Dissecting Squares into Similar Rectangles

Byung-Kyu Chun (student)
Harry Ainlay Composite High School
Andy Liu
University of Alberta
and
Daniel van Vliet (student)
University of Alberta

1 Introductory Remarks.

Karl Scherer and Martin Gardner [1] have proposed the following three-part problem. Cut a square into three similar pieces, where

(1) all three are congruent;

(2) exactly two are congruent;

(3) no two are congruent.

Note that (2) really consists of two sub-problems, where the congruent pieces are

(2a) smaller than the third;

(2b) larger than the third.

All their solutions to (2) are to (2a). It appears that, alas, (2b) is not to be. Also considered in [1] are related dissections of an equilateral triangle, but they do not concern us here.

We generalize the problem of Scherer and Gardner for the square as follows. Given any integer $m > 1$ and any of its 2^{m-1} compositions, or ordered partitions, $m = a_1 + a_2 + \cdots + a_n$, dissect a square into m similar pieces so that there are a_1 congruent pieces of the largest size, a_2 congruent pieces of the next largest size, and so on. In the original problem, $m = 3$ and the compositions are: (1) 3; (2a) $1 + 2$; (2b) $2 + 1$; (3) $1 + 1 + 1$.

Our main result is that the dissection problem always admits a solution using rectangular pieces if and only if the composition is not of the form $k+1$, where k is any positive integer. These solvable cases are covered by two constructions which are only slightly different.

2 Construction for the case $m = a_1 + a_2 + \cdots + a_n, a_n > 1$.

Suppose the composition is $m = a_1 + a_2 + \cdots + a_n$, $a_n > 1$. We start with a rectangle R and divide it into n rectangles $R_1, R_2, \ldots, R_n$ as follows. R_1 is the right half of R, R_2 is the bottom half of $R - R_1$, R_3 is the right half of $R - R_1 - R_2$, R_4 is the bottom half of $R - R_1 - R_2 - R_3$, and so on, except that $R_n = R - R_1 - R_2 - \cdots - R_{n-1}$. The dimensions of these rectangles will be adjusted later. Divide R_i into a_i rectangular pieces, using vertical lines if i is odd and horizontal lines if i is even. Let the horizontal and vertical dimensions of each piece in R_i be x_i and y_i respectively, as shown in Figure 1 for the case $6 = 1 + 3 + 2$.

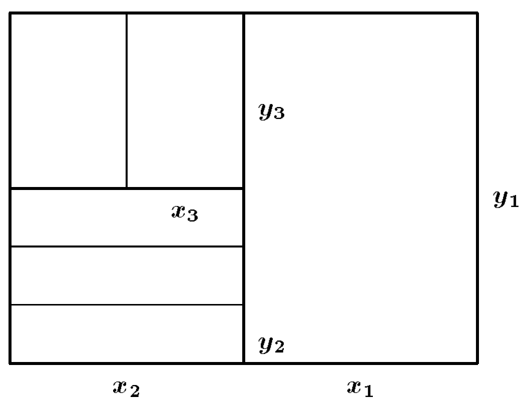

Figure 1

Next, we make all the pieces similar to one another by setting $y_i = x_i t$ for all i, where t is a given positive number. It follows from the assumption $a_n > 1$ that a piece in R_i is larger than a piece in R_j if $i > j$. We choose $x_n = 1$, so that $y_n = t$. If n is even, $y_{n-1} = a_n y_n = a_n t$ and $x_{n-1} = a_n$. If $n > 1$ is odd, $x_{n-1} = a_n x_n = a_n$ and $y_{n-1} = a_n t$.

Going backwards step by step on the basis of the recursive formulae $y_{i-1} = a_i y_i + y_{i+1}$ for i even and $x_{i-1} = a_i x_i + x_{i+1}$ for i odd, we can compute the dimensions of the pieces in $R_{n-2}, R_{n-3}, \ldots, R_1$ and R. Figure 2 is obtained by applying this process to Figure 1. Note that the x_i are always integers while the y_i are always integral multiples of t.

Finally, we want to find a t so that R is a square. In other words, t is a solution of $y_1 = a_1 x_1 + x_2$. Since y_1 is linear in t while $a_1 x_1 + x_2$ is a constant, $y_1 > a_1 x_1 + x_2$ for sufficiently large t. On the other hand, if $t = 1$, then the pieces will be square pieces, so that $y_1 = x_1 \le a_1 x_1 \le a_1 x_1 + x_2$. Since $m > 1$, either $a_1 > 1$ or $x_2 > 0$, so that $y_1 < a_1 x_1 + x_2$. It follows that there exists a real number $t > 1$ for which $y_1 = a_1 x_1 + x_2$.

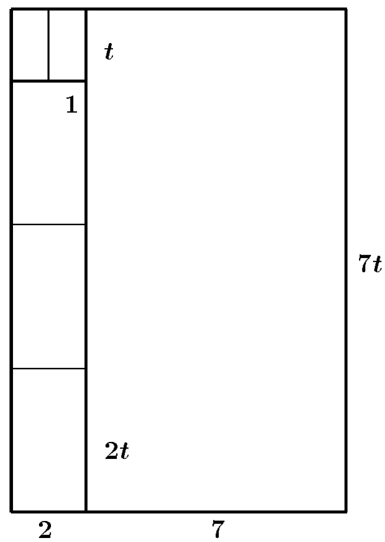

Figure 2

From Figure 2, we have $7t = 9$ or $t = \frac{9}{7}$. In general, since t is the solution of a linear equation with integral coefficients, it is a rational number. Hence we can apply a uniform magnification to make all pieces have integral sides. Figure 3 is a magnification of Figure 2 by a factor of 7.

3 Construction for the case $m = a_1 + a_2 + \cdots + a_n, a_n = 1$ and $n \geq 3$.

Suppose the composition is $m = a_1 + a_2 + \cdots + a_n, n \geq 3$ and $a_n = 1$. The first step is identical to the construction in the last Section. We make all the pieces similar to one another by setting $y_i = x_i t$, except that $x_i = y_i t$ for the last odd i. We shall prove that the equation $y_1 = a_1 x_1 + x_2$ has a real solution. Note that, as before, we have $y_1 < a_1 x_1 + x_2$ if $t = 1$.

Suppose $n \geq 4$ is even. Let $x_n = 1$ and $y_n = t$. Then $y_{n-1} = t, x_{n-1} = t^2, x_{n-2} = t^2 + 1$ and $y_{n-2} = t^3 + t$. By induction, x_i is of the form $at^2 + b$ while y_i is of the form $ct^3 + dt$ for all i, where a, b, c and d are integers dependent only on i. Thus $y_1 > a_1 x_1 + x_2$ for sufficiently large t.

Suppose $n \geq 3$ is odd. Let $x_n = t$ and $y_n = 1$. Then $x_{n-1} = t, y_{n-1} = t^2, y_{n-2} = t^2 + 1$ and $x_{n-2} = t + \frac{1}{t}$. By induction, x_i is of the form $at + \frac{b}{t}$ while y_i is of the form $ct^2 + d$ for all i, where a, b, c and d are integers dependent only on i. Thus $y_1 > a_1 x_1 + x_2$ for sufficiently large t.

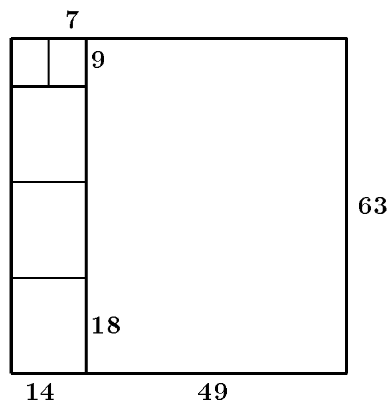

Figure 3

Figure 4 illustrates the cases $6 = 1 + 2 + 2 + 1$ and $6 = 2 + 3 + 1$. In many cases covered by this construction, we can also use pieces with integral sides in the same orientation. Figure 5 illustrates one such example, namely, $11 = 4 + 6 + 1$.

4 Impossibility proof for the case $m = k + 1$.

We now consider the remaining cases, compositions of the form $m = k + 1$, where k is any positive integer. We shall prove that the dissection problem is not solvable with rectangular pieces. Our approach is indirect. We assume that a solution exists, with a "1 by t" piece and k "s by st" pieces, where $t \geq 1$ and $s > 1$ are real numbers.

Let the side length of the square be ℓ. Consider a horizontal or vertical segment from one side to the other, not running along any side of a piece. Then $\ell = as + bst$ for some integers a and b if the segment does not cut the small piece. If it does, we have either $\ell = cs + dst + 1$ or $\ell = es + fst + t$ for some integers c, d, e and f.

From $as + bst = cs + dst = 1$, it follows that s is rational if and only if t is. We first dispose of the case where s and t are rational. Let $s = \frac{g}{h}$ and $t = \frac{i}{j}$, where g and h are relatively prime integers, as are i and j. Then $agj + bgi = cgj + dgi + hj = egj + fgi + hi$. Hence g divides hi and hj. Since g and h are relatively prime, g divides i and j. Since i and j are relatively prime, we must have $g = 1$. However, $s = \frac{1}{h} \leq 1$, which is a contradiction.

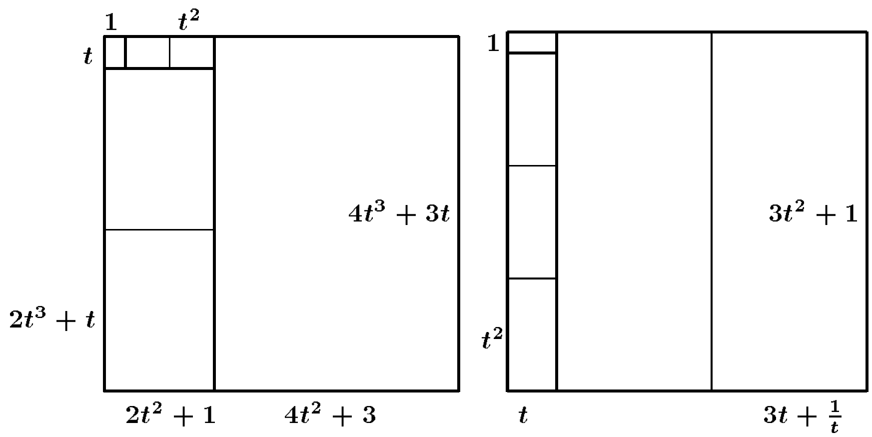

Figure 4

We now consider the case where s and t are irrational, so that $t > 1$. Consider a horizontal or vertical segment from one side of the square to the other, not running along any side of a piece and not cutting the small piece. As before, we have $\ell = as + bst$ for some integers a and b.

We make the important observation that a and b are independent of the choice of such a segment. If we have $\ell = a's + b'st$ for some integers $a' \neq a$ and $b' \neq b$, then $t = \frac{a-a'}{b'-b}$ will be rational. Moreover, since the segment can be horizontal or vertical, we must have $a > 0$ and $b > 0$.

We now use this to prove that the dissected square does not have a fault-line. This is defined as a segment which divides the square into two rectangles, each containing only complete pieces. Suppose to the contrary that there is a horizontal fault-line. Then one of the rectangles is dissected into only large pieces. Divide it into horizontal strips of width s from the top. Each strip contains b "st by s" pieces and a or $2a$ squares of side s, each of which contains parts of one or two s by st pieces. It follows that the height of the rectangle is equal to ps for some positive integer p.

We can rearrange the pieces within this rectangle so that all the horizontal ones are to the left and all the vertical ones are to the right. We then have $ps = qst$ for some positive integer q. However, t will then be rational, which is a contradiction. It follows that the dissected square has no fault-lines.

We now combine the large pieces into rectangles until the union of any two of them is not a rectangle. The small piece is not part of any rectangle. If this combination is not unique, we choose the one for which the number r of rectangles is minimum. If $r \leq 3$, there will be a fault-line. Hence $r \geq 4$.

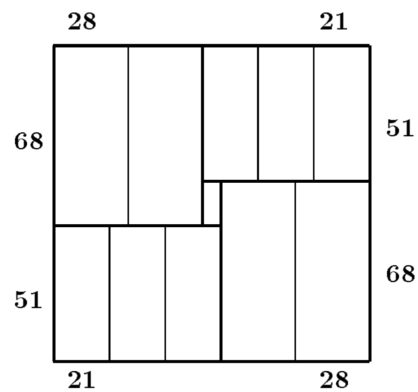

Figure 5

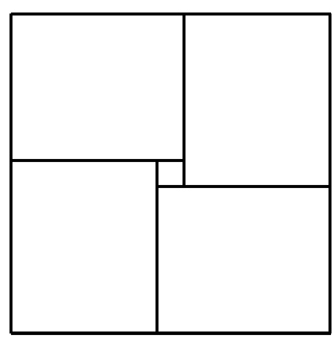

Figure 6

Suppose $r = 4$. The only possible configuration has the four rectangles at the corners and the small piece in the middle, as shown in Figure 6. Now each of the rectangles is dissected into large pieces all of which are in the same orientation. Since $\ell = as + bst$ with constants $a > 0$ and $b > 0$, two opposite rectangles must be dissected into a columns and b rows of s by st pieces, with the other two into a rows and b columns of st by s pieces. However, the small piece will then be a square, and $t = 1$ is a contradiction.

Suppose $r \geq 5$. Then there is a rectangle at a corner of the square which is not adjacent to the small piece. At least one of its two sides within the square is the union of the sides of at least two other rectangles. If the vertical side is not, as illustrated in Figure 7, then the horizontal side must be.

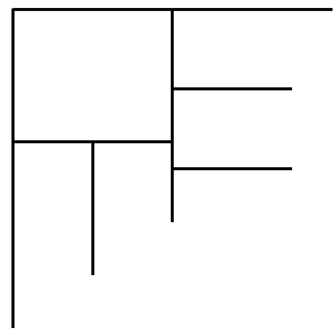

Figure 7

Now this corner rectangle and those immediately below it are dissected into large pieces in the same orientation within each. In fact, it must be the same in all of them, as otherwise we can show that t is rational. Now we can expand the corner rectangle until it swallows up a rectangle below it. However, this reduction in r contradicts its minimality assumption.

5 Concluding Remarks.

We conjecture that for the compositions $m = k + 1$, the dissection problem is not solvable even if non-rectangular polygonal pieces are permitted. This is true for the simplest case, namely, $2 = 1 + 1$. We give a proof using an indirect argument.

Suppose that a solution exists. Clearly, the smaller polygon P cannot contain two opposite corners of the square S. Hence the larger polygon P' contains a side of S. We claim that this is the longest sides of P'. Otherwise, the longest side will be inside S, and it separates P' from P. Hence it will also be the longest side of P, and the two polygons are in fact congruent. This contradiction justifies the claim. It follows that the longest side of P is shorter than a side of S. Now P and P' have the same number of sides inside S, but P' has more sides than P on the boundary of S. Hence P and P' cannot be similar.

On the other hand, Figure 8 shows a "solution" using fractal-like pieces.

For the cases covered by our two constructions, we have many solutions using polygonal pieces that are not rectangles. Figure 9 illustrates one such example, namely, $7 = 1 + 1 + 1 + 1 + 1 + 1 + 1$, based on a solution to a problem in [2].

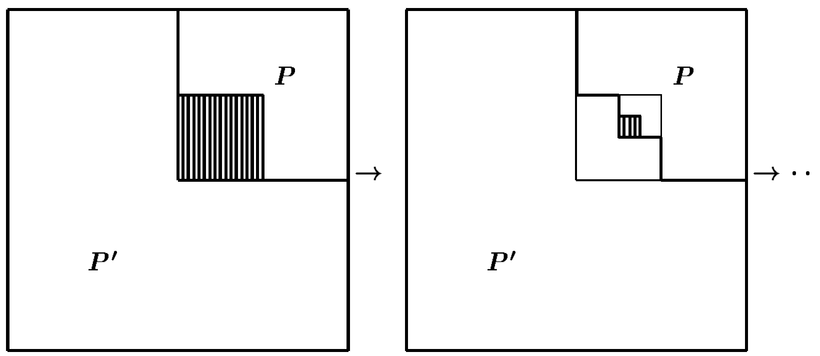

Figure 8

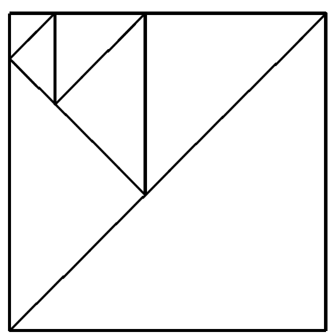

Figure 9

6 References:

[1] Martin Gardner, *Six Challenging Dissection Tasks*, Quantum, Volume 4, Issue 5 (1994) 26-27.

[2] Peter Taylor, "International Mathematics Tournament of the Towns (1980-1984)", Australian Mathematics Trust, Canberra (1993) 106.