow{C}$, where t varies from 0 to 1 as M moves from C to B. Suppose that $\overrightarrow{K} = \kappa\overrightarrow{B}$ and that $\overrightarrow{L} = \lambda\overrightarrow{C}$. Because M is the mid-point of KL, we have

$$\frac{\kappa\overrightarrow{B} + \lambda\overrightarrow{C}}{2} = t\overrightarrow{B} + (1-t)\overrightarrow{C},$$

so that $\kappa = 2t$ and $\lambda = 2(1-t)$ (since $\overrightarrow{B}$ and $\overrightarrow{C}$ are linearly independent). Since $ALKK'$ is a parallelogram, it follows that

$$\begin{aligned} \overrightarrow{K'} &= \overrightarrow{K} - \overrightarrow{L} \\ &= t(2\overrightarrow{B} + (1-t)(-2\overrightarrow{C}). \end{aligned}$$

Thus the locus of K' is the segment joining $-2\overrightarrow{C}$ (where $t = 0$) to $2\overrightarrow{B}$ where $t = 1$).

Editor's comment: The algebra makes clear that it is not necessary to restrict M to the segment BC; as t ranges over the real numbers, K' moves along its line, while M moves along the line BC.

Also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; CRISTÓBAL SÁNCHEZ–RUBIO, I.B. Penyagolosa, Castellón, Spain; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer. One anonymous solution was received - see note at the start of this section. There was one incorrect solution received.

2070. [1995: 236] *Proposed by Joaquín Gómez Rey, IES Luis Buñuel, Alcorcón, Madrid, Spain.*

For which positive integers n is the Catalan number

$$\frac{1}{n+1}\binom{2n}{n}$$

odd?

I *Virtually identical solutions by Toby Gee, student, the John of Gaunt School, Trowbridge, England; Douglas E. Jackson, Eastern New Mexico University, Portales, New Mexico, USA; Thomas Leong, Staten Island, NY, USA; Andy Liu, University of Alberta, Edmonton, Alberta; Waldemar Pompe, student, University of Warsaw, Poland; and the proposer.*

It is well-known that the n^{th} Catalan number, $C_n = \frac{1}{n+1}\binom{2n}{n}$, satisfies the recurrence relation

$$C_{n+1} = C_0 C_n + C_1 C_{n-1} + \ldots + C_n C_0 \qquad n \geq 0.$$

We show that C_n is odd if and only if $n = 2^k - 1$ for some integer $k \geq 0$. We use induction on n.

Since $C_0 = 1$, the assertion is true for $n = 0$. Assume that among the numbers C_0, C_1, ..., C_n, only those with n of the form $2^k - 1$ are odd. If $n+1$ is even, then

$$C_{n+1} = 2 \sum_{k=0}^{(n-1)/2} C_k C_{n-k},$$

which is even.

On the other hand, if $n+1$ is odd, then

$$C_{n+1} = 2 \sum_{k=0}^{(n-1)/2} C_k C_{n-k} \;+\; C_{n/2}^2,$$

showing that C_{n+1} is odd if and only if $C_{n/2}$ is odd. But $C_{n/2}$ is odd if and only if $\frac{n}{2} = 2^k - 1$ for some integer $k \geq 0$. Thus C_{n+1} is odd if and only if $n + 1 = 2(2^k - 1) + 1 = 2^{k+1} - 1$, which completes the proof.

II *Solution by Hans Engelhaupt, Franz–Ludwig–Gymnasium, Bamberg, Germany.*

Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ denote the n^{th} Catalan number, $n = 1, 2, 3, \ldots$, and let $F(n)$ denote the highest power of 2 that divides n. Then it is well known that

$$F(n!) \;=\; \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n}{8} \right\rfloor + \ldots .$$

From this, it is easy to see that $F(n!) \leq n - 1$, with equality if and only if $n = 2^k$ for some integer $k \geq 0$. [More generally, it is known, and easy to show, that if p is prime and $n = a_r a_{r-1} \ldots a_1 a_0$ is the base-p representation of n, then

$$h = \frac{n - \sum_{i=0}^{r} a_i}{p-1},$$

where $p^h \| n!$ [See, for example, Theorem 2.30 in *Elementary Introduction to Number Theory*, 3rd edition, by Calvin T. Long — Ed.]

Since

$$C_n = \frac{(2n)!}{(n+1)!\,n!} = \frac{2^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)}{(n+1)!},$$

C_n is odd if and only if $F((n+1)!) = n$, which is true if and only if $n+1 = 2^k$ or $n = 2^k - 1$ for some integer $k \geq 1$. We can also allow $k = 0$ since $C_0 = 1$ is odd.

Solutions similar or equivalent to II *above were submitted by ŠEFKET ARSLANAGIĆ, Berlin, Germany; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; SABIN CAUTIS, student, Earl Haig Secondary School, North York, Ontario; EMERIC DEUTSCH, Brooklyn, NY, USA; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; SOLOMON W. GOLOMB, Univ of Southern California, Los Angeles, CA, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOE HOWARD, New Mexico Highlands University, Las Vegas, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; HARRY SEDINGER, St. Bonaventure University, St. Bonaventure, NY, USA; KEE-WAI LAU, Hong Kong; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; SKIDMORE COLLEGE PROBLEM GROUP Saratoga Springs, NY, USA; CHRIS WILDHAGEN, Rotterdam, the Netherlands; KENNETH M. WILKE, Topeka, Kansas, USA; PAUL YIU, Florida Atlantic University, Boca Raton, Florida, USA (two solutions).*

Several solvers pointed out the well-known fact that the highest power of 2 *that divides* $\binom{2n}{n}$ *is equal to the number of ones in the binary representation of* n. *This is, of course, an immediate consequence of the more general fact mentioned in the editor's comment in solution* II *above.*

This problem is certainly not new. Deutsch supplied the reference: "The Parity of the Catalan Numbers via Lattice Paths" *by Ömer Eğecioğlu,* Fibonacci Quarterly, (**21**), *1983, 65–66. Godin gave the reference:* "Time Travel and other Mathematical Bewilderments" *by Martin Gardner, W.H. Freeman and Co. In fact, the result of this problem was also mentioned by Martin Gardner in his article* "Mathematical Games; Catalan number: an integer sequence that materializes in unexpected places", Scientific American, **234**(*6*), *1976, 120–125, but he did not indicate any proof.*

2071. [1995: 277] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

P is an interior point of an equilateral triangle ABC so that $PB \neq PC$, and BP and CP meet AC and AB at D and E respectively. Suppose that $PB : PC = AD : AE$. Find angle BPC.

Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

We claim that $\angle BPC = 120°$. Let the side of the triangle be 1. Further, let $x = AE, y = AD, \eta = \angle BCE$ and $\varphi = \angle CBD$. By the law of cosines (for $\triangle ABD$ and $\triangle AEC$ respectively) we get (since $\cos 60^\circ = 1/2$): $BD = \sqrt{1+y^2-y}$ and $CE = \sqrt{1+x^2-x}$. Via the law of sines applied to

$$\triangle BCD: \quad \sin\varphi = \frac{1-y}{BD}\sin 60^\circ = \frac{1-y}{\sqrt{1+y^2-y}} \cdot \frac{\sqrt{3}}{2}$$

$$\triangle EBC: \quad \sin\eta = \frac{1-x}{CE}\sin 60^\circ = \frac{1-x}{\sqrt{1+x^2-x}} \cdot \frac{\sqrt{3}}{2}$$

$$\triangle BCP: \quad \frac{\sin\varphi}{\sin\eta} = \frac{PC}{PB} = \frac{x}{y} \text{ [by assumption for the problem!]}$$

Hence, $\frac{y-y^2}{\sqrt{1+y^2-y}} = \frac{x-x^2}{\sqrt{1+x^2-x}}$, that is (with $X = x - x^2$ and $Y = y - y^2$)

$$\frac{X}{\sqrt{1-X}} = \frac{Y}{\sqrt{1-Y}}. \tag{1}$$

Here, since $0 < x, y < 1,\ 0 < X, Y \le 1/4$.

Now, (1) $\Longleftrightarrow X^2(1-Y) = Y^2(1-X) \Longleftrightarrow (X-Y)(XY-X-Y) = 0$. Thus, two cases are to be considered.

(a) $XY - X - Y = 0 \Longleftrightarrow (X-1)(Y-1) = 1$, which is impossible.

(b) $X = Y \Longleftrightarrow x - x^2 = y - y^2 \Longleftrightarrow (y-x)(x+y-1) = 0 \Longleftrightarrow y = 1 - x$ (since $x \ne y$).

Hence $\triangle ABD$ is congruent to $\triangle BCE \Rightarrow \angle ABD = \eta \Rightarrow \eta + \varphi = 60^\circ \Rightarrow \angle BPC = 180^\circ - (\eta + \varphi) = 120^\circ$.

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JORDI DOU, Barcelona, Spain; HANS ENGELHAUPT, Franz–Ludwig–Gymnasium, Bamberg, Germany; RICHARD I. HESS, Rancho Palos Verdes, California, USA; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; P. PENNING, Delft, the Netherlands; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; HOE TECK WEE, student, Hwa Chong Junior College, Singapore; one anonymous solver and the proposer. There were two incorrect solutions.

2073*. [1995: 277] *Proposed by Jan Ciach, Ostrowiec Świętokrzyski, Poland.*

Let P be an interior point of an equilateral triangle $A_1A_2A_3$ with circumradius R, and let $R_1 = PA_1$, $R_2 = PA_2$, $R_3 = PA_3$. Prove or disprove that

$$R_1R_2R_3 \le \frac{9}{8}R^3.$$

Equality holds if P is the mid-point of a side. [Compare this problem with *Crux* 1895 [1995: 204].]

I *Solution by Kee-Wai Lau, Hong Kong.*

Without loss of generality we assume that $R = 2$, so that the length of each side of the triangle is $2\sqrt{3}$. Let $\angle A_1PA_2 = \theta, \angle A_2PA_3 = \phi$, so that $\angle A_1PA_3 = 2\pi - \theta - \phi$. Applying the cosine law to $\Delta A_1PA_2, \Delta A_2PA_3$ and ΔA_1PA_3, respectively, we obtain

$$\cos\theta = ({R_1}^2 + {R_2}^2 - 12)/(2R_1R_2),$$

$$\cos\phi = ({R_2}^2 + {R_3}^2 - 12)/(2R_2R_3) \text{ and}$$

$$\cos(\theta+\phi) = ({R_1}^2 + {R_3}^2 - 12)/(2R_1R_3).$$

Since $[\cos\theta\cos\phi - \cos(\theta+\phi)]^2 = (1-\cos^2\theta)(1-\cos^2\phi)$, we obtain after some simplification that

$$\begin{aligned} {R_1}^4 + {R_2}^4 + {R_3}^4 - {R_1}^2{R_2}^2 - {R_2}^2{R_3}^2 - {R_3}^2{R_1}^2 - & \\ 12({R_1}^2 + {R_2}^2 + {R_3}^2) + 144 &= 0. \end{aligned}$$

Thus

$$R_3{}^2 = \frac{1}{2}(z + 24 \pm (12y - 3z^2)^{1/2}), \qquad (1)$$

where $y = {R_1}^2{R_2}^2, z = {R_1}^2 + {R_2}^2 - 12$ and $z^2 \leq 4y$.

By considering $R_1 = R_2 = R_3 = 2$, we see that we should choose in (1) the $-$ sign from $\pm$. It follows that

$$f(y,z) := {R_1}^2{R_2}^2{R_3}^2 = \frac{1}{2}y[z + 24 - (12y - 3z^2)^{1/2}].$$

From $df/dz = 0$, we obtain $z = -\sqrt{y}$ and $f(y, -\sqrt{y}) = y(12 - 2\sqrt{y}) \leq 64$.

Now at the boundary $z^2 = 4y$, we have $|R_1 - R_2| = \sqrt{12}$ or $R_1 + R_2 = \sqrt{12}$, so that $-6 \leq z \leq 0$. Hence $f = z^2(z+24)/8 \leq 81$ and equality holds when $z = -6$ or P is the mid-point of a side. This proves the required inequality.

II *Solution by Manuel Benito Munoz and Emilio Fernández Moral, I.B. Sagasta, Logroño, Spain*

The proposed inequality holds (equality only on the mid-point of a side), as a particular case ($n = 3$) of the following:

$$R_1R_2\ldots R_n \leq (1 + \cos^n\frac{\pi}{n})\cdot R^n$$

when P is any point from a closed regular n-gon, $A_1A_2\ldots A_n$, with circumradius R and $R_i = PA_i$. (Equality holds only on the mid-point of a side.)

Without loss of generality, let us suppose that $R = 1$ and that the vertices of the n-gon are the points $\alpha_1, \alpha_2, \ldots, \alpha_n$ of the complex plane, where $\alpha_k = e^{\frac{2\pi i}{n}(k-1)}$. Let $z = \rho e^{i\theta}$ represent the point P.

By the maximum modulus principle applied to the function $\Pi_{k=1}^{n}(z-\alpha_k)$, the modulus of that function, i.e., $PA_1 \cdot PA_2 \dots PA_n = \Pi_{k=1}^{n}|z-\alpha_k|$, assumes its greatest value on the boundary of the n-gon.

Now, geometrical considerations permit us to avoid much analytical treatment and we conclude that the maximum value is attained on the mid-point of any side A_iA_{i+1}. Let P be a point on the side A_1A_2 of the n-gon; the vertices $A_3, \dots, A_n$ of the n-gon are pairwise arranged symmetrically with respect to the mediatrix of the segment A_1A_2 (except for the single point $A_{\frac{n+1}{2}+1}$ when n is odd).

Let A_k, A_l be one of such pairs of vertices; if M is the mid-point of A_1A_2 we have, by the GA-means inequality and "shortest path principle", that

$$PA_k \cdot PA_l \le \left(\frac{PA_k + PA_l}{2}\right)^2 \le \left(\frac{MA_k + MA_l}{2}\right)^2 = MA_k \cdot MA_l$$

(the latter because $MA_k = MA_l$).

When n is even, as $PA_1 \cdot PA_2 \le MA_1 \cdot MA_2$, we are done.

When n is odd, let A_m be the unpaired vertex and put $A_1A_2 = 2l$, $PM = x$ and $MA_m = a$. Therefore,

$$PA_1 \cdot PA_2 \cdot PA_m = (l^2 - x^2) \cdot \sqrt{a^2 + x^2},$$

and the maximum value of that function for $x \in [0, l]$ is attained at $x = 0$; that is, when $P = M$. (There are no other critical points because $a > l$.) So that, for every n, we have shown

$$PA_1 \cdot PA_2 \dots PA_n \le MA_1 \cdot MA_2 \dots MA_n,$$

where M is the mid-point of a side.

Now finally

$$\begin{aligned}(PA_1 \cdot PA_2 \dots PA_n)^2 \\ &= \Pi_{k=1}^{n}|z-\alpha_k|^2 = \Pi_{k=1}^{n}(z-\alpha_k) \cdot \Pi_{k=1}^{n}(\overline{z}-\overline{\alpha}_k) \\ &= (z^n - 1) \cdot (\overline{z}^n - 1) = \rho^{2n} - 2\rho^n \cos n\theta + 1,\end{aligned}$$

and if P is the mid-point of A_1A_2 ($\rho = \cos\left(\frac{\pi}{n}\right), \theta = \frac{\pi}{n}$), that value is $\left(\cos\left(\frac{\pi}{n}\right)\right)^{2n} - 2\cos^n\left(\frac{\pi}{n}\right) \cdot \cos\pi + 1 = \left(\cos^n\left(\frac{\pi}{n}\right) + 1\right)^2$, as required.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WOLFGANG GMEINER, Bundesgymnasium Spittal/Drau, Spittal, Austria (also found the generalization of π); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta(with a generalization from triangles to tetrahedra); VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, *Japan; JOHANNES WALDMANN, Friedrich-Schiller-Universität, Jena, Germany; and the proposer.*

2074. [1995: 277] *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

The number 3774 is divisible by 37, 34 and 74 but not by 77. Find another four-digit integer $abcd$ that is divisible by the two-digit numbers ab, ac, ad, bd and cd but is not divisible by bc.

Solution by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.

The two possible solutions are

$$1995 \quad \text{and} \quad 2184.$$

Let $N = abcd = 100 \cdot (ab) + cd$ be divisible by ab. Then cd must be divisible by ab as well; say, $cd = k \cdot (ab)$ where $k < 10$. Since N is divisible by cd, so is $100 \cdot (ab)$. It follows that k is a divisor of 100. As such, it must be one of 1, 2, 4, 5. Now, a direct computer search reveals that only 1995, 2184 and 3774 satisfy the requirement.

Note: apart from numbers of the form $aaaa$, 1155 is the only four-digit number divisible by all two-digit numbers "contained in it".

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; KEITH EKBLAW, Walla Walla, Washington, USA; HANS ENGELHAUPT, Franz-Ludwig-Gymnasium, Bamberg, Germany; JEFFREY K. FLOYD, Newnan, Georgia, USA; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California, USA; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; J.A. MCCALLUM, Medicine Hat, Alberta; STEWART METCHETTE, Gardena, California, USA; P. PENNING, Delft, the Netherlands; ROBERT P. SEALY, Mount Allison University, Sackville, New Brunswick; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; DAVID R. STONE, Georgia Southern University, Statesboro, Georgia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; CHRIS WILDHAGEN, Rotterdam, the Netherlands; KENNETH M. WILKE, Topeka, Kansas, USA; and the proposer.

A bit over half of the solvers found both solutions, often by computer, as there does not appear to be a short way to do the problem by hand. The proposer also used a computer, but it seems to have let him down, as he thought that 1995 was the only answer!

Metchette also gave the "solution" 0315, which works, but is not a true four-digit number. He wonders why all the solutions are multiples of 3; can any reader see an easy explanation?

2075. [1995: 278] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK*

ABC is a triangle with $\angle A < \angle B < \angle C$, and I is its incentre. BCL, ACM, ABN are the sides of the triangle with L on BC produced, etc., and the points L, M, N chosen so that

$$\angle CLI = \frac{1}{2}(\angle C - \angle B), \quad \angle AMI = \frac{1}{2}(\angle C - \angle A), \quad \angle BNI = \frac{1}{2}(\angle B - \angle A).$$

Prove that L, M, N are collinear.

Solution by Gottfried Perz, Pestalozzigymnasium, Graz, Austria.

We have

$$\begin{aligned} \angle AIN &= 180^\circ - (\angle BAI + \angle BNI) \\ &= 180^\circ - \left(\frac{\angle A}{2} + \frac{\angle B - \angle A}{2}\right) \\ &= 180^\circ - \frac{\angle B}{2} \; = \; \angle NBI, \end{aligned}$$

which implies that triangles $\triangle ANI$ and $\triangle INB$ are similar, or

$$AN : NI : AI = NI : BN : BI$$

Hence it follows (via $AN \cdot BN = NI^2$) that

$$\frac{AN}{BN} = \frac{NI^2}{BN^2} = \frac{AI^2}{BI^2}.$$

Analogously we get

$$\frac{BL}{CL} = \frac{BI^2}{CI^2}, \quad \frac{CM}{AM} = \frac{CI^2}{AI^2}$$

and finally

$$\frac{AN}{BN} \cdot \frac{BL}{CL} \cdot \frac{CM}{AM} = \frac{AI^2}{BI^2} \cdot \frac{BI^2}{CI^2} \cdot \frac{CI^2}{AI^2} = 1$$

whence (by the converse of Menelaus' Theorem) M, N and L are collinear.

Perz also remarks that, more generally, L, M, N are collinear, if P is a point inside or outside $\triangle ABC$ and L, M, N are such that

$$\begin{aligned} \angle CLP &= \angle BCP - \angle PBC, \; \angle AMP \\ &= \angle CAP - \angle PCA, \; \angle BNI \\ &= \angle ABP - \angle PAB, \end{aligned}$$

where all angles are oriented angles.

Also solved by TOBY GEE, student, the John of Gaunt School, Trowbridge, England; CYRUS HSIA, student, University of Toronto, Toronto, Ontario; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, the Netherlands; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; and the proposer.

2077. [1995: 278] *Proposed by Joseph Zaks, University of Haifa, Israel.*

The determinant

$$\begin{vmatrix} z_1\overline{z_1} & z_1 & \overline{z_1} & 1 \\ z_2\overline{z_2} & z_2 & \overline{z_2} & 1 \\ z_3\overline{z_3} & z_3 & \overline{z_3} & 1 \\ z_4\overline{z_4} & z_4 & \overline{z_4} & 1 \end{vmatrix}$$

equals 0 if and only if the four complex numbers z_1, z_2, z_3, z_4 satisfy what simple geometric property?

Solution by Václav Konečný, Ferris State University, Big Rapids, Michigan, USA. [*modified slightly by the editor*.]

Let D denote the given determinant. If any two (or more) of the four complex numbers are equal, $z_1 = z_2$, say, then $D = 0$, and the points z_2, z_3, z_4 must be collinear (or concyclic). We thus assume that the four numbers are all distinct.

It is well-known that an equation of the circle passing through three non-collinear points (x_2, y_2), $(x_3, y_3,$ (x_4, y_4), can be written as

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \\ x_4^2 + y_4^2 & x_4 & y_4 & 1 \end{vmatrix} = 0.$$

Letting $z_1 = x + \imath y$ and $z_k = x_k + \imath y_k$ for $k = 2, 3, 4$, we have

$$D = \begin{vmatrix} z_1\overline{z_1} & x & \overline{z_1} & 1 \\ z_2\overline{z_2} & x_2 & \overline{z_2} & 1 \\ z_3\overline{z_3} & x_3 & \overline{z_3} & 1 \\ z_4\overline{z_4} & x_4 & \overline{z_4} & 1 \end{vmatrix} = -2\imath \begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \\ x_4^2 + y_4^2 & x_4 & y_4 & 1 \end{vmatrix}.$$

Thus $D = 0$ if and only if z_1 lies on the circle passing through the three points z_2, z_3 and z_4. There the sought property is that z_1, z_2, z_3, z_4 are either concyclic or collinear (which can be viewed as a degenerate case).

[Editor's note: This is certainly a known result. The references supplied by several readers are listed below. The number, if there is one, after a solver's name corresponds to the reference given by that solver.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, Berlin, Germany; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain [3]; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; WOLFGANG GMEINER, Millstatt, Austria; RICHARD I. HESS, Rancho Palos Verdes, California, USA [2]; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany [1]; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer. One anonymous solution [4] and one incomplete solution were also received.

References:

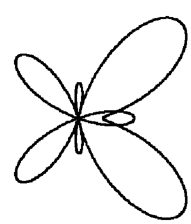
[1] Conway, J.B., *Functions of One Complex Variable*, Springer-Verlag, 2nd ed., 1978, p. 49, Proposition 3.10.

[2] Dodge, Clayton W., *Complex Numbers*, 1993, p. 135 and p. 199.

[3] Krzyz, *Problems in Complex Variable Theory*, Elsevier, PWN, 1971, P. 5 and p. 142.

[4] Schwerdtfeger, H., *Geometry for Complex Numbers*, Dover, NY, 1962.

In the May 1996 issue of ***CRUX*** [1996: 168], we asked "Do you know the equation of this curve?"



Here is a hint – it is known as the "butterfly"!