On The Generalized Ptolemy Theorem

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Introduction. The following note describes a few uses of a relatively less known result in plane geometry, the Generalized Ptolemy Theorem (**GPT**, for short), also known as Casey's Theorem (see Johnson[1]). Featured will be two proofs of the problem proposed by India for the 33rd IMO in Moscow [1993: 255; 1995: 86].

Theorem 1. Circles Ω_1 and Ω_2 are externally tangent at a point I, and both are enclosed by and tangent to a third circle Ω. One common tangent to Ω_1 and Ω_2 meets Ω in B and C, while the common tangent at I meets Ω in A on the same side of BC as I. Then I is the incentre of triangle ABC.

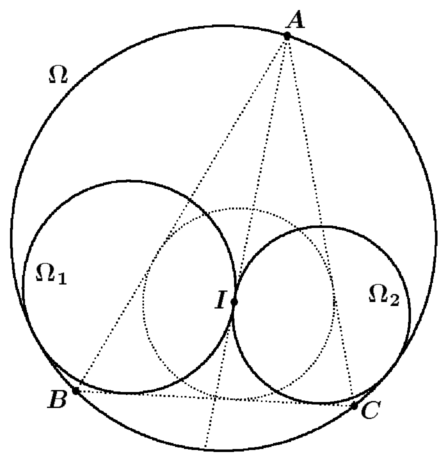

Figure 1.

Proof 1.
Notation: Let t_{ij} refer to the length of the external common tangent to circles i and j (thus the two circles lie on the same side of the tangent). We use the **GPT**, which we state in the following manner.

(**The GPT**) *Let circles α, β, γ, δ all touch the circle Γ, the contacts being all internal or all external and in the cyclical order α, β, γ, δ. Then:*

$$t_{\alpha\beta} \cdot t_{\gamma\delta} + t_{\beta\gamma} \cdot t_{\delta\alpha} = t_{\alpha\gamma} \cdot t_{\beta\delta} \, .$$

Moreover, a converse also holds: if circles α, β, γ, δ are located such that

$$\pm\, t_{\alpha\beta} \cdot t_{\gamma\delta} \;\pm\; t_{\alpha\delta} \cdot t_{\beta\gamma} \;\pm\; t_{\alpha\gamma} \cdot t_{\beta\delta} = 0$$

for some combination of $+$, $\Leftrightarrow$ signs, then there exists a circle that touches all four circles, the contacts being all internal or all external.

For a proof of the **GPT** and its converse, please refer to [1].

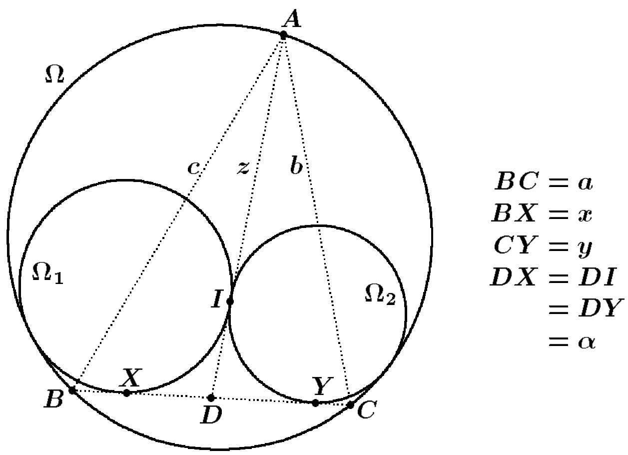

Figure 2.

Consider the configuration shown in Figure 2, where x and y are, respectively, the lengths of the tangents from B and C to Ω_1 and Ω_2; D is $AI \cap BC$; $z = |AI|$; $u = |ID|$; and a, b, c are the sides of $\triangle ABC$.

We apply the **GPT** to the two 4-tuples of circles $(A,\, \Omega_1,\, B,\, C)$ and $(A,\, \Omega_2,\, C,\, B)$. We obtain:

$$az + bx = c(2u + y) \tag{1}$$

$$az + cy = b(2u + x)\,. \tag{2}$$

Subtracting (2) from (1) yields $bx \Leftrightarrow cy = u(c \Leftrightarrow b)$, so $(x + u)/(y + u) = c/b$, that is, $BD/DC = AB/AC$, which implies that AI bisects $\angle BAC$ and that $BD = ac/(b + c)$. Adding (1) and (2) yields $az = u(b + c)$, so $z/u = (b+c)/a$, that is, $AI/ID = AB/BD$, which implies that BI bisects $\angle ABC$. This proves the result. ■

Proof 2.
Lemma. Let BC be a chord of a circle Γ, and let S_1, S_2 be the two arcs of Γ cut off by BC. Let M be the midpoint of S_2, and consider all possible circles Ω that touch S_1 and BC. Then the length $t_{M\Omega}$ of the tangent from M to Ω is constant for all such Ω. (See Figure 3.)

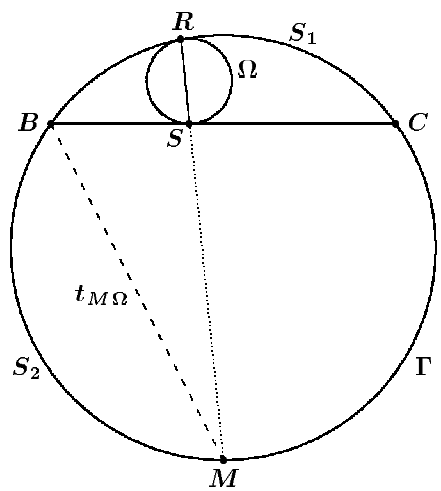

Figure 3.

Proof of Lemma. Let $\Omega \cap \Gamma = R$, $\Omega \cap BC = S$. Applying the **GPT** to the 4-tuple (B, Ω, C, M), we find: $BS \cdot CM + CS \cdot BM = t_{M\Omega} \cdot BC$. Since $BM = CM$, we obtain: $t_{M\Omega} = BM$, a constant. ■

Proof of Theorem 1. (See figure 4.) Let S_1, S_2 be the two arcs of Γ cut off by chord BC, S_1 being the one containing A, and let M denote the midpoint of S_2. Using the above lemma,

$$t_{M\Omega_1} = MB = MI = MC = t_{M\Omega_2} .$$

Therefore M has equal powers with respect to Ω_1 and Ω_2 and lies on their radical axis, namely AI. It follows that AI bisects $\angle A$ of $\triangle ABC$.

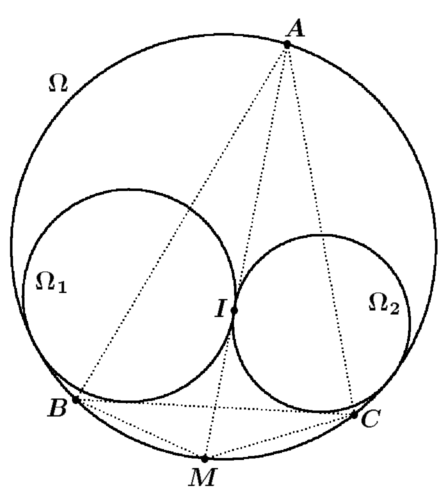

Figure 4.

Next, $\triangle IBM$ is isosceles, so $\angle IBM = \pi/2 \Leftrightarrow C/2$. Also, $\angle CBM = A/2$, so $\angle IBC = \pi/2 \Leftrightarrow C/2 \Leftrightarrow A/2 = B/2$, that is, IB bisects $\angle B$ of $\triangle ABC$. It follows that I is the incentre of $\triangle ABC$. ■

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For non-believers, here are two more illustrations of the power and economy of the **GPT**.

Theorem 2. Let $\triangle ABC$ have circumcircle Γ, and let Ω be a circle lying within Γ and tangent to it and to the sides AB (at P) and AC (at Q). Then the midpoint of PQ is the incentre of $\triangle ABC$. (See Figure 5.)

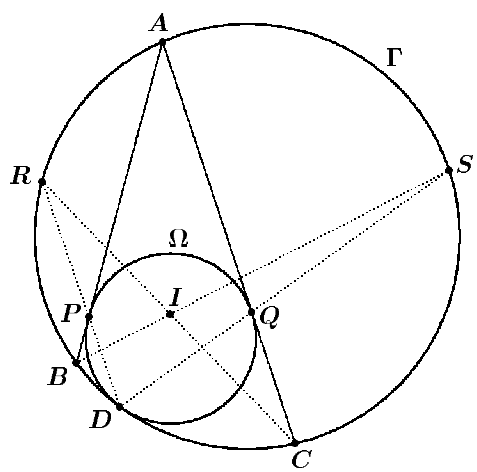


Figure 5.

Proof. Let the **GPT** be applied to the 4-tuple of circles (A, B, Ω, C). Let $AP = x = AQ$. Then:

$$t_{AB} = c, \;\; t_{A\Omega} = AP = x, \;\; t_{AC} = b,$$
$$t_{B\Omega} = BP = c \Leftrightarrow x, \;\; t_{BC} = a, \;\; t_{\Omega C} = CQ = b \Leftrightarrow x\,.$$

The **GPT** now gives: $c(b \Leftrightarrow x) + (c \Leftrightarrow x)b = ax$, so $x = bc/s$, where $s = (a+b+c)/2$ is the semi-perimeter of $\triangle ABC$. Let I denote the midpoint of PQ; then $IP = x \sin A/2 = (bc/s) \sin A/2$, and the perpendicular distance from I to AB is $IP \cos A/2$, which equals $(bc/s)(\sin A/2)(\cos A/2) = ((1/2)bc \sin A)/s$. But this is just the radius of the incircle of $\triangle ABC$. Since I is equidistant from AB and AC, it follows that I is the incentre of the triangle. ■

The next illustration concerns one of the most celebrated discoveries in elementary geometry made during the last two centuries.

Theorem 3. (**Feuerbach's Theorem**) The incircle and nine-point circle of a triangle are tangent to one another.

Proof. Let the sides BC, CA, AB of $\triangle ABC$ have midpoints D, E, F respectively, and let Ω be the incircle of the triangle. Let a, b, c be the sides of $\triangle ABC$, and let s be its semi-perimeter. We now consider the 4-tuple of circles (D, E, F, Ω). Here is what we find:

$$
\begin{aligned}
t_{DE} &= \frac{c}{2}, \quad t_{DF} = \frac{b}{2}, \quad t_{EF} = \frac{a}{2}, \\
t_{D\Omega} &= \left|\frac{a}{2} \Leftrightarrow (s \Leftrightarrow b)\right| = \left|\frac{b \Leftrightarrow c}{2}\right|, \\
t_{E\Omega} &= \left|\frac{b}{2} \Leftrightarrow (s \Leftrightarrow c)\right| = \left|\frac{a \Leftrightarrow c}{2}\right|, \\
t_{F\Omega} &= \left|\frac{c}{2} \Leftrightarrow (s \Leftrightarrow a)\right| = \left|\frac{b \Leftrightarrow a}{2}\right|.
\end{aligned}
$$

We need to check whether, for some combination of $+$, $\Leftrightarrow$ signs, we have

$$\pm\, c(b \Leftrightarrow a) \;\pm\; a(b \Leftrightarrow c) \;\pm\; b(a \Leftrightarrow c) = 0\,.$$

But this is immediate! It follows from the converse to the **GPT** that there exists a circle that touches each of D, E, F and Ω. Since the circle passing through D, E, F is the nine-point circle of the triangle, it follows that Ω and the nine-point circle are tangent to one another. ∎

One would surmise that the **GPT** should provide a neat proof of the following theorem due to Victor Thebault:

> *Let $\triangle ABC$ have circumcircle Γ, let D be a point on BC, and let Ω_1 and Ω_2 be the two circles lying within Γ that are tangent to Γ and also to AD and BC. Then the centres of Ω_1 and Ω_2 are collinear with the incentre of $\triangle ABC$.*

I have, however, not been able to find such a proof, and I leave the problem to the interested reader. We note in passing that Thebault's theorem provides yet another proof of Theorem 1.

References:

[1] R.A. Johnson, *Advanced Euclidean Geometry*, Dover, 1960.

Acknowledgements:

I thank the referee for making several valuable comments that helped tidy up the presentation of the paper.

THE SKOLIAD CORNER

No. 12

R.E. Woodrow

This issue we feature Part I of the 1995–96 Alberta High School Mathematics Competition, written November 21, 1995. My thanks go to Professor T. Lewis, the University of Alberta for forwarding me a copy. Students have 90 minutes to complete their contest. It is mostly written by students in Grade XII, but has often been won by students in earlier grades.

ALBERTA HIGH SCHOOL MATHEMATICS COMPETITION

Part I

November 21, 1995 (Time: 90 minutes)

1. A circle and a parabola are drawn on a piece of paper. The number of regions they divide the paper into is at most

A. 3 **B.** 4 **C.** 5 **D.** 6 **E.** 7.

2. The number of different primes $p > 2$ such that p divides $71^2 \Leftrightarrow 37^2 \Leftrightarrow 51$ is

A. 0 **B.** 1 **C.** 2 **D.** 3 **E.** 4.

3. Suppose that your height this year is 10% more than it was last year, and last year your height was 20% more than it was the year before. By what percentage has your height increased during the last two years?

A. 30 **B.** 31 **C.** 32 **D.** 33 **E.** none of these.

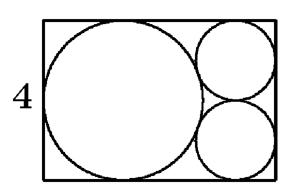
4. Multiply the consecutive even positive integers together until the product $2 \cdot 4 \cdot 6 \cdot 8 \cdots$ becomes divisible by 1995. The largest even integer you use is

A. between 1 and 21 **B.** between 21 and 31

C. between 31 and 41 **D.** bigger than 41

E. non-existent, since the product never becomes divisible by 1995.

5. A rectangle contains three circles as in the diagram, all tangent to the rectangle and to each other. If the height of the rectangle is 4, then the width of the rectangle is



A. $3 + 2\sqrt{2}$ **B.** $4 + \frac{4\sqrt{2}}{3}$ **C.** $5 + \frac{2\sqrt{2}}{3}$

D. 6 **E.** $5 + \sqrt{10}$.

6. Mary Lou works a full day and gets her usual pay. Then she works some overtime hours, each at 150% of her usual hourly salary. Her total pay that day is equivalent to 12 hours at her usual hourly salary. The number of hours that she usually works each day is

A. 6 **B.** 7.5 **C.** 8 **D.** 9 **E.** not uniquely determined by the given information.

7. A fair coin is tossed $10,000$ times. The probability p of obtaining at least three heads in a row satisfies

A. $0 \le p < \frac{1}{4}$ **B.** $\frac{1}{4} \le p < \frac{1}{2}$ **C.** $\frac{1}{2} \le p < \frac{3}{4}$ **D.** $\frac{3}{4} \le p < 1$ **E.** $p = 1$.

8. In the plane, the angles of a regular polygon with n sides add up to less than n^2 degrees. The smallest possible value of n satisfies:

A. $n < 40$ **B.** $40 \le n < 80$ **C.** $80 \le n < 120$ **D.** $120 \le n < 160$ **E.** $n \ge 160$.

9. A cubic polynomial P is such that $P(1) = 1$, $P(2) = 2$, $P(3) = 3$ and $P(4) = 5$. The value of $P(6)$ is

A. 7 **B.** 10 **C.** 13 **D.** 16 **E.** 19.

10. The positive numbers x and y satisfy $xy = 1$. The minimum value of $\frac{1}{x^4} + \frac{1}{4y^4}$ is

A. $\frac{1}{2}$ **B.** $\frac{5}{8}$ **C.** 1 **D.** $\frac{5}{4}$ **E.** no minimum.

11. Of the points $(0,0)$, $(2,0)$, $(3,1)$, $(1,2)$, $(3,3)$, $(4,3)$ and $(2,4)$, at most how many can lie on a circle?

A. 3 **B.** 4 **C.** 5 **D.** 6 **E.** 7.

12. The number of different positive integer triples (x, y, z) satisfying the equations

$$x^2 + y \Leftrightarrow z = 100 \quad \text{and} \quad x + y^2 \Leftrightarrow z = 124 \quad \text{is:}$$

A. 0 **B.** 1 **C.** 2 **D.** 3 **E.** none of these.

13. Which of the following conditions does *not* guarantee that the convex quadrilateral $ABCD$ is a parallelogram?

A. $AB = CD$ and $AD = BC$ **B.** $\angle A = \angle C$ and $\angle B = \angle D$

C. $AB = CD$ and $\angle A = \angle C$ **D.** $AB = CD$ and AB is parallel to CD

E. none of these.

14. How many of the expressions

$$x^3 + y^4, \quad x^4 + y^3, \quad x^3 + y^3, \quad \text{and} \quad x^4 \Leftrightarrow y^4,$$

are positive for all possible numbers x and y for which $x > y$?

A. 0 **B.** 1 **C.** 2 **D.** 3 **E.** 4.

15. In triangle ABC, the altitude from A to BC meets BC at D, and the altitude from B to CA meets AD at H. If $AD = 4$, $BD = 3$ and $CD = 2$, then the length of HD is

A. $\frac{\sqrt{5}}{2}$ **B.** $\frac{3}{2}$ **C.** $\sqrt{5}$ **D.** $\frac{5}{2}$ **E.** $\frac{3\sqrt{5}}{2}$.

16. Which of the folowing is the best approximation to

$$\frac{(2^3 \Leftrightarrow 1)(3^3 \Leftrightarrow 1)(4^3 \Leftrightarrow 1)}{(2^3+1)(3^3+1)(4^3+1)} \cdots \frac{(100^3 \Leftrightarrow 1)}{(100^3+1)}?$$

A. $\frac{3}{5}$ **B.** $\frac{33}{50}$ **C.** $\frac{333}{500}$ **D.** $\frac{3,333}{5,000}$ **E.** $\frac{33,333}{50,000}$.

Last issue we gave the **SHARP** U.K. Intermediate Mathematical Challenge, written February 2, 1995. Here are answers.

1.	E	**2.**	C	**3.**	D	**4.**	E	**5.**	A
6.	B	**7.**	C	**8.**	B	**9.**	D	**10.**	A
11.	E	**12.**	D	**13.**	C	**14.**	D	**15.**	B
16.	A	**17.**	A	**18.**	C	**19.**	C	**20.**	D
21.	D	**22.**	E	**23.**	B	**24.**	B	**25.**	A

That completes this month's Skoliad Corner. I need materials of a suitable level to build up a bank of contests. Please send me suitable materials as well as comments, criticisms, and suggestions. I would like to have some feedback too about how your students do with these materials.

THE OLYMPIAD CORNER

No. 172

R.E. Woodrow

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada. T2N 1N4.

Because we are now publishing eight numbers of the Corner rather than ten, I am giving two Olympiad Sets for your pleasure. Besides, here in Canada it is winter, and quite cold, so having a stock of problems to contemplate in a warm spot is a good idea. Both of the sets we give this number were collected by Georg Gunther, Sir Wilfred Grenfell College, Corner Brook, when he was Canadian Team Leader to the IMO at Istanbul. Many thanks to him for gathering a wide sample of contests. We begin with the Telecom 1993 Australian Mathematical Olympiad.

TELECOM 1993 AUSTRALIAN MATHEMATICAL OLYMPIAD
Paper 1

Tuesday, 9th February, 1993

(Time: 4 hours)

1. In triangle ABC, the angle ACB is greater than $90°$. Point D is the foot of the perpendicular from C to AB; M is the midpoint of AB; E is the point on AC extended such that $EM = BM$; F is the point of intersection of BC and DE; moreover $BE = BF$. Prove that $\angle CBE = 2\angle ABC$.

2. For each function f which is defined for all real numbers and satisfies

$$f(x, y) = x \cdot f(y) + f(x) \cdot y \qquad (1)$$

and

$$f(x + y) = f(x^{1993}) + f(y^{1993}) \qquad (2)$$

determine the value $f(\sqrt{5753})$.

3. Determine all triples (a_1, a_2, a_3), $a_1 \geq a_2 \geq a_3$, of positive integers in which each number divides the sum of the other two numbers.

4. For each positive integer n, let

$$f(n) = [2\sqrt{n}] \Leftrightarrow [\sqrt{n \Leftrightarrow 1} + \sqrt{n + 1}].$$

Determine all values n for which $f(n) = 1$.
Note: If x is a real number, then $[x]$ is the largest integer not exceeding x.

Paper 2

Wednesday, 10th February, 1993

(Time: 4 hours)

5. Determine all integers x and y that satisfy

$$(x+2)^4 \Leftrightarrow x^4 = y^3.$$

6. In the acute-angled triangle ABC, let D, E, F be the feet of altitudes through A, B, C, respectively, and H the orthocentre. Prove that

$$\frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF} = 2.$$

7. Let n be a positive integer, $a_1, a_2, \ldots, a_n$ positive real numbers and $s = a_1 + a_2 + \cdots + a_n$. Prove that

$$\sum_{i=1}^{n} \frac{a_i}{s \Leftrightarrow a_i} \geq \frac{n}{n \Leftrightarrow 1} \quad \text{and} \quad \sum_{i=1}^{n} \frac{s \Leftrightarrow a_i}{a_i} \geq n(n \Leftrightarrow 1).$$

8. The vertices of triangle ABC in the $x \Leftrightarrow y$ plane have integer coordinates, and its sides do not contain any other points having integer coordinates. The interior of ABC contains only one point, G, that has integer coordinates. Prove that G is the centroid of ABC.

Next we give the Final Round of the Japan Mathematical Olympiad.

JAPAN MATHEMATICAL OLYMPIAD
Final Round — 11 February, 1993

(Time: 4.5 hours)

1. Suppose that two words A and B have the same length $n > 1$ and that the first letters of them are different while the others are the same. Prove that A or B is not periodic.

2. Let $d(n)$ be the largest odd number which divides a given number n. Suppose that $D(n)$ and $T(n)$ are defined by

$$D(n) = d(1) + d(2) + \cdots + d(n),$$

$$T(n) = 1 + 2 + \cdots + n.$$

Prove that there exist infinitely many positive numbers n such that $3D(n) = 2T(n)$.

3. In a contest, x students took part, and y problems were posed. Each student solved $y/2$ problems. For every problem, the number of students who solved it was the same. For each pair of students, just three problems were solved by both of them. Determine all possible pairs (x, y). Moreover, for each (x, y), give an example of the matrix (a_{ij}) defined by $a_{ij} = 1$ if the ith student solved the jth problem and $a_{ij} = 0$ if not.

4. Five radii of a sphere are given so that no three of them are in a common plane. Among the 32 possible choices of an end point from each segment, find out the number of choices for which the 5 points are in a hemisphere.

5. Prove that there exists a positive constant C (independent of n, a_j) which satisfies the inequality

$$\max_{0\leq x\leq 2} \prod_{j=1}^{n} |x \Leftrightarrow a_j| \leq C^n \max_{0\leq x\leq 1} \prod_{j=1}^{n} |x \Leftrightarrow a_j|$$

for any positive integer n and any real numbers $a_1, \ldots, a_n$.

Last issue we gave a set of six Klamkin Quickies. Here are his "quick" solutions. Many thanks go to Murray Klamkin, the University of Alberta, for sending them to me.

SIX KLAMKIN QUICKIES

1. Which is larger

$$(\sqrt[3]{2} \Leftrightarrow 1)^{1/3} \qquad \text{or} \qquad \sqrt[3]{1/9} \Leftrightarrow \sqrt[3]{2/9} + \sqrt[3]{4/9}?$$

Solution. That they are equal is an identity of Ramanujan.
Letting $x = \sqrt[3]{1/3}$ and $y = \sqrt[3]{2/3}$, it suffices to show that

$$(x + y)(\sqrt[3]{2} \Leftrightarrow 1)^{1/3} = x^3 + y^3 = 1,$$

or equivalently that

$$(\sqrt[3]{2} + 1)^3(\sqrt[3]{2} \Leftrightarrow 1) = 3,$$

which follows by expanding out the left hand side.

For other related radical identities of Ramanujan, see Susan Landau, *How to tangle with a nested radical*, Math. Intelligencer, 16 (1994), pp. 49–54.

2. Prove that

$$3\min\left\{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}, \frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right\} \geq (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right),$$

where a, b, c are sides of a triangle.

Solution. Each of the inequalities

$$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),$$

$$3\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right) \geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),$$

follow from their equivalent forms (which follow by expansion):

$$(b+a-c)(c-a)^2+(c+b-a)(a-b)^2+(a+c-b)(b-c)^2 \geq 0,$$

$$(b+c-a)(c-a)^2+(c+a-b)(a-b)^2+(a+b-c)(b-c)^2 \geq 0.$$

3. Let $\omega = e^{i\pi/13}$. Express $\frac{1}{1-\omega}$ as a polynomial in ω with integral coefficients.

Solution. We have

$$\frac{2}{(1-\omega)} = \frac{(1-\omega^{13})}{(1-\omega)} = 1+\omega+\omega^2+\cdots+\omega^{12},$$

$$0 = \frac{(1+\omega^{13})}{(1+\omega)} = 1-\omega+\omega^2-\cdots+\omega^{12}.$$

Adding or subtracting, we get

$$\frac{1}{(1-\omega)} = 1+\omega^2+\omega^4+\cdots+\omega^{12}$$

$$= \omega+\omega^3+\cdots+\omega^{11}.$$

More generally, if $\omega = e^{i\pi/(2n+1)}$,

$$\frac{1}{(1-\omega)} = 1+\omega^2+\omega^4+\cdots+\omega^{2n}.$$

4. Determine all integral solutions of the simultaneous Diophantine equations $x^2+y^2+z^2 = 2w^2$ and $x^4+y^4+z^4 = 2w^4$.

Solution. Eliminating w we get

$$2y^2z^2+2z^2x^2+2x^2y^2-x^4-y^4-z^4 = 0$$

or

$$(x+y+z)(y+z-x)(z+x-y)(x+y-z) = 0,$$

so that in general we can take $z = x+y$. Note that if (x, y, z, w) is a solution, so is $(\pm x, \pm y, \pm z, \pm w)$ and permutations of the x, y, z. Substituting back, we get

$$x^2+xy+y^2 = w^2.$$

Since $(x, y, w) = (1, -1, 1)$ is one solution, the general solution is obtained by the method of Desboves, that is, we set $x = r + p$, $y = -r + q$ and $w = r$. This gives $r = \frac{(p^2+pq+q^2)}{(q-p)}$. On rationalizing the solutions (since the equation is homogeneous), we get

$$\begin{aligned} x &= p^2 + pq + q^2 + p(q - p) = q^2 + 2pq, \\ -y &= p^2 + pq + q^2 - q(q - p) = p^2 + 2pq, \\ w &= p^2 + pq + q^2, \\ z &= q^2 - p^2. \end{aligned}$$

5. Prove that if the line joining the incentre to the centroid of a triangle is parallel to one of the sides of the triangle, then the sides are in arithmetic progression and, conversely, if the sides of a triangle are in arithmetic progression then the line joining the incentre to the centroid is parallel to one of the sides of the triangle.

Solution. Let **A**, **B**, **C** denote vectors to the respective vertices A, B, C of the triangle from a point outside the plane of the triangle. Then the incentre I and the centroid G have the respective vector representations **I** and **G**, where

$$\mathbf{I} = \frac{(a\mathbf{A} + b\mathbf{B} + c\mathbf{C})}{(a + b + c)}, \qquad \mathbf{G} = \frac{(\mathbf{A} + \mathbf{B} + \mathbf{C})}{3},$$

(where a, b, c are sides of the triangle). If $\mathbf{G} - \mathbf{I} = k(\mathbf{A} - \mathbf{B})$, then by expanding out

$$(b + c - 2a - k')\mathbf{A} + (a + c - 2b + k')\mathbf{B} + (a + b - 2c)\mathbf{C} = 0,$$

where $k' = 3k(a + b + c)$. Since A, B, C are linearly independent, the coefficient of **C** must vanish so that the sides are in arithmetic progression. Also then $k' = b + c - 2a = 2b - a - c$.

Conversely, if $2c = a + b$, then $\mathbf{G} - \mathbf{I} = \frac{3(\mathbf{A}-\mathbf{B})(b-a)}{6(a+b+c)}$, so that GI is parallel to the side AB.

6. Determine integral solutions of the Diophantine equation

$$\frac{x - y}{x + y} + \frac{y - z}{y + z} + \frac{z - w}{z + w} + \frac{w - x}{w + x} = 0$$

(joint problem with Emeric Deutsch, Polytechnic University of Brooklyn).

Solution. It follows by inspection that $x = z$ and $y = w$ are two solutions. To find the remaining solution(s), we multiply the given equation by the least common denominator to give

$$P(x, y, z, w) = 0,$$

where P is the 4th degree polynomial in x, y, z, w which is skew symmetric in x and z and also in y and w. Hence,

$$P(x,y,z,w) = (x \Leftrightarrow z)(y \Leftrightarrow w)Q(x,y,z,w),$$

where Q is a quadratic polynomial. On calculating the coefficient of x^2 in P, we get $2z(y \Leftrightarrow w)$. Similarly the coefficient of y^2 is $\Leftrightarrow 2w(x \Leftrightarrow z)$, so that

$$P(x,y,z,w) = 2(x \Leftrightarrow z)(y \Leftrightarrow w)(xz \Leftrightarrow yw).$$

Hence, the third and remaining solution is given by $xz = yw$.

Next we turn to the readers' solutions of problems from earlier numbers of the *Corner*. Let me begin by thanking Beatriz Margolis, Paris France; Bob Prielipp, University of Wisconsin-Oshkosh, USA; Toshio Seimiya, Kawasaki, Japan; D.J. Smeenk, Zaltbommel, the Netherlands and Chris Wildhagen, Rotterdam, the Netherlands, for sending in nice solutions to some of the problems of the 1994 Canadian and 1994 U.S.A. Mathematical Olympiads. As we publish "official" solutions to the former, and refer readers to an MAA publication for the latter, I normally do not publish these solutions.

Next, correspondence from Murray Klamkin, the University of Alberta, filed under the September number of the *Corner*, contains comments about problems from several numbers.

3. [1993: 5, 1994: 69] 1991 *British Mathematical Olympiad*.

$ABCD$ is a quadrilateral inscribed in a circle of radius r. The diagonals AC, BD meet at E. Prove that if AC is perpendicular to BD then

$$EA^2 + EB^2 + EC^2 + ED^2 = 4r^2. \qquad (*)$$

Is it true that, if $(*)$ holds, then AC is perpendicular to BD? Give a reason for your answer.

Klamkin's Comment. A simpler solution than that published, plus a generalization, is given in *Crux*, 1989, p. 243, #1.

3. [1994: 184] 1st *Mathematical Olympiad of the Republic of China (Taiwan)*.

If $x_1, x_2, \dots, x_n$ are n non-negative numbers, $n \geq 3$ and $x_1 + x_2 + \cdots + x_n = 1$ prove that $x_1^2x_2 + x_2^2x_3 + \cdots + x_n^2x_1 \leq 4/27$.

Klamkin's Comment. This problem appeared as problem 1292 in *The Math. Magazine*, April 1988.

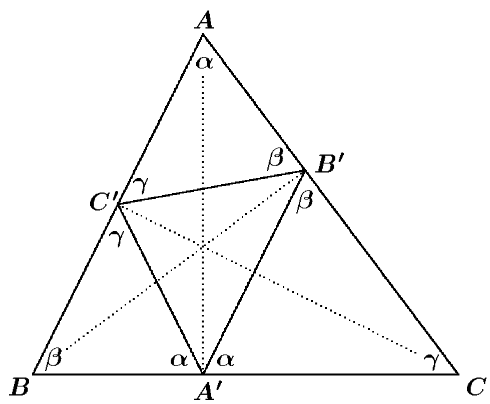
Now we turn to readers' solutions of problems proposed to the Jury, but not used, at the 34th International Mathematical Olympiad at Istanbul. A booklet of "official solutions" was issued by the organizers. Because I do not have formal permission to reproduce these solutions, I will only discuss readers' solutions that are different.

2. [1994: 216] *Proposed by Canada.*

Let triangle ABC be such that its circumradius $R = 1$. Let r be the inradius of ABC and let p be the inradius of the orthic triangle $A'B'C'$ of triangle ABC. Prove that $p \leq 1 \Leftrightarrow 1/3(1+r)^2$.

Solution by D.J. Smeenk, Zaltbommel, The Netherlands.

In the first instance I misread the problem: I read $\dfrac{1}{3(1+r)^2}$ instead of $\frac{1}{3}(1+r)^2$!



Now $r_{\triangle ABC} = R(\cos\alpha + \cos\beta + \cos\gamma \Leftrightarrow 1)$. From $R = 1$, we obtain $r = \cos\alpha + \cos\beta + \cos\gamma \Leftrightarrow 1$. Similarly,

$$\begin{aligned} p &= \frac{1}{2}[\cos(\pi \Leftrightarrow 2\alpha) + \cos(\pi \Leftrightarrow 2\beta) + \cos(\pi \Leftrightarrow 2\gamma) \Leftrightarrow 1] \\ &= 2\cos\alpha\cos\beta\cos\gamma. \end{aligned}$$

We are to show that $p + \frac{1}{3}(1+r)^2 \leq 1$. (1)
Equivalently,

$$2\cos\alpha\cos\beta\cos\gamma + \frac{1}{3}(\cos\alpha + \cos\beta + \cos\gamma)^2 \leq 1. \qquad (2)$$

Now

$$\left.\begin{aligned} \Leftrightarrow 1 < \cos\alpha\cos\beta\cos\gamma &\leq \tfrac{1}{8}, \\ 1 < \cos\alpha + \cos\beta + \cos\gamma &\leq \tfrac{3}{2}. \end{aligned}\right\} \qquad (3)$$

[Bothema, *Geom. Inequalities*, 2.24, resp. 2.16]
Now $2 \cdot \frac{1}{8} + \frac{1}{3}(\frac{3}{2})^2 = 1$, so it is clear that (1) holds. ■

3. [194: 216] *Proposed by Spain.*

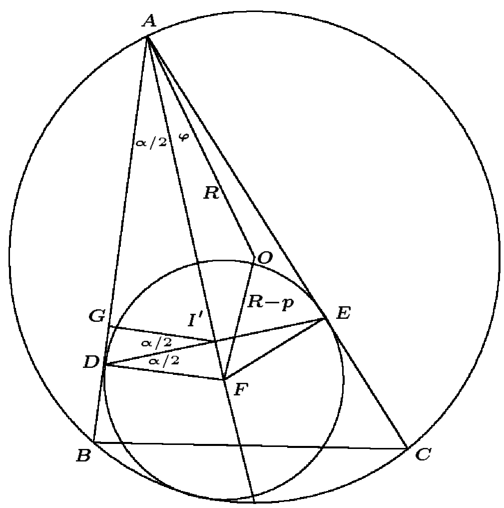
Consider the triangle ABC, its circumcircle k of centre O and radius R, and its incircle of centre I and radius r. Another circle K_c is tangent to the sides CA, CB at D, E, respectively, and it is internally tangent to k. Show that I is the midpoint of DE.

Solution by D.J. Smeenk, Zaltbommel, The Netherlands.

Assume (see figure) that $\beta > \gamma$. Let F be the centre of K_c and ρ its radius. Then F lies on the production of AI. Now $FD = \rho$, so

$$AF = \frac{\rho}{\sin\frac{\alpha}{2}}.$$

Also $AO = R$, $OF = R \Leftrightarrow \rho$, and $\angle FAO = \varphi = \frac{1}{2}(\beta \Leftrightarrow \gamma)$.



Apply the Law of Cosines to ΔAFO:

$$OF^2 = AF^2 + AO^2 \Leftrightarrow 2AF \cdot AO \cos\varphi,$$

or

$$(R \Leftrightarrow \rho)^2 = \frac{\rho^2}{\sin^2\frac{\alpha}{2}} + R^2 \Leftrightarrow \frac{2R\rho\cos\left(\frac{\beta-\gamma}{2}\right)}{2\sin\frac{\alpha}{2}}; \quad \rho \neq 0.$$

From this we obtain

$$\Leftrightarrow 2R + \rho = \frac{\rho}{\sin^2\frac{\alpha}{2}} \Leftrightarrow \frac{2R\cos\left(\frac{\beta-\gamma}{2}\right)}{\cos\left(\frac{\beta+\gamma}{2}\right)},$$

or

$$\frac{\rho\cos^2\frac{\alpha}{2}}{\sin^2\alpha/2} = \frac{4R\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\sin\alpha/2}.$$

Thus

$$\rho = \frac{4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos^2\frac{\alpha}{2}} = \frac{r}{\cos^2\frac{\alpha}{2}} = FD.$$

Now let DE intersect AF at I'. Then $DI' = DF\cos\frac{\alpha}{2} = \frac{r}{\cos\frac{\alpha}{2}}$.
Let G be the foot of the perpendicular from I' to AB.
Then $I'G = I'D\cos\frac{\alpha}{2} = \frac{r}{\cos\frac{\alpha}{2}}\cos\frac{\alpha}{2} = r$. So I' coincides with I. ∎

7. [1994: 217] *Proposed by Israel.*

The vertices D, E, F of an equilateral triangle lie on the sides BC, CA, AB respectively of a triangle ABC. If a, b, c are the respective lengths of these sides and S is the area of ABC, prove that

$$DE \geq 2\sqrt{2}\,S \cdot \{a^2 + b^2 + c^2 + 4\sqrt{3}\,S\}^{-1/2}.$$

Comment by Murray S. Klamkin, The University of Alberta.

This problem is equivalent to problem #624 of *Crux* [1982: 109–110].

11. [1994: 241] *Proposed by Spain.*

Given the triangle ABC, let D, E be points on the side BC such that $\angle BAD = \angle CAE$. If M and N are respectively, the points of tangency with BC of the incircles of the triangles ABD and ACE, show that

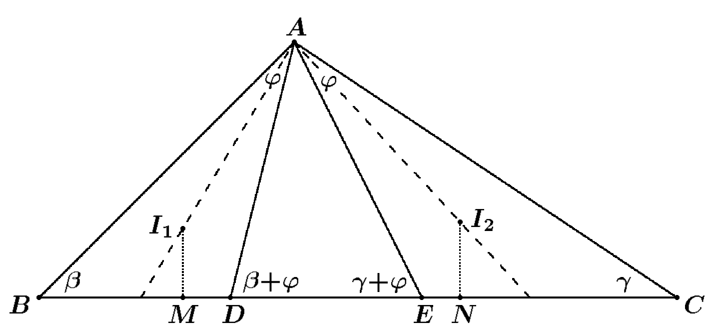
$$\frac{1}{BM} + \frac{1}{MD} = \frac{1}{NC} + \frac{1}{NE}.$$

Solution by D.J. Smeenk, Zaltbommel, The Netherlands.

We are to show $\frac{1}{MB} + \frac{1}{MD} = \frac{1}{NC} + \frac{1}{NE}$, or equivalently

$$BD \cdot NC \cdot NE = CE \cdot MB \cdot MD. \tag{1}$$

We denote $\angle BAD = \angle CAE = \varphi$.



Applying the law of sines to ΔABD we obtain

$$BD = \frac{c\sin\varphi}{\sin(\beta+\varphi)}, \qquad AD = \frac{c\sin\beta}{\sin(\beta+\varphi)}. \tag{2}$$

The sine law for ΔACE gives

$$CE = \frac{b\sin\varphi}{\sin(\gamma+\varphi)}, \qquad AE = \frac{b\sin\gamma}{\sin(\gamma+\varphi)}. \tag{2'}$$

From ΔABD:

$$2MB = AB + BD \Leftrightarrow AD$$
$$2MD = \Leftrightarrow AB + BD + AD,$$

and from ΔACE:

$$2NC = AC + CE \Leftrightarrow AE$$
$$2NE = \Leftrightarrow AC + CE + AE.$$

From this we see that to show (1) we must show

$$\begin{aligned} & BD(AC + CE \Leftrightarrow AE)(\Leftrightarrow AC + CE + AE) \\ = \quad & CE(AB + BD \Leftrightarrow AD)(\Leftrightarrow AB + BD + AD), \end{aligned}$$

or equivalently

$$\begin{aligned} & BD(CE^2 \Leftrightarrow AC^2 \Leftrightarrow AE^2 + 2AC \cdot AE) \\ = \quad & CE(BD^2 \Leftrightarrow AB^2 \Leftrightarrow AD^2 + 2AB \cdot AD). \end{aligned} \tag{3}$$

Now use the law of cosines in ΔACE, and in ΔABD to obtain

$$\left.\begin{aligned} CE^2 \Leftrightarrow AC^2 \Leftrightarrow AE^2 = \Leftrightarrow 2AC AE \cos\varphi \\ BD^2 \Leftrightarrow AB^2 \Leftrightarrow AD^2 = \Leftrightarrow 2AB AD \cos\varphi. \end{aligned}\right\} \tag{4}$$

Combining (3) and (4) we see that we must show

$$BD \cdot AC \cdot AE(1 \Leftrightarrow \cos\varphi) = CE \cdot AB \cdot AD(1 \Leftrightarrow \cos\varphi).$$

As $1 \Leftrightarrow \cos\varphi \neq 0$, we find with (2) and (2′) that we must verify that

$$\frac{c \sin\varphi}{\sin(\beta+\gamma)} \cdot b \cdot \frac{b \sin\gamma}{\sin(\gamma+\varphi} = \frac{b \sin\varphi}{\sin(\gamma+\varphi)} \cdot c \cdot \frac{c \sin\beta}{\sin(\beta+\varphi)},$$

and as $b \sin\gamma = c \sin\beta$, this holds. ∎

13. [1994: 241] *Proposed by India.*

A natural number n is said to have the property P if, whenver n divides $a^n \Leftrightarrow 1$ for some integer a, n^2 also necessarily divides $a^n \Leftrightarrow 1$.

(a) Show that every prime number has property P.

(b) Show there are infinitely many composite numbers n that possess property P.

Solution by E.T.H. Wang, Sir Wilfrid Laurier University, Waterloo, Ontario.

(a) Suppose $n = p$ is a prime such that $a^p \equiv 1 \bmod p$. Since $a^p \equiv a \bmod p$ by Fermat's Little Theorem, we have $a \equiv 1 \bmod p$. Thus $a = kp+1$ for some integer k. Hence, by the Binomial Theorem $a^p \Leftrightarrow 1 = (kp+1)^p \Leftrightarrow 1 = (kp)^p + \binom{p}{1}(kp)^{p-1} + \cdots + \binom{p}{p-1}kp$. Since $\binom{p}{p-1} = p$ and $p \geq 2$, it follows that $p^2 \mid a^p \Leftrightarrow 1$.

(b) We show that all composite numbers of the form $n = 2p$, where p is an odd prime have property P. Suppose $2p \mid a^{2p} \Leftrightarrow 1$. Then $p \mid (a^2)^p \Leftrightarrow 1$

which, in view of (a), implies $p^2 \mid a^{2p} \Leftrightarrow 1$. On the other hand, $2 \mid a^{2p} \Leftrightarrow 1$ implies $2 \mid (a^p \Leftrightarrow 1)(a^p + 1)$. Since $a^p \Leftrightarrow 1$ and $a^p + 1$ have the same parity, they must both be even, and hence $4 \mid (a^p \Leftrightarrow 1)(a^p + 1)$. Since $\gcd(4, p^2) = 1$, $4p^2 \mid a^{2p} \Leftrightarrow 1$ follows.

Remarks. (1) The fact that all prime numbers satisfy property P is well known. In fact, using exactly the same argument as in the proof of (a) above, one can show easily that if $a^p \equiv b^p \bmod p$, then $a^p \equiv b^p \bmod p^2$. (See e.g. Ex. 13 on page 190 of *Elementary Number Theory and its Applications* by Kenneth H. Rosen, 3rd Edition).

(2) $n = 4$ also has property P. For suppose that $4 \mid a^4 \Leftrightarrow 1$. Then $4 \mid (a^2 \Leftrightarrow 1)(a^2 + 1)$, which implies that a is odd. Since any odd square is congruent to 1 modulo 8, we have $8 \mid a^2 \Leftrightarrow 1$, which, together with $2 \mid a^2 + 1$, yields $16 \mid (a^2 \Leftrightarrow 1)(a^2 + 1)$.

(3) In view of (2) and our proof above, $n = 8$ is the first natural number which need not possess property P. Indeed, it does not since $3^8 \Leftrightarrow 1 = 6560$ is divisible by 8 but not by 64.

(4) It might be of interest to characterize *all* natural numbers with property P.

We finish this number of the Corner with a solution to one of the IMO problems from the 35th IMO in Hong Kong.

2. [1994: 244]

ABC is an isosceles triangle with $AB = AC$. Suppose that

(i) M is the midoint of BC and D is the point on the line AM such that OB is perpendicular to AB;

(ii) Q is an arbitrary point on the segment BC different from B and C;

(iii) E lies on the line AB and F on the line AC such that E, Q and F are distinct and collinear.

Prove that OQ is perpendicular to EF if and only if $QE = QF$.

Solution by D. J. Smeenk, Zaltbommel, the Netherlands.

$(\Longrightarrow)$. See Figure 1. Assume $OQ \perp EF$. We want to show $QE = QF$.

Note that quadrilateral $OBEQ$ is inscribed on the circle with diameter OE. Thus $\angle OEQ = \angle OBQ = \angle OQB = \frac{\alpha}{2}$. Also $OFCQ$ is cyclic and thus $\angle OFQ = \angle OCQ = \angle OAC = \frac{\alpha}{2}$. Together these give $\angle OEQ = \angle OFQ$ and $OQ \perp EF$, so $QE = QF$.

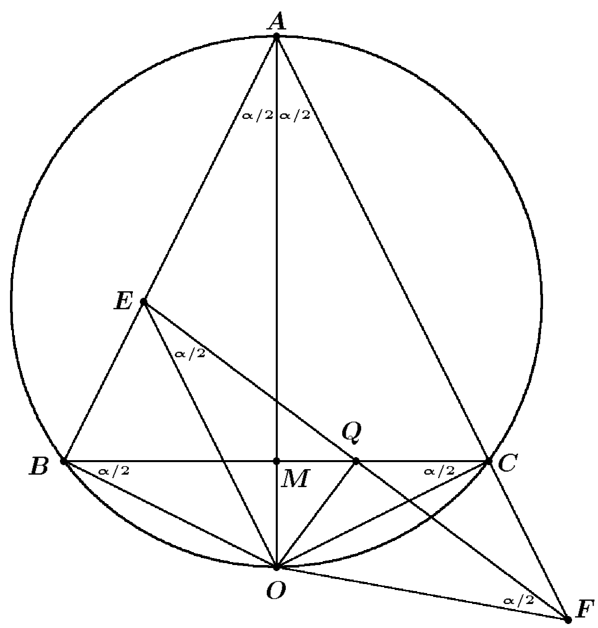


Figure 1.

($\Longleftarrow$). See Figure 2. Assume that $QE = QF$. We want to show that $OQ \perp EF$.

Let D lie on AC with QD parallel to AB. As $QE = QF$ we see that D is the midpoint of AF. As $AB = AC$ we have $DQ = DC$. (1)
Also,

$$QE = 2DQ = 2CD \tag{2}$$

$$BE = AB \Leftrightarrow AE. \tag{3}$$

From (2) and (3)

$$\begin{aligned} BE &= AC \Leftrightarrow 2CD = (AC \Leftrightarrow CD) \Leftrightarrow CD \\ &= AD \Leftrightarrow CD = DF \Leftrightarrow CD. \end{aligned}$$

Thus $BE = CF$. (4)

Draw OB, OE, OC, and OF.

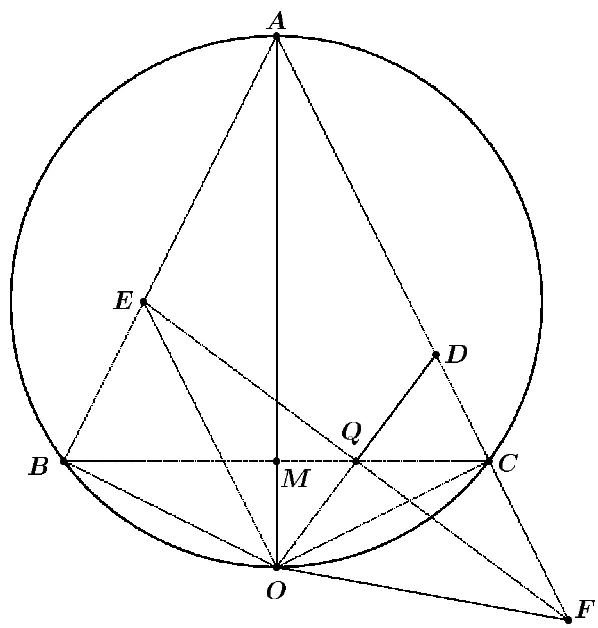


Figure 2.

Then

$$\Delta OBE \cong \Delta COF[BE = CF,\ OB = OC,\ \angle OBE = \angle OCF = n/2]. \quad (5)$$

Now this implies that $OE = OF$, $QE = QF$ giving $OQ \perp EF$.

Remark. This direction could be shortened by using the law of sines in triangles BQF and CQF, but the given solution is, I think, more elementary, and therefore more elegant.

That completes the material we have available for this number. The Olympiad Season is fast approaching. Please collect your contests and send them to me. Also send me your nice solutions to problems posed in the *Corner*.

THE ACADEMY CORNER

No. 2

This will appear in a future issue.

IMO95 PUZZLES

In the September 1995 issue of *CRUX*, two puzzles were given about the logo for the 1995 IMO. The second puzzle, attributed there to Mike Dawes of the University of Western Ontario, asked:

> *Suppose that you take a physical model of this logo, and manipulate it to turn the infinity sign into a circle. What does the circle turn into?*

It turns out that the same question occurred to several other people at the same time, including some members of the Netherlands team at IMO 95.

> Less than half an hour prior to the start of the competition, the 1995 IMO logo inspired one of the contestants (RvL) to pose the following problem:
>
> > Consider the logo as a knot composed of a blue string forming the infinity-sign and a red string forming the zero. Can the knot be rearranged so that the blue string forms the zero and the red string forms the infinity-sign (without using scissors)?
>
> The contestant solved the problem shortly after the competition and, upon returning home, wrote a program together with one of the other members of his team (DG).
>
> Ronald van Luijk (rmluijk@cs.ruu.nl) and Dion Gijswijt.

The program was sent by the Leader of the Netherlands team (Johannes Notenboom) to the Editor-in-Chief (in his role as Chief Operating Officer of the 1995 IMO). We were so impressed by the program that we have had it made available on the IMO95 WWW site. You may download this program (zipped) by going to: **http://camel.math.ca/IMO/IMO95/**.

Ronald has pointed out the the IMO 95 logo is, in fact, the Whitehead Link. See, for example, L.H. Kauffman's book: *On Knots*, Annals of Mathematics Studies, 115, Princeton University Press, 1987, p. 14.

BOOK REVIEWS

Edited by ANDY LIU

Assessing Calculus Reform Efforts : A Report to the Community, edited by James Leitzel and Alan C. Tucker, 1995. Paperback, 100+ pages, US$18.00, ISBN 0-88385-093-1.

Preparing for a New Calculus, edited by Anita Solow, 1994. Paperback, 250+ pages, US$24.00, ISBN 0-88385-092-3.

Both published by The Mathematical Association of America, Washington, DC 20036. *Reviewed by* **Jack Macki**, *University of Alberta.*

Anyone who attended the January 1995 joint meetings in San Francisco could not help but be struck by the contrast between the evident malaise on the research side of the meeting and the enthusiasm, energy and high quality of the discourse on the educational side.

These two books, very different in nature and content, together represent enormously persuasive documentation of the on-going revolution in teaching and curriculum design at the University/College level. They describe the past and present of reform in undergraduate calculus, and evaluate the situation in a thorough and realistic manner.

Assessing Calculus Reform Efforts is one of the best presentations ever of the history, present activities and future plans of the movement for reform in the design and teaching of the undergraduate mathematics curriculum.

The history begins with the founding of the Committee on the Undergraduate Program (CUP) in 1953. The authors describe the debate between Anthony Ralston and Ron Douglas which led to the famous Tulane workshop in 1986 (with 25 participants, four of whom were research mathematicians). They follow the entire development of the reform movement, carefully describing the key players, and the roles of NRC and NSF. They present all kinds of surprising (at least to this reviewer) information along the way — did you know that in 1985 the IEEE, deeply dissatisfied with existing texts, published a calculus book? In the nine years since that seminal Tulane conference, reform has clearly moved into a central position in the mathematical life of North America.

The effort of key individuals to convert the research community to believers (and activists) makes fascinating reading, beginning with a widely disseminated article by Peter Lax (UME Trends, May, 1990) which was highly critical of the attitudes of the majority of research mathematicians. Very determined department chairs at Stony Brook and Michigan convinced their departments to adopt reform texts. Said Don Lewis, chair at Michigan, "An NSF grant providing two months of summer salary is icing on the cake. The cake is calculus."

The editors are very tactful in their presentation, but the lack of interest in education of the majority of research mathematicians in the United States (and Canada?) comes through quite clearly. They present the results of four surveys carried out to date, and discuss the issue of "cosmetic" versus "real" change. The bottom line:

- By Spring 1994, three-quarters of responding departments (1048 respondents) had some reform under way, one-quarter of the respondents were conducting major reforms.
- The Graduate Record Exam has been changed to reflect reform.
- The NSF continues to fund reform initiatives at the level of about $2.7 million per year.
- Sales of reform texts have increased to 108,700 annually (552 institutions) in Fall, 1994 (not counting the 375 high schools which use them).

The main part of this book has only 44 pages of text, and then concludes with 50 pages of important and highly readable appendices: data on enrolments, copies of surveys, text-by-text description of reform materials presently available, and a very detailed list of NSF-funded projects.

If you have colleagues who pooh-pooh the reform movement (and who doesn't?), get them started on the path to rightness with this book. If they can read it and remain unwilling to look at change, order them coffins, they probably died a long time ago!

Preparing for a New Calculus is the proceedings of a conference held at the University of Illinois in April, 1993. Eighty individuals attended, 40 of them associated with projects at colleges or universities, 25 of them associated with projects at the high school level, seven from community college projects, and eight from organizations like the MAA, NSF, NCTM, etc., and publishers.

Part I contains seven background papers, which were provided to participants before the conference. The first paper cites some impressive figures: well over 100 schools use the Harvard Consortium Project materials; 27% of all college-level calculus students in the State of Washington are in a reform course. All seven papers offer several important observations based on experience, for example:

- There is no correct way to do a reform course, there are many ways to treat calculus effectively.
- Reform courses have fewer topics with much richer content.
- The "rule of three" (graphical, numerical, analytical) should be replaced by the "rule of four" (writing) or "five" (oral).
- It is mindless to ask a student who has access to a calculator to approximate $\sqrt{17}$ using a differential, but one can instead study Euler's method for solving the logistic differential equation before the students know any integration theory.

- Reform makes *appropriate* use of technology; technology does not equal reform.
- While most mathematics professors believe that learning is transmitted (I call this the "I am God" theory of learning), most evidence points to the correctness of the constructive theory of learning, which emphasizes that most learning is constructed by the learner in response to challenges to refine or revise what he/she already knows in order to cope with new situations. Sally Berenson, North Carolina State University:

  "Telling is not teaching, listening is not learning".

- Never underestimate the importance and difficulty of educating faculty for real change. Teaching reform courses is very hard work.
- Creating a cooperative learning environment is not easy, but pays off for all students, especially minorities.
- It is very useful to carry open-ended long-term applied problems through one or, even better, several courses. At the U.S. Military Academy at West Point, they begin with a few core applied problems and proceed to attack them with non-calculus methods, leading to a semester's study of difference equations and the associated linear algebra. Calculus starts in term two! K. Stroyan of the University of Iowa starts his course by asking "Why is it we can eradicate polio and smallpox, but not measles and rubella?"

You get the picture, there is just a tremendous amount of experience and insight available in these seven articles — and wonderful quotes: The chairman of physics at Duke, Larry Evans, when asked to comment on changes in the teaching of calculus:

"There is nowhere to go but up."

Other important themes: downgrading of exams, strong emphasis on (and evaluation of) writing, emphasis on cooperative learning. And don't expect all students to love the changes — students who are successful in standard courses often react negatively to being saddled with lab partners and being evaluated in unfamiliar ways. In fact, several authors emphasize that student questionnaires cannot be the sole means of teaching evaluation — one needs to talk to students who have gone down the road a way in order to get a fair picture.

The second part of *Preparing for a New Calculus* reports on the workshops, one each on the topics of content, teaching strategies, and institutional context. Each report has a long and useful list of suggestions and observations. Among them:

- Characterize and reward that which constitutes effective teaching.

- Allocate time as a resource to support involved faculty.
- Every person involved with evaluating staff should read Ernest Boyer's Carnegie Foundation Report "Scholarship Reconsidered".

The third section consists of five contributed papers of very high quality on topics ranging from calculator courses, to the gateway exam at Michigan. The final piece is a thoughtful article by Peter Renz of Academic Press on Publishers, Innovation and Technology, which should be required reading for all university mathematics professors. Many of the articles in this book have extensive lists of very timely and useful references.

All in all, *Preparing for a New Calculus* is an excellent detailed introduction to calculus reform, to be read after *Assessing Calculus Reform Efforts*. Anyone who can read them both and not be interested in and excited about reform doesn't need a coffin, they've been dead too long!

Introducing the new Associate Editor-in-Chief

For those of you who do not know Colin, here is a short profile:

Born:	Bedford, England [1]
Educated	Bedford Modern School University of Birmingham, England University of London, England
Employment	Queen's College, Nassau, Bahamas Sir John Talbot's Grammar School, Whitchurch, England Memorial University of Newfoundland, Canada
Mathematical Interests	Mathematical Education History and Philosophy of Science Nineteenth century Astrophysics

[1] *Bedford is halfway between Oxford and Cambridge (just an average place).*

PROBLEMS

Problem proposals and solutions should be sent to Bruce Shawyer, Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. A1C 5S7. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a submission is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk ($\star$) after a number indicates that a problem was submitted without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later that **1 October 1996**. *They may also be sent by email to* **cruxeditor@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in LaTeX, preferably in LaTeX2e*). *Graphics files should be in epic format, or plain postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication.*

2114. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

$ABCD$ is a square with incircle Γ. A tangent ℓ to Γ meets the sides AB and AD and the diagonal AC at P, Q and R respectively. Prove that

$$\frac{AP}{PB}+\frac{AR}{RC}+\frac{AQ}{QD}=1.$$

2115. *Proposed by Toby Gee, student, The John of Gaunt School, Trowbridge, England.*

Find all polynomials f such that $f(p)$ is a prime for every prime p.

2116. *Proposed by Yang Kechang, Yueyang University, Hunan, China.*

A triangle has sides a, b, c and area F. Prove that

$$a^3b^4c^5 \geq \frac{25\sqrt{5}(2F)^6}{27}.$$

When does equality hold?

2117. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with $AB > AC$, and the bisector of $\angle A$ meets BC at D. Let P be an interior point of the side AC. Prove that $\angle BPD < \angle DPC$.

2118. *Proposed by Paul Yiu, Florida Atlantic University, Boca Raton, Florida, USA.*

The primitive Pythagorean triangle with sides 2547 and 40004 and hypotenuse 40085 has area 50945094, which is an 8-digit number of the form $abcdabcd$. Find another primitive Pythagorean triangle whose area is of this form.

2119. *Proposed by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore.*

(a) Show that for any positive integer $m \geq 3$, there is a permutation of m 1's, m 2's and m 3's such that

  (i) no block of consecutive terms of the permutation (other than the entire permutation) contains equal numbers of 1's, 2's and 3's; and

  (ii) there is no block of m consecutive terms of the permutation which are all equal.

(b) For $m = 3$, how many such permutations are there?

2120. *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

Let $A_1A_3A_5$ and $A_2A_4A_6$ be nondegenerate triangles in the plane. For $i = 1, \ldots, 6$ let ℓ_i be the perpendicular from A_i to line $A_{i-1}A_{i+1}$ (where of course $A_0 = A_6$ and $A_7 = A_1$). If ℓ_1, ℓ_3, ℓ_5 concur, prove that ℓ_2, ℓ_4, ℓ_6 also concur.

2121. *Proposed by Krzysztof Chelmiński, Technische Hochschule Darmstadt, Germany; and Waldemar Pompe, student, University of Warsaw, Poland.*

Let $k \geq 2$ be an integer. The sequence (x_n) is defined by $x_0 = x_1 = 1$ and

$$x_{n+1} = \frac{x_n^k + 1}{x_{n-1}} \qquad \text{for} \quad n \geq 1.$$

(a) Prove that for each positive integer $k \geq 2$ the sequence (x_n) is a sequence of integers.

(b) If $k = 2$, show that $x_{n+1} = 3x_n \Leftrightarrow x_{n-1}$ for $n \geq 1$.

(c)* Note that for $k = 2$, part (a) follows immediately from (b). Is there an analogous recurrence relation to the one in (b), not necessarily linear, which would give an immediate proof of (a) for $k \geq 3$?

2122. *Proposed by Shawn Godin, St. Joseph Scollard Hall, North Bay, Ontario.*

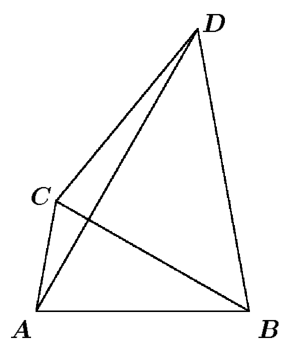
Little Sam is a unique child and his math marks show it. On four tests this year his scores out of 100 were all two-digit numbers made up of eight different non-zero digits. What's more, the average of these scores is the same as the average if each score is reversed (so 94 becomes 49, for example), and this average is an integer none of whose digits is equal to any of the digits in the scores. What is Sam's average?

2123. *Proposed by Sydney Bulman–Fleming and Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

It is known (e.g., exercise 23, page 78 of Kenneth H. Rosen's *Elementary Number Theory and its Applications*, Third Edition) that every natural number greater than 6 is the sum of two relatively prime integers, each greater than 1. Find all natural numbers which can be expressed as the sum of *three* pairwise relatively prime integers, each greater than 1.

2124. *Proposed by Catherine Shevlin, Wallsend, England.*

Suppose that $ABCD$ is a quadrilateral with $\angle CDB = \angle CBD = 50^\circ$ and $\angle CAB = \angle ABD = \angle BCD$. Prove that $AD \perp BC$.



Mathematical Literacy

1. Who said: "To be able to read the great book of the universe, one must first understand its language, which is that of mathematics".
2. In referring to "the unreasonable effectiveness of mathematics in the natural sciences", who wrote: "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve".

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

1827. [1993: 78; 1994:57] *Proposed by Šefket Arslanagić, Berlin, Germany, and D.M. Milošević, Pranjani, Yugoslavia.*

In commenting on the solutions submitted, the editor asked for a short proof of

$$\sum \frac{bc}{s \Leftrightarrow a} = s + \frac{(4R+r)^2}{s}.$$

Two very nice and very different solutions have been received.

I. *Solution by TOSHIO SEIMIYA, Kawasaki, Japan.*

Since $\tan(A/2) = r_1/s$, and using a result in R.A. Johnson, *Advanced Euclidean Geometry*, p. 189, we obtain

$$\sum \tan(A/2) = \sum r_1/s = (4R+r)/s.$$

Since $A + B + C = \pi$, we then have $\sum \tan(B/2)\tan(C/2) = 1$, which leads to

$$\left(\sum \tan(A/2)\right)^2 = \Leftrightarrow 1 + \sum \sec^2(A/2).$$

Together, we now have $\sum \sec^2(A/2) = 1 + \left(\dfrac{4R+r}{s}\right)^2$.

Since $\dfrac{bc}{s \Leftrightarrow a} = s\left(\dfrac{bc}{s(s \Leftrightarrow a)}\right) = s\ \sec^2(C/2)$, we have

$$\sum \frac{bc}{s \Leftrightarrow a} = s\left(\sum \sec^2(A/2)\right) = s + \frac{(4R+r)^2}{s}. \quad \blacksquare$$

II. *Solution by Waldemar Pompe, student, University of Warsaw, Poland.*

Since $\sum a = 2s$, $\sum bc = s^2 + r^2 + 4Rr$, and $abc = 4sRr$, we get:

$$\sum (s \Leftrightarrow b)(s \Leftrightarrow c) = 3s^2 \Leftrightarrow 4s^2 + \sum bc = r^2 + 4Rr.$$

Using this, we obtain

$$\sum \frac{1}{s \Leftrightarrow a} = \frac{r^2 + 4Rr}{(s \Leftrightarrow a)(s \Leftrightarrow b)(s \Leftrightarrow c)} = \frac{r^2 + 4Rr}{sr^2} = \frac{r + 4R}{sr}.$$

Therefore

$$\begin{aligned}\sum \frac{bc}{s \Leftrightarrow a} &= abc \sum \frac{1}{a(s \Leftrightarrow a)} = 4Rr \sum \left(\frac{1}{a} + \frac{1}{s \Leftrightarrow a}\right) \\ &= 4Rr \left(\frac{s^2 + r^2 + 4Rr}{4sRr} + \frac{r + 4R}{sr}\right) \\ &= s + \frac{r^2 + 4Rr}{s} + \frac{4Rr + 16R^2}{s} = s + \frac{(4R + r)^2}{s}. \quad \blacksquare\end{aligned}$$

2000. [1994: 286] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

A 1000–element set is randomly chosen from $\{1, 2, \ldots, 2000\}$. Let p be the probability that the sum of the chosen numbers is divisible by 5. Is p greater than, smaller than, or equal to $1/5$?

Comment by Stan Wagon, Macalester College, St. Paul, Minnesota, USA.

The answer is that

$$p = \frac{1}{5} + \frac{4}{5}\frac{\binom{400}{200}}{\binom{2000}{1000}} = \frac{1}{5} + 410^{-482}.$$

See Stan Wagon and Herbert S. Wilf, *When are subset sums equidistributed modulo m?*, Electronic Journal of Combinatorics 1 (1994).

In particular, we obtained a necessary and sufficient condition that the t–subsets of $[n]$ be equidistributed $\mod m$. That condition is:

> $t* > n* \mod d$ for all d dividing m (except $d = 1$), where $*$ refers to the least non-negative residue.

In the case at hand, $m = 5$, so only $d = 5$ need be considered, and then $t* = n* = 0$, so equidistribution fails. While this does not directly answer problem 2000 (since it gives no information about the specific remainder $0 \mod 5$), the paper does discuss many intriguing open questions related to the $\mod m$ distribution of subset sums. Thus it strikes me that, because some of your readers were successful at generalizing the problem as stated, they would be interested in this reference. Indeed, perhaps some characterization of the quadruples (i, t, n, m), such that the set of t–subsets of $[n]$ whose $\mod m$ sum is i has less than average size, is possible.

The problem arose from lottery considerations. When are the tickets in a lottery equidistributed with respect to the $\mod m$ value of their sums?

2011. [1995: 52] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is a triangle with incentre I. BI and CI meet AC and AB at D and E respectively. P is the foot of the perpendicular from I to DE, and IP meets BC at Q. Suppose that $IQ = 2IP$. Find angle A.

Solution by the proposer.

Let X and Y be the feet of perpendiculars from I to BC and AB, then $IX = IY = r$, where r is the inradius of $\triangle ABC$.

We put $\angle ABI = \angle IBC = \beta$, $\angle ACI = \angle ICB = \gamma$, $\angle EIB = \alpha$, $\angle IDE = x$, and $\angle IED = y$. Then we have $\beta + \gamma = \alpha$ and $x + y = \alpha$. As $\angle IQC = \angle IBQ + \angle BIQ = \angle IBQ + \angle DIP = \beta + \frac{\pi}{2} \Leftrightarrow x$, we get $r = IX = IQ \sin \angle IQC = IQ \sin(\beta + \frac{\pi}{2} \Leftrightarrow x)$, so that

$$r = IQ \cos(\beta \Leftrightarrow x) \tag{1}$$

As $\angle AEI = \angle EBI + \angle EIB = \alpha + \beta$, we get

$$r = IY = IE \sin \angle AEI = IE \sin(\alpha + \beta). \tag{2}$$

Because $PI = IE \sin y$, and $IQ = 2PI$, we have from (1)

$$r = 2PI \cos(\beta \Leftrightarrow x) = 2IE \sin y \cos(\beta \Leftrightarrow x). \tag{3}$$

From (2) and (3) we have (cancelling IE),

$$\begin{aligned} \sin(\alpha + \beta) = 2 \sin y \cos(\beta \Leftrightarrow x) &= \sin(\beta + y \Leftrightarrow x) + \sin(x + y \Leftrightarrow \beta) \\ &= \sin(\beta + y \Leftrightarrow x) + \sin(\alpha \Leftrightarrow \beta). \end{aligned}$$

Therefore $\sin(\beta + y \Leftrightarrow x) = \sin(\alpha + \beta) \Leftrightarrow \sin(\alpha \Leftrightarrow \beta) = 2 \cos \alpha \sin \beta$, thus we obtain

$$\sin \beta \cos(y \Leftrightarrow x) + \cos \beta \sin(y \Leftrightarrow x) = 2 \cos \alpha \sin \beta. \tag{4}$$

Similarly we have (exchanging β for γ, and x for y, y for x, simultaneously) $\sin \gamma \cos(x \Leftrightarrow y) + \cos \gamma \sin(x \Leftrightarrow y) = 2 \cos \alpha \sin \gamma$, or

$$\sin \gamma \cos(y \Leftrightarrow x) \Leftrightarrow \cos \gamma \sin(y \Leftrightarrow x) = 2 \cos \alpha \sin \gamma. \tag{5}$$

Multiplying (4) by $\sin \gamma$ and (5) by $\sin \beta$, we get

$$(\sin \gamma \cos \beta + \cos \gamma \sin \beta) \sin(y \Leftrightarrow x) = 0$$

or $\sin(\beta + \gamma) \sin(y \Leftrightarrow x) = 0$, that is $\sin \alpha \sin(y \Leftrightarrow x) = 0$.

Since $\sin \alpha > 0$, we have $\sin(y \Leftrightarrow x) = 0$, therefore $y = x$, and $\cos(y \Leftrightarrow x) = 1$. Hence we get from (4) $\sin \beta = 2 \cos \alpha \sin \beta$. Because $\sin \beta > 0$ we get $\cos \alpha = \frac{1}{2}$. Thus we have $\alpha = 60°$. As $\alpha = 90° \Leftrightarrow \frac{1}{2} \angle A$, we obtain $\angle A = 60°$. ■

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UKand KEE-WAI LAU, Hong Kong. There was one incorrect solution.

2016. [1995: 53] *Proposed by N. Kildonan, Winnipeg, Manitoba.*

Recall that $0.\overline{19}$ stands for the repeating decimal $0.19191919\ldots$, for example, and that the period of a repeating decimal is the number of digits in the repeating part. What is the period of

$$\text{(a) } 0.\overline{19} + 0.\overline{199}\,, \qquad \text{(b) } 0.\overline{19} \times 0.\overline{199}\ ?$$

Solution by Christopher J. Bradley, Clifton College, Bristol, UK

(a) We have

$$0.\overline{19} + 0.\overline{199} = \frac{19}{99} + \frac{199}{999} = \frac{19 \times 10101 + 199 \times 1001}{999999} = \frac{391118}{999999}$$
$$= 0.\overline{391118},$$

so the period is **6**.

(b) Since

$$0.\overline{19} \times 0.\overline{199} = \frac{19 \times 199}{99 \times 999}\,,$$

we look for a number with all nines which is a multiple of 999×99. In order to be a multiple of 999 it must have $3k$ nines for some integer k. When this number is divided by 999, the quotient has k ones interspersed with pairs of zeros: $1001001\ldots1001$. In order that this quotient be divisible by 99 as well, k must be divisible by 9 *and* must be even (using the well known rules for divisibility by 9 and 11). Hence the required number will have $3 \times 9 \times 2 = 54$ nines, so the period is **54**. That the number has the full period of 54 results from the fact that 19 and 199 are both coprime to 99 and 999 ensuring no fortuitous cancellations. ∎

Also solved by HAYO AHLBURG, Benidorm, Spain; ŠEFKET ARSLANAGIĆ, Berlin, Germany; NIELS BEJLEGAARD, Stavanger, Norway; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LÓPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; TIM CROSS, Wolverley High School, Kidderminster, UK; KEITH EKBLAW, Walla Walla, Washington, USA; JEFFREY K. FLOYD, Newnan, Georgia, USA; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; DAVID E. MANES, State University of New York, Oneonta, NY, USA; J. A. MCCALLUM, Medicine Hat, Alberta; P. PENNING, Delft, the Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; HEINZ-JURGËN SEIFFERT, Berlin, Germany; DAVID R. STONE, Georgia Southern University, Statesboro; PANOS E. TSAOUSSOGLOU, Athens, Greece; EDWARD T. H. WANG, Wilfrid Laurier University, Waterloo, Ontario; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer.

Part (a) *only was solved by CHARLES ASHBACHER, Cedar Rapids, Iowa, USA; TOBY GEE, student, The John of Gaunt School, Trowbridge, England; and JOHN S. VLACHAKIS, Athens, Greece. Part* (b) *only was solved by KEE-WAI LAU, Hong Kong.*

Konečný and Perz had solutions which were comparable to Bradley's in their simplicity and minimal dependence on calculation.

Cross notes that the product in (b) *equals*

$$\frac{3781}{98901} = \frac{3780}{98901} + \frac{1}{98901},$$

where the first fraction is $140/3663 = 0.\overline{038220}$, *and the second fraction is the "rather interesting" recurring decimal*

$$0.\overline{000010\ 111121\ 222232\ 333343\ 444454\ 555565\ 666676\ 777787\ 888899}\ !$$

Bellot and López mention the following general result of J. E. Oliver which is quoted (for an arbitrary number of fractions) on page 166 *of Dickson's* History of the Theory of Numbers, Volume 1*: if* x'/x *and* z'/z *are periodic fractions, with periods* a *and* b *respectively, then* $(x'/x)(z'/z)$ *has a period of*

$$\frac{xz}{[x, z]} \cdot [a, b],$$

where $[a, b]$ *is the least common multiple of* a *and* b. *This formula, which can be more simply written as* $(x, z)[a, b]$ *where* (x, z) *is the greatest common divisor of* x *and* z, *gives precisely* 54 *in our case. However, perhaps "period" here was intended to mean* **any** *period rather than just the smallest period, because the formula fails in many cases. For example, for the fractions* $11/3$ *and* $1/11$, *which have periods* 1 *and* 2 *respectively, the formula gives* $(3, 11)[1, 2] = 1 \cdot 2 = 2$ *as the period for the product* $(11/3)(1/11) = 1/3$, *which has period only* 1 *of course. Unfortunately, the reference given for Oliver's result is page* 295 *of* Math. Monthly, *Vol. 1, 1859; this cannot be the familiar* American Math. Monthly, *which only started publishing in 1894. Can any reader supply us with more information?*

2017. [1995: 53] *Proposed by D. J. Smeenk, Zaltbommel, the Netherlands.*

We are given a fixed circle κ and two fixed points A and B not lying on κ. A variable circle through A and B intersects κ in C and D. Show that the ratio

$$\frac{AC \cdot AD}{BC \cdot BD}$$

is constant. [This is not a new problem. A reference will be given when the solution is published.]

Solution by Toshio Seimiya, Kawasaki, Japan.

Let γ be a fixed circle passing through A, B and intersecting κ at X and Y. Then XY, CD, and AB are all parallel or concurrent at the radical centre of the three circles.

Case 1. $XY \| AB$.

If $XY \| AB$ then $CD \| AB$, so $AC = BD$ and $AD = BC$. Hence

$$\frac{AC \cdot AD}{BC \cdot BD} = 1 \text{ (which is a constant).}$$

Case 2. $XY \not\| AB$.

Let P be the intersection of XY with AB, then P is a fixed point (the radical centre) and CD passes through P.

Since either $\angle CAD = \angle CBD$, or $\angle CAD + \angle CBD = 180°$, we get

$$\frac{[ACD]}{[BCD]} = \frac{AC \cdot AD}{BC \cdot BD}, \tag{1}$$

where $[UVW]$ denotes the area of triangle UVW.

Let A', B' be the feet of perpendiculars from A, B to CD, then $AA' \| BB'$, and

$$\frac{AA'}{BB'} = \frac{AP}{BP}. \tag{2}$$

Because $\dfrac{[ACD]}{[BCD]} = \dfrac{AA'}{BB'}$, we have from (1) and (2)

$$\frac{AC \cdot AD}{BC \cdot BD} = \frac{AP}{BP} = \text{constant.} \quad \blacksquare$$

The proposer tells us that the problem comes from a 1939 book by Dr. P. Molenbroek.

(*Note that* $\angle CAD$ *and* $\angle DBC$ *might be supplementary rather than equal, so* $\sin \angle CAD = \sin \angle DBC$ *still holds, but the triangles* CAD *and* DBC *are not necessarily similar.*)

Also solved by: CLAUDIO ARCONCHER, Jundiaí, Brazil; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAN PEDOE, Minneapolis, Minnesota, USA; P. PENNING, Delft, the Netherlands; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; there were four incorrect solutions.

2018. [1995: 53] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

How many permutations $(x_1, \ldots, x_n)$ of $\{1, \ldots, 2\}$ are there such that the cyclic sum $\sum_{i=1}^{n} |x_1 \Leftrightarrow x_{i+1}|$ (with $x_{n+1} = x_1$) is (a) a minimum, (b) a maximum?

Solution by Carl Bosley, student, Washburn Rural High School, Topeka, Kansas, USA.

(a) Let j and k be such that $x_j = 1$ and $k_k = n$. For all m, define $x_{n+m} = x_m$. Then we have

$$\begin{aligned} \sum_{i=1}^{n} |x_i \Leftrightarrow x_{i+1}| &= \sum_{i=j}^{k-1} |x_i \Leftrightarrow x_{i+1}| + \sum_{i=k}^{j-1} |x_i \Leftrightarrow x_{i+1}| \\ &\geq |x_k \Leftrightarrow x_j| + |x_j \Leftrightarrow x_k| \quad \text{by the triangle inequality} \\ &= 2n \Leftrightarrow 2. \end{aligned}$$

Equality holds if and only if both sequences

$$\begin{aligned} &\{x_k = n,\ x_{k+1},\ \ldots,\ x_{j-1},\ x_j = 1\} \\ &\{x_k,\ x_{k-1},\ \ldots,\ x_{j+1},\ x_j = 1\} \end{aligned}$$

are monotonic decreasing. Each element of $\{n \Leftrightarrow 1,\ n \Leftrightarrow 2,\ \ldots,\ 3,\ 2\}$ may be in either the first or second sequence, but not both. Choosing which elements are in these sequences determines the positions of all of $\{n \Leftrightarrow 1, n \Leftrightarrow 2, \ldots, 2, 1\}$ uniquely. Thus there are $n2^{n-2}$ permutations with minimal sum $2n \Leftrightarrow 2$. ∎

(b) Consider the effect of a single element x_i on the sum. If $x_i > x_{i+1}$ and $x_i > x_{i-1}$, then the two terms in the sum involving x_i are $(x_i \Leftrightarrow x_{i+1})$ and $(x_i \Leftrightarrow x_{i-1})$. Thus the term x_i contributes $2x_i$ to the sum. Similarly, if $x_i < x_{i+1}$ and $x_i < x_{i-1}$, then the term x_i contributes $\Leftrightarrow 2x_i$ to the sum. Further, if x_i is less than one of x_{i-1}, x_{i+1}, and greater than the other, then x_i has no contribution to the sum.

Suppose that j elements have no contribution to the sum. There are n pairs of elements x_i, x_{i+1}. In each pair, one number is greater and one is less than the other. Thus there are $\frac{n-j}{2}$ numbers x_i which contribute $2x_i$ to the sum, and $\frac{n-j}{2}$ numbers x_i which contribute $\Leftrightarrow 2x_i$ to the sum. If we can arrange the numbers x_i which contribute $2x_i$ to the sum to be the largest $\frac{n-j}{2}$ numbers, and the numbers x_i which contribute $\Leftrightarrow 2x_i$ to the sum to be the smallest $\frac{n-j}{2}$ numbers, then a maximum value is clearly attained with j as small as possible.

This is indeed possible. If n is even ($n = 2k$) and $j = 0$, we must have that $\{x_i,\ x_{i+2},\ \ldots,\ x_{i-4},\ x_{i-2}\}$ is a permutation of $\{k+1,\ k+2,\ \ldots,\ 2k\}$ for $i = 0$ or $i = 1$, and the other k numbers must be a permutation of $\{1,\ 2,\ \ldots,\ k\}$. Otherwise, for some x_i, we would have $x_i < x_{i+1} < x_{i+2}$ which is a contradiction. This gives a same maximal sum of $2(k + 1 + k + 2 + \ldots + 2k) \Leftrightarrow 2(1 + 2 + \ldots + k) = 2(k^2) = n^2/2$, for any permutation of $\{k+1,\ k+2\ \dots,\ 2k\}$ and $\{1,\ 2,\ \dots,\ k\}$. So there are $2(k!)^2$ possible permutations with maximal sum if $n = 2k$.

If n is odd ($n = 2k+1$), then j must be odd, so j is at least 1. Placing the middle element $k+1$ in one of the $2k+1$ possible positions gives permutations of $\{1,\ 2,\ \dots,\ k\}$ and $\{k+2,\ k+3, \dots,\ 2k+1\}$ in alternating positions. This gives the same maximal sum, $(n^2 \Leftrightarrow 1)/2$, for every such permutation. Hence there are $2(2k+1)(k!)^2$ possible permutations with maximal sum of $n = 2k+1$. ∎

Also solved by NEILS BEJLEGAARD, Stavanger, Norway; TOBY GEE, student, The John of Gaunt School, Trowbridge, England, part (a) only; P. PENNING, Delft, the Netherlands; and the proposer.

2019. [1995: 53] *Proposed by P. Penning, Delft, the Netherlands.*

In a plane are given a circle C with diameter ℓ, and a point P within C but not on ℓ. Construct the equilateral triangles that have one vertex at P, one on C, and one on ℓ.

Solution by Carl Bosley, student, Washburn Rural High School, Topeka, Kansas, USA.

Assume that we have a solution triangle, and consider the effect of rotating the circle C by $60°$ about P. We can do this in two directions, clockwise or counterclockwise. The vertex of the equilateral triangle on C is rotated to the vertex on the original diameter. Since each rotated circle intersects the diameter only once (or twice, if ℓ is to be interpreted as the line containing the given diameter), this vertex is now known and we can easily find the vertex on C of the equilateral triangle. We see that there are two such equilateral triangles (four if the extension of the diameter is allowed). To construct the pair of rotated circles, find their centres as the third vertices of the equilateral triangles that have PO as base, where O is the centre C; their radius is half the length of the given diameter. ∎

Also solved by CLAUDIO ARCONCHER, Jundiaí, Brazil; NIELS BEJLEGAARD, Stavanger, Norway; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; PETER HURTHIG, Columbia College, Burnaby, BC; D. J. SMEENK, Zaltbommel, the Netherlands; and the proposer. One incorrect solution was received.

Most of the solvers used an argument similar to the featured solution. Bradley remarked that he has seen the rotation technique used to solve the analogous problem in which a line parallel to ℓ is given instead of the circle C. Bejlegaard went even further, pointing out that the given triangle could have any prescribed shape (instead of equilateral) and that C and ℓ could be any two curves for which the points of intersection of ℓ with the rotated image of C could be constructed.

2020. [1995: 53] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Let a, b, c, d be **distinct** real numbers such that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}=4 \quad \text{and} \quad ac=bd.$$

Find the maximum value of

$$\frac{a}{c}+\frac{b}{d}+\frac{c}{a}+\frac{d}{b}.$$

Solution by Carl Bosley, student, Washburn Rural High School, Topeka, Kansas, USA.

Let $x=\frac{a}{b}$ and $y=\frac{b}{c}$. Then by $ac=bd$ we have $\frac{c}{d}=\frac{b}{a}=\frac{1}{x}$, $\frac{d}{a}=\frac{c}{b}=\frac{1}{y}$, $\frac{a}{c}=xy$ and $\frac{b}{d}=\frac{b}{c}\cdot\frac{c}{d}=\frac{y}{x}$. Thus we are to find the maximum value of $xy+\frac{y}{x}+\frac{1}{xy}+\frac{x}{y}$ subject to the conditions that a, b, c, d are distinct and

$$(*) \quad x+y+\frac{1}{x}+\frac{1}{y}=4.$$

Let $e=x+\frac{1}{x}$ and $f=y+\frac{1}{y}$. Then $xy+\frac{z}{x}+\frac{1}{xy}+\frac{x}{y}=ef$.

By the Arithmetic Mean-Geometric Mean Inequality, we have $t+\frac{1}{t}\geq 2$ if $t>0$ and $t+\frac{1}{t}\leq \Leftrightarrow 2$ if $t<0$.

By $(*)$, we have that x and y cannot both be negative. If both are positive, then $(*)$ implies $x=y=1$ or $a=b=c$, a contradiction. Hence exactly one of x and y is negative.

Assume, without loss of generality, that $x>0$, $y<0$. Then we get $f\leq \Leftrightarrow 2$, $e=4\Leftrightarrow f\geq 6$ and so $ef\leq \Leftrightarrow 12$. Equality holds, for example, when $a=3+2\sqrt{2}$, $b=1$, $c=\Leftrightarrow 1$ and $d=\Leftrightarrow(3+2\sqrt{2})$. ∎

Also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; NIELS BEJLEGAARD, Stavanger, Norway; ADRIAN CHAN, student, Upper Canada College, Toronto, Ontario; F.J. FLANIGAN, San Jose State University, San Jose, California, USA; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; PETER HURTHIG, Columbia College, Burnaby, BC; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, Michigan, USA; KEE-WAI LAU, Hong Kong; DAVID E. MANES, SUNY at Oneonta, Oneonta, New York, USA; BEATRIZ MARGOLIS, Paris, France; P. PENNING, Delft, the Netherlands; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; JOHN VLACHAKIS, Athens, Greece; and the proposer. Ten incorrect or incomplete solutions were also received. (*Is this a record?*)

Many of them showed that $\frac{a}{c}+\frac{b}{d}+\frac{c}{a}+\frac{d}{b} \leq 4$ *and then claimed erroneously that* 4 *is actually the maximum value.*

From Bosley's solutions above, it is not difficult to show that the upper bound $\Leftrightarrow 12$ *is attained if and only if* (a, b, c, d) *equals* $(k, (3 \pm 2\sqrt{2})k, \Leftrightarrow (3 \pm 2\sqrt{2})k, \Leftrightarrow k)$ *or* $(k, \Leftrightarrow k, \Leftrightarrow (3 \pm 2\sqrt{2})k, (3 \pm 2\sqrt{2})k)$ *for some* $k \neq 0$. *This was shown by Flanigan and Mane.*

Arslanagić conjectured that the minimum value of the given sum is $\Leftrightarrow 27\frac{661}{900}$*!*

2021. [1995: 89] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

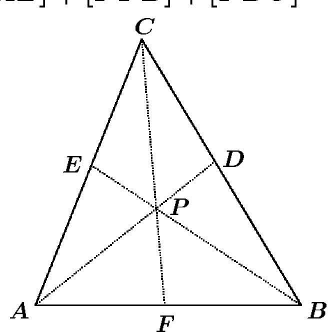
P is a variable interior point of a triangle ABC, and AP, BP, CP meet BC, CA, AB at D, E, F respectively. Find the locus of P so that

$$[PAF] + [PBD] + [PCE] = \frac{1}{2}[ABC],$$

where $[XYZ]$ denotes the area of triangle XYZ.

I. *Solution by P. Penning, Delft, the Netherlands.*

First, note that $[PAE] + [PFB] + [PDC] = \frac{1}{2}[ABC]$.



Let $AF/AB = \frac{1}{2} \Leftrightarrow x$, $BD/BC = \frac{1}{2} \Leftrightarrow y$, $CE/CA = \frac{1}{2} \Leftrightarrow z$.
From Ceva's Theorem, we get:

$$\left(\tfrac{1}{2} \Leftrightarrow x\right)\left(\tfrac{1}{2} \Leftrightarrow y\right)\left(\tfrac{1}{2} \Leftrightarrow z\right) = \left(\tfrac{1}{2} + x\right)\left(\tfrac{1}{2} + y\right)\left(\tfrac{1}{2} + z\right)$$

or

$$x + y + z = 4xyz. \qquad (1)$$

From the figure, we can see that

$$\left(\tfrac{1}{2} \Leftrightarrow x\right)[ABC] = [PAF]+[PAE]+[PEC] = [PAE]+\tfrac{1}{2}[ABC] \Leftrightarrow [PBD].$$

Thus

$$x[ABC] = [PBD] \Leftrightarrow [PAE]. \qquad (2)$$

Similarly, we have

$$y[ABC] \;=\; [PCE] \Leftrightarrow [PFB], \qquad (3)$$
$$z[ABC] \;=\; [PAF] \Leftrightarrow [PDC]. \qquad (4)$$

Adding (2), (3), and (4) gives

$$(x+y+z)[ABC] \;=\; \tfrac{1}{2}[ABC] \Leftrightarrow \tfrac{1}{2}[ABC],$$

so that $x+y+z=0$. This, together with (1), gives $xyz=0$, so that $x=0$ or $y=0$ or $z=0$.
Thus the locus consists of the three medians of the triangle (excluding the end points). ∎

II. *Solution by the Austin Academy Problem Solvers, Austin, Texas, USA.*

Assign masses a, b, c to the vertices A, B, C, respectively, so that P is the centre of mass and $a+b+c=2$.
Then $\dfrac{[PAF]}{[ABC]} = \dfrac{FP}{CF} \cdot \dfrac{AF}{AB} = \dfrac{c}{(a+b+c)} \cdot \dfrac{b}{(a+b)}$.
There are similar expression for PBD and PCE.
So we need to find the locus of P such that

$$\frac{bc}{(a+b)} + \frac{ac}{(b+c)} + \frac{ab}{(a+c)} \;=\; \frac{(a+b+c)}{2} \quad (\,=\,1).$$

Cross multiplying and simplifying leads to

$$(a \Leftrightarrow b)(b \Leftrightarrow c)(c \Leftrightarrow a)(a+b+c) = 0.$$

But $a+b+c=2$, so we must have at least one of $a=b$, $b=c$ and $c=a$.
Now, this is exactly when P lies on a median of triangle ABC (excluding the end points). ∎

Also solved by ŠEFKET ARSLANAGIĆ, Berlin, Germany; CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL AMENGUAL COVAS, Cala Figuera, Mallora, Spain; PETER HURTHIG, Columbia College, Burnaby, BC; KEE-WAI LAU, Hong Kong; ASHISH KR. SINGH, Student, Kanpur, India; D. J. SMEENK, Zaltbommel, the Netherlands; HOE TECK WEE, student, Hwa Chong Junior College, Singapore; and the proposer.

2022. [1995: 89] *Proposed by K. R. S. Sastry, Dodballapur, India.*
Find the smallest integer of the form

$$\frac{A \star B}{B},$$

where A and B are three-digit integers and $A \star B$ denotes the six-digit integer formed by placing A and B side by side.

Solution by M. Parmenter, Memorial University of Newfoundland, St. John's, Newfoundland.

I claim that the answer to the problem is 121.

Note that when $A = 114$ and $B = 950$, then $\frac{A \star B}{B} = 121$, so this number is actually obtained. We must therefore show that in any other case when $D = \frac{A \star B}{B}$ is an integer, then $D \geq 121$.

Observe that $\frac{A \star B}{B} = \frac{(1000A + B)}{B} = \frac{1000A}{B} + 1$, so we must now show that whenever $E = \frac{1000A}{B}$ is an integer, then $E \geq 120$.

Let $B = GX$ where $G = \gcd(B, 1000)$. We will show that whenever $\frac{A}{X}$ is an integer, then $F = \left(\frac{1000}{G}\right)\left(\frac{A}{X}\right) \geq 120$. Note that since $B < 1000$, we have $X < \frac{1000}{G}$. Also, recall that we are only interested in cases when $\frac{A}{X}$ is an integer.

If $G = 1, 2, 4, 5$ or 8, then $\frac{1000}{G} > 120$, and we are done.

If $G = 10$, then $X < 100$, so $\frac{A}{X} \geq 2$ (since $A \geq 100$). In this case, $F \geq 200$, and we are done.

If $G = 20$, then $X < 50$ and $\frac{A}{X} \geq 3$. Thus $F \geq 150$, and we are done.

If $G = 25$, then $X < 40$ and $\frac{A}{X} \geq 3$. Thus $F \geq 120$, and we are done.

If $G = 40$, then $X < 25$ and $\frac{A}{X} \geq 5$. Thus $F \geq 125$, and we are done.

If $G = 50$, then $X < 20$ and $\frac{A}{X} \geq 6$. Thus $F \geq 120$, and we are done.

If $G = 100$, then $X < 10$ and $\frac{A}{X} \geq 12$. (Note that $x \leq 9$). Thus $F \geq 120$, and we are done.

If $G = 125$, then $X < 8$ and $\frac{A}{X} \geq 15$. Thus $F \geq 120$, and we are done.

If $G = 200$, then $X < 5$ and $\frac{A}{X} \geq 25$. (Note that $x \leq 4$). Thus $F \geq 125$, and we are done.

If $G = 250$, then $X < 4$ and $\frac{A}{X} \geq 34$. Thus $F \geq 136$, and we are done.

If $G = 500$, then $B = 500$ and $\frac{1000A}{B} = 2A \geq 200$, and we are now all done. ∎

Note that I am assuming that A is an "honest" three-digit integer, that is $100 \leq A$. Otherwise, there is a trivial solution: $A = 001$, $B = 500$, and $\frac{A \star B}{B} = 3$.

Also solved by CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; MIGUEL ANGEL CABEZÓN OCHOA, Logroño, Spain; JEFFREY R. FLOYD, Newnan, Georgia, USA; TOBY GEE, student, the John of Gaunt School, Trowbridge, England; SHAWN GODIN, St. Joseph Scollard Hall, North Bay, Ontario; RICHARD I. HESS, Rancho Palos Verdes, California, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; J. A. MCCALLUM, Medicine Hat, Alberta; STEWART METCHETTE, Culver City, California, USA; P. PENNING, Delft, the Netherlands; the SCIENCE ACADEMY PROBLEM SOLVERS; Austin, Texas, USA; DAVID STONE and BILL MEISEL, Georgia Southern University, Statesboro, Georgia, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; HOE TECK WEE, student, Hwa Chong Junior College, Singapore; CHRIS WILDHAGEN, Rotterdam, the Netherlands; and the proposer. Some solvers made use of computers to search for all possible solutions. One incorrect solution was submitted. Janous suggested an extension to:

Let $\lambda \in \mathbb{N}$. Determine the integer-minimum, λ_n of $\frac{\lambda A}{B}$, where A and B are n-digit numbers.

2023. [1995: 89] *Proposed by Waldemar Pompe, student, University of Warsaw, Poland.*

Let a, b, c, d, e be positive numbers with $abcde = 1$.

(a) Prove that

$$\frac{a+abc}{1+ab+abcd}+\frac{b+bdc}{1+bc+bcde}+\frac{c+cde}{1+cd+cdea}+\frac{d+dea}{1+de+deab}+\frac{e+eab}{1+ea+eabc}\geq\frac{10}{3}.$$

(b) Find a generalization!

Solution to (a) by Carl Bosley, student, Washburn Rural High School, Topeka, Kansas, USA.

Let $x_1 = a$, $x_2 = ab$, $x_3 = abc$, $x_4 = abcd$ and $x_5 = abcde = 1$. Multiply the second, the third, the fourth and the fifth fractions on the left by $\frac{a}{a}$, $\frac{ab}{ab}$, $\frac{abc}{abc}$ and $\frac{abcd}{abcd}$ respectively. Then the expression on the left becomes:

$$\frac{x_1+x_3}{x_5+x_2+x_4}+\frac{x_2+x_4}{x_1+x_3+x_5}+\frac{x_3+x_5}{x_2+x_4+x_1}+\frac{x_4+x_1}{x_3+x_5+x_2}+\frac{x_5+x_2}{x_4+x_1+x_3}.$$

Add 1 to each of the five fractions. Then the desired inequality is equivalent to:

$$\left(\sum_{i=1}^{5}x_i\right)\left(\frac{1}{x_5+x_2+x_4}+\frac{1}{x_1+x_3+x_5}+\frac{1}{x_2+x_4+x_1}+\frac{1}{x_3+x_5+x_2}+\frac{1}{x_4+x_1+x_3}\right)\geq\frac{25}{3},$$

which follows by applying the Arithmetic Mean – Harmonic Mean inequality to the second factor on the left. Equality holds if and only if the five denominators are all equal, which happens if and only if $x_1 = x_2 = x_3 = x_4 = x_5$, which is true if and only if $a = b = c = d = e = 1$. ∎

Solution to (b) by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore.

Suppose that $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1a_2\ldots a_n = 1$, where $n > 1$. For each $i = 1, 2, \ldots, n$, define $a_{i+n} = a_i$, and for $i, j = 1, 2, \ldots, n$, let $A_{i,j} = \prod_{r=i}^{i+j-1} a_r$. Further, for each $i = 1, 2, \ldots, n$, define $A_{i,j+n} = A_{i,j}$.

Let I be any non-empty subset of $S = \{0, 1, 2, \ldots, n \Leftrightarrow 1\}$. Then the generalized inequality is:

$$\sum_{i=1}^{n}\left(\frac{\sum\limits_{j\in S-I} A_{i,j}}{\sum\limits_{k\in I} A_{i,k}}\right) \;\geq\; \frac{n\,(n \Leftrightarrow |I|)}{|I|}.$$

The proof is as follows: let $u = \sum\limits_{r=0}^{n-1} A_{1,r}$. Then

$$\begin{aligned}\frac{\sum\limits_{j\in I} A_{i,j}}{\sum\limits_{k=0}^{n-1} A_{i,k}} &= \frac{\sum\limits_{j\in I} A_{1,i-1}A_{i,j}}{\sum\limits_{k=0}^{n-1} A_{1,i-1}A_{i,k}} = \frac{\sum\limits_{j\in I} A_{1,j+i-1}}{\sum\limits_{k=0}^{n-1} A_{1,k+i-1}} \\ &= \frac{1}{u}\sum_{j\in I} A_{1,j+i-1},\end{aligned}$$

so that

$$\begin{aligned}\sum_{i=1}^{n}\left(\frac{\sum\limits_{j\in I} A_{i,j}}{\sum\limits_{k=0}^{n-1} A_{i,k}}\right) &= \sum_{i=1}^{n}\frac{1}{u}\sum_{j\in I} A_{1,j+i-1} \;=\; \frac{1}{u}\sum_{j\in I}\sum_{i=1}^{n} A_{1,j+i-1} \\ &= \frac{1}{u}\sum_{j\in I} u \;=\; |I|.\end{aligned}$$

By the Arithmetic Mean – Harmonic Mean inequality, we have:

$$\sum_{i=1}^{n}\left(\frac{\sum\limits_{j\in S} A_{i,j}}{\sum\limits_{k\in I} A_{i,k}}\right) \;\geq\; \frac{n^2}{\sum\limits_{i=1}^{n}\dfrac{\sum\limits_{j\in I} A_{i,j}}{\sum\limits_{k\in S} A_{i,k}}} \;=\; \frac{n^2}{|I|},$$

and thus

$$\sum_{i=1}^{n}\left(\frac{\sum\limits_{j\in S-I} A_{i,j}}{\sum\limits_{k\in I} A_{i,k}}\right) \;\geq\; \frac{n^2}{|I|} \Leftrightarrow n \;=\; \frac{n\,(n \Leftrightarrow |I|)}{|I|}.$$

Note that part (a) is the special case when $n = 5$ and $I = \{0,\ 2,\ 4\}$. ■

Besides Bosley and Wee, both parts were also solved by VEDULA N. MURTY, Andhra University, Visakhapatnam, India; and the proposer. Part (a) only was solved by SABIN CAUTIS, Earl Haig Secondary School, North York, Ontario.

2024. [1995: 90] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta.*

It is a known result that if P is any point on the circumcircle of a given triangle ABC with orthocentre H, then $(PA)^2 + (PB)^2 + (PC)^2 \Leftrightarrow (PH)^2$ is a constant. Generalize this result to an n-dimensional simplex.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

Let O be the centre of a circumhypersphere, let R be the radius, and let A_1, A_2, ... , A_n be the vertices. Define H by the vector expression:

$$\overrightarrow{OH} = \sum_{i=1}^{n} \overrightarrow{OA}_i \,.$$

Also, let P be any point on the surface of the circumhypersphere such that $|\overrightarrow{OP}|^2 = R^2$.

Then we claim that $\displaystyle\sum_{i=1}^{n}(PA_i)^2 \Leftrightarrow (PH)^2 = \text{constant}$.

Let $\overrightarrow{OP} = \vec{x}$. Then

$$\begin{aligned}
\sum_{i=1}^{n}(PA_i)^2 \Leftrightarrow (PH)^2 &= \sum_{i=1}^{n}(\vec{x} \Leftrightarrow \overrightarrow{OA}_i)\,.\,(\vec{x} \Leftrightarrow \overrightarrow{OA}_i) \\
&\quad \Leftrightarrow \left(\vec{x} \Leftrightarrow \sum_{i=1}^{n}\overrightarrow{OA}_i\right).\left(\vec{x} \Leftrightarrow \sum_{i=1}^{n}\overrightarrow{OA}_i\right) \\
&= n\,|\vec{x}|^2 \Leftrightarrow 2\vec{x}.\sum_{i=i}^{n}\overrightarrow{OA}_i + \sum_{i=1}^{n} OA_i^2 \\
&\quad \Leftrightarrow |\vec{x}|^2 + 2\vec{x}.\sum_{i=1}^{n}\overrightarrow{OA}_i \Leftrightarrow \sum_{i=1}^{n}\sum_{j=1}^{n}\overrightarrow{OA}_i.\overrightarrow{OA}_j \\
&= (n \Leftrightarrow 1)R^2 \Leftrightarrow \sum_{i=1}^{n}\sum_{j=1}^{n}\overrightarrow{OA}_i.\overrightarrow{OA}_j \\
&= \text{constant}. \quad \blacksquare
\end{aligned}$$

CARL BOSLEY, student, Washburn Rural High School, Topeka, Kansas, USA, WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria, and the proposer had essentially the same solution.

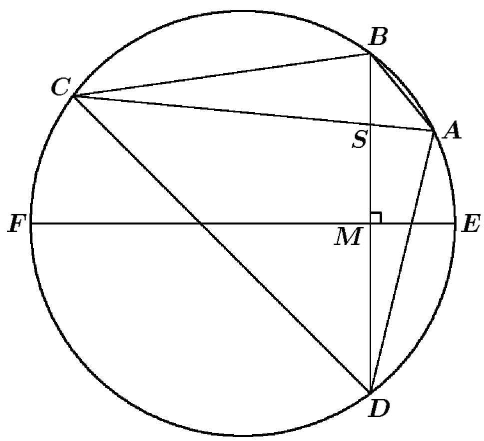
2027. [1995: 90] *Proposed by D. J. Smeenk, Zaltbommel, the Netherlands.*

Quadrilateral $ABCD$ is inscribed in a circle Γ, and has an incircle as well. EF is a diameter of Γ with $EF \perp BD$. BD intersects EF in M and AC in S. Show that $AS : SC = EM : MF$.

Solution by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore.

Note: It is necessary that A and E lie on the same side of the line BD, otherwise the result is false.

Since $ABCD$ has an incircle, we have $AB + CD = BC + AD$, or, equivalently, $BC \Leftrightarrow CD = AB \Leftrightarrow AD$. Thus



$$\begin{aligned}
\frac{AS}{AC} &= \frac{\text{Area}\Delta ABD}{\text{Area}\Delta BCD} = \frac{AB\,AD\sin(\angle BAD)}{BC\,CD\sin(\angle BCD)} \\
&= \frac{AB\,AD}{BC\,CD} \qquad \text{(since } \angle BAD + \angle BCD = 180^\circ) \\
&= \frac{2AB\,AD\ \left((BC \Leftrightarrow CD)^2 \Leftrightarrow BD^2\right)}{2BC\,CD\ \left((AB \Leftrightarrow AD)^2 \Leftrightarrow BD^2\right)} \\
&= \frac{AB\,AD\ \left(BC^2 + CD^2 \Leftrightarrow 2BC\,CD \Leftrightarrow BD^2\right)}{2BC\,CD\ \left(AB^2 + AD^2 \Leftrightarrow 2AB\,AD \Leftrightarrow BD^2\right)} \\
&= \frac{\dfrac{BC^2 + CD^2 \Leftrightarrow BD^2 \Leftrightarrow 2BC\,CD}{2BC\,CD}}{\dfrac{AB^2 + AD^2 \Leftrightarrow BD^2 \Leftrightarrow 2AB\,AD}{2AB\,AD}} \\
&= \frac{\cos(\angle BCD) \Leftrightarrow 1}{\cos(\angle BAD) \Leftrightarrow 1},
\end{aligned}$$

using the cosine rule, applied to triangles ABD and BCD.

Since EF is a diameter, we have $BE + DF = DE + BF$, and so, by an argument similar to the above, we get

$$\frac{EM}{MF} = \frac{\cos(\angle BFD) \Leftrightarrow 1}{\cos(\angle BED) \Leftrightarrow 1}.$$

Since $ABCD$ is a cyclic quadrilateral, we have $\angle BED = \angle BAD$ and $\angle BFD = \angle BCD$, and hence that $AS : SC = EM : MF$. ■

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE-WAI LAU, Hong Kong; P. PENNING, Delft, the Netherlands; SCIENCE ACADEMY PROBLEM SOLVERS, Austin, Texas, USA; TOSHIO SEIMIYA, Kawasaki, Japan; ASHISH KR. SINGH, student, Kanpur, India; and the proposer. These solvers assumed implicitly that A and E lay on the same side of the line BD.

2028. [1995: 90] *Proposed by Marcin E. Kuczma, Warszawa, Poland.*

If $n \geq m \geq k \geq 0$ are integers such that $n + m \Leftrightarrow k + 1$ is a power of 2, prove that the sum $\binom{n}{k} + \binom{m}{k}$ is even.

Solution by Hoe Teck Wee, student, Hwa Chong Junior College, Singapore.

For any non-negative integer t, let the binary representation of t be $(a_r a_{r-1} \cdots a_0)_2$, where $a_i \in \{0, 1\}$, $i = 0, 1, \ldots, r$. Let $f(t)$ be the largest integer α such that 2^α divides t. Let $g(t)$ be the sum of the digits in the binary representation of t. Then

$$\begin{aligned} f(t!) &= \sum_{j=1}^{\infty} \left[\frac{t}{2^j}\right] = \sum_{j=1}^{r} (a_r a_{r-1} \cdots a_j)_2 \\ &= \sum_{j=1}^{r} \sum_{s=0}^{j-1} 2^s = \sum_{j=1}^{r} a_j (2^j \Leftrightarrow 1) = t \Leftrightarrow g(t). \end{aligned}$$

From this we get

$$f\left(\binom{n}{k}\right) = g(n \Leftrightarrow k) + g(k) \Leftrightarrow g(n).$$

Similarly, $f\left(\binom{m}{k}\right) = g(m \Leftrightarrow k) + g(k) \Leftrightarrow g(m)$. Let $n + m \Leftrightarrow k + 1 = 2^\beta$, that is, $n + m \Leftrightarrow k = 2^\beta \Leftrightarrow 1$. Then, by considering the binary representations of n, $m \Leftrightarrow k$, $2^\beta \Leftrightarrow 1$, m, and $n \Leftrightarrow k$, it is clear that

$$g(n) + g(m \Leftrightarrow k) = \beta = g(n \Leftrightarrow k) + g(m).$$

Thus $g(n \Leftrightarrow k) + g(k) \Leftrightarrow g(n) = g(m \Leftrightarrow k) + g(k) \Leftrightarrow g(m)$, that is, $f\left(\binom{n}{k}\right) = f\left(\binom{m}{k}\right)$. Hence $2 | \binom{n}{k} \Leftrightarrow 2 | \binom{m}{k}$, so $\binom{n}{k} + \binom{m}{k}$ is even. ■

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; SOFIYA VASINA, student, University of Arizona, Tucson, USA; and the proposer. Kuczma notes that the problem can be solved without any computation as follows:

> Look at Pascal's triangle, rows 0 through 2^{β}, modulo 2, and visualize it geometrically as an equilateral triangle with black and white spots (zeros and ones). It has three (geometric) symmetry axes, and each one of these symmetries preserves the spot design. This fact can be considered as known. And, if not, it suffices to draw the "row 0 through 7 triangle" and notice that it consists of three copies of the (twice) smaller triangle formed by rows 0 through 3, the remaining quarter in the centre of the figure being filled with zeros. The base row is all ones. These observations provide a scheme for an induction proof of the claim for rows 0 through $2^{\beta} - 1$. The symmetry with respect to one of the oblique axes is precisely the contents of the problem, expressed in algebraic terms.

Comment by the Editor-in-Chief

Choosing a solution to print is not an easy task. Each collection of submitted solutions is assigned to a member of the Editorial Board, and these editors work independently of one another. The choice of which solution to highlight is left entirely to that editor. Sometimes it happens, as in this issue, that several solutions are chosen from the same solver. I would like to assure all subscribers that every submission is very important to *CRUX*, and that we encourage everyone to submit solutions, as well as proposals for problems, articles for publication, and contributions to the other corners. Every subscriber is very important to us and we really value all contributions.

And while I am on the subject of contributions, please continue to send in proposals for problems. We publish 100 per year, and we do not have too many in reserve at this time. Without your contributions, there would be no *CRUX*.