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CONTENTS

FIELD PLANES

R.B. KILLGROVE

1. *Introduction.*

The author intends to write a text on the foundations of geometry some day. Field planes will play a central role in this proposed text. We wish here to outline that role, giving references when details are not supplied. An important reference is Artin [1].

2. *Affine planes.*

Let Σ be a set of elements called *points*, and σ a collection of distinguished nonempty subsets of Σ called *lines*. If $k,m \in \sigma$, and if $k = m$ or $k \cap m = \emptyset$, then k is said to be *parallel* to m ($k \parallel m$). The pair (Σ,σ) is said to constitute an *affine plane* if the points and lines satisfy the following three axioms:

(i) For all distinct points A and B, there is a unique line k such that $A \in k$ and $B \in k$. (This line will be denoted by AB.)

(ii) There are three points A,B,C such that, for any line k, we have $A \notin k$ or $B \notin k$ or $C \notin k$. (There are three noncollinear points.)

(iii) For any point A and any line k, there is a unique line m such that $A \in m$ and $k \parallel m$.

We then have the

THEOREM. Every line of an affine plane has at least two points.

Proof. Suppose, on the contrary, that a line m (already known to be nonempty) contains only one point, say $m = \{P\}$; and let A,B,C be the three points of Axiom (ii), where we assume without loss of generality that $P \neq A$ and $P \notin AC$. If also $P \notin AB$, then AB and AC are both parallel to m, contrary to the uniqueness condition of Axiom (iii). Hence $P \in AB$. Now there is a line k through C parallel to AB, and $k \cap AB = \emptyset$. But then AB and $m \neq AB$ are both lines through P parallel to k, violating the uniqueness condition.

3. *Field planes.*

We assume that the reader knows the axioms of a field (which will be in the proposed text); if not, McCoy [6] is a convenient reference.

Let $(K,+,\cdot)$ be a field. A *field plane over K* is a pair (Σ,σ), where the *points* are the elements of $\Sigma = K \times K$, and the *lines* are the elements of σ, which are subsets of Σ defined as follows: for each equation

$$Ax + By + C = 0,$$

where $A,B,C \in K$ and A,B are not both zero, there is a corresponding line consisting of all points (u,v) such that

$$Au + Bv + C = 0.$$

The *real plane* is then the field plane over the field of real numbers; the *rational plane* is the field plane over the field of rational numbers. We can even define an *integer plane* in a similar way, but to avoid the requisite number theory (not to be avoided in the proposed text) we will not describe it further. But unfortunately, the terms "Argand diagram", "Gaussian plane", and "complex plane" are synonyms for an object which is not our field plane over the field of complex numbers. Since we are about to show that every field plane is an affine plane, we will refer to the field plane over the field of complex numbers as the *complex affine plane*.

4. *Field affine planes.*

We now establish that every field plane is an affine plane by showing that the axioms of an affine plane are theorems for a field plane.

LEMMA 1. Every line of a field plane has at least two points.

Proof. Take a line whose equation is $Ax + By + C = 0$. If $A,B,C \neq 0$, then $(0, -C/B)$ and $(-C/A, 0)$ are distinct points of the line; if $A = 0$, then we can take $(0, -C/B)$ and $(1, -C/B)$; if $B = 0$, then we take $(-C/A, 0)$ and $(-C/A, 1)$; and if $C = 0$, then $(0,0)$ and $(B,-A)$ will do. (We write $-C/B$, etc. for any field, even though the notation $-B^{-1}C$ would be more usual for finite fields.)

We will need the following

Definition. For the pair of lines

$$Ax + By + C = 0 \quad \text{and} \quad Dx + Ey + F = 0, \tag{1}$$

the *determinant of coefficients* is the field element $AE - BD$.

LEMMA 2. If the determinant of coefficients for two lines is not zero, then the intersection of those two lines consists of a single point.

Proof. If (u,v) is a common point of the lines (1), then

$$Au + Bv + C = 0, \quad Du + Ev + F = 0,$$

and we have uniquely

$$u = \frac{BF-CE}{AE-BD}, \quad v = \frac{CD-AF}{AE-BD}. \tag{2}$$

Conversely, it is easily verified that the point (u,v) given by (2) lies on both lines (1).

LEMMA 3. If the determinant of coefficients for the lines (1) is zero, then there is a field element $R \neq 0$ such that $D = AR$ and $E = BR$.

Proof. Observe that $A = 0 \iff D = 0$ and $B = 0 \iff E = 0$. If $A = 0$, take $R = E/B$; if $B = 0$, take $R = D/A$; otherwise take $R = D/A = E/B$. In all cases we have $R \neq 0$ and $D = AR$, $E = BR$.

LEMMA 4. The lines (1) are parallel (i.e., identical or disjoint sets) if and only if their determinant of coefficients is zero.

Proof. If the determinant of coefficients is not zero, then the lines have exactly one point in common by Lemma 2, so they are not disjoint sets. But then the lines are distinct by Lemma 1, so they are not parallel. Now suppose the determinant of coefficients is zero. With the $R \neq 0$ of Lemma 3, the lines (1) also have the equations

$$ARx + BRy + CR = 0, \qquad ARx + BRy + F = 0. \tag{3}$$

Suppose the lines are not disjoint sets but have the point (u,v) in common; then, from (3),

$$CR = -ARu - BRv = F,$$

so the lines are identical and hence parallel.

LEMMA 5. Two distinct lines do not join the same pair of distinct points.

Proof. Suppose there are such lines. Being neither disjoint nor identical, they are not parallel. By Lemma 4, their determinant of coefficients is not zero; then, by Lemma 2, they have only one point in common, contrary to the assumption. □

We are now ready to show that the axioms of an affine plane are theorems for a field plane, so that every field plane is an affine plane.

THEOREM 1. There is a unique line containing two given distinct points.

Proof. Let (a,b) and (c,d) be distinct points, so that $b-d$ and $c-a$ are not both zero. It is easy to verify by substitution that these two points lie on the line

$$(b-d)x + (c-a)y + (ad-bc) = 0.$$

That the two points lie on no other line follows from Lemma 5.

THEOREM 2. There are three noncollinear points.

Proof. Consider the three distinct points $(0,0)$, $(1,0)$, $(0,1)$. Exactly two of them lie on each of the distinct lines

$$x = 0, \qquad y = 0, \qquad x + y - 1 = 0.$$

By Lemma 5, no other line joins any pair of the points, much less all three.

THEOREM 3. For any given point and any given line, there is a unique line through the given point which is parallel to the given line.

Proof. Let (u,v) and $Ax + By + C = 0$ be the given point and line. It is clear that the line

$$Ax + By + (-Au-Bv) = 0 \tag{4}$$

contains the given point and is parallel to the given line. If

$$Dx + Ey + F = 0 \tag{5}$$

is also such a line, then, by Lemmas 4 and 3, equation (5) can be written

$$ARx + BRy + F = 0 \tag{6}$$

and we have $ARu + BRv + F = 0$. Thus $F = -ARu - BRv$, and (6) is equivalent to (4).

5. *Applications*.

We can study Euclid's *Elements* and see that the rational plane is a satisfactory model for his definitions, postulates, and common notions. We will then discover that the construction in Euc. I-1 is not possible in the rational plane since it would involve $\sqrt{3}$ which is not rational, so there are "holes" in rational circles. The question then arises: how many points are on rational circles? On the circle $x^2 + y^2 = 1$, for example, the rational points are $(a/c, b/c)$, where (a,b,c) is any Pythagorean triple, of which there are infinitely many.

We can find intersections of two circles, or of a circle and a line, in field planes by solving quadratic equations. But the standard quadratic formula is valid only in fields where $1 + 1 \neq 0$. In any case, the solution of the quadratic exists either in the field in which the problem was posed or in a quadratic extension of that field.

With this background, we can then show the impossibility of duplicating the cube and of trisecting the angle. To do this we can follow Perlis [7] with degrees of extension, or Courant and Robbins [4] with conjugates, or use the method of descent. This last method was discovered independently by the author, but he has been informed that others have used it in the past. As an illustration, we apply the method to the duplication of the cube. The trisection problem can be treated in a similar way.

We assume that we have a finite sequence of fields

$$K_0 \subset K_1 \subset K_2 \subset \ldots \subset K_n,$$

where K_{i+1} is a quadratic extension of K_i, and K_0 is the field of rational numbers. Suppose that $\sqrt[3]{2} \in K_n$. Since $\sqrt[3]{2} \notin K_0$, there is an i such that $\sqrt[3]{2}$ is in K_{i+1} but not in K_i. Let the elements of K_{i+1} be of the form $u + v\sqrt{c}$, where $u,v,c \in K_i$ but $\sqrt{c} \notin K_i$. If $\sqrt[3]{2} = a + b\sqrt{c}$, then

$$(a^3+3ab^2c-2) + (3a^2b+b^3c)\sqrt{c} = 0 = 0 + 0\sqrt{c}.$$

Since $u + v\sqrt{c} = s + t\sqrt{c}$ implies $u = s$ and $v = t$ (otherwise $\sqrt{c} = (u-s)/(t-v) \in K_i$), we get

$$a^3 + 3ab^2c - 2 = 0 \quad \text{and} \quad 3a^2b + b^3c = 0.$$

Now $b \neq 0$, so $b^2c = -3a^2$ and $a^3 + 3a(-3a^2) - 2 = 0$, that is, $(-2a)^3 = 2$. Since there is only one real $\sqrt[3]{2}$, we must have $\sqrt[3]{2} = -2a \in K_i$, a contradiction.

6. *Transformations in field planes.*

In preparation for this more abstract topic, we first note that translations, rotations, and the reflection $x' = x$, $y' = -y$ are isometries (distance-preserving maps) of the real plane. Then, observing that all circles sharing a pair of distinct points have collinear centers, we can show that all isometries fixing three non-collinear points coincide with the identity. So, if we know the images of $(0,0)$, $(1,0)$, and $(0,1)$ under any isometry, we can obtain the identity by composing a translation and a rotation and, when necessary, a reflection. This in turn shows that all isometries are compositions of these three types of maps.

Now we begin a development based very much on Artin [1]. Under the *translation*

$$x' = x + a, \qquad y' = y + b,$$

the line

$$Ax + By + C = 0 \tag{7}$$

goes onto the line

$$Ax + By + (C - Aa - Bb) = 0, \tag{8}$$

since each (u,v) which satisfies (7) has the image $(u+a,\ b+v)$ which satisfies (8), and each (r,s) which satisfies (8) has the preimage $(r-a,\ s-b)$ which satisfies (7). In summary, the image of a line is a line parallel to the original line. In particular, when a and b are not both zero (thus excluding the identity map), the lines $bx - ay + C = 0$ are fixed. These lines are called the *traces* of the translation. (All lines are traces of the identity map.) Finally, given any pair of points (r,s) and (u,v), there is a translation that takes the first into the second, namely

$$x' = x + u - r, \qquad y' = y + v - s.$$

These ideas are used to prove the following theorem about field planes.

VECTOR THEOREM. If A,B,C,A',B',C' are distinct points of a field plane, and if

$$AA' \parallel BB' \parallel CC', \qquad AB \parallel A'B', \qquad BC \parallel B'C',$$

then AC $\parallel$ A'C'.

Proof. Suppose a translation sends A into A'; then the traces are the lines parallel to AA'. Now the image of B must lie on BB', the trace through B, and on A'B', since AB $\|$ A'B'. Since B' is the unique point common to BB' and A'B', it must be the image of B under the translation. Similarly, the image of C is on trace CC' and on B'C', the image of BC. Thus C' is the image of C, and so AC $\|$ A'C'. □

A *scalar map* is a transformation of the type

$$x' = \lambda x, \quad y' = \lambda y, \quad \lambda \neq 0.$$

It can be shown that a scalar map also sends lines into parallel lines and that its traces, when $\lambda \neq 1$, consist of all lines through the origin. Furthermore, given three distinct collinear points of which the first, say, is the origin, there is a scalar map which sends the second point into the third. A proof similar to the above would then establish the

SCALAR THEOREM. If A,B,C,A',B',C' are distinct points in a field plane, none of which is the origin O; if {O,A,A'}, {O,B,B'}, and {O,C,C'} are collinear sets; and if

$$AB \parallel A'B' \quad \text{and} \quad AC \parallel A'C',$$

then BC $\|$ B'C'.

The only other important theorem we will need to mention is

PAPPUS' THEOREM. If A,B,C,A',B',C' are distinct points in a field plane, none of which is the origin O; if {O,A,B,C} and {O,A',B',C'} are collinear sets; and if

$$AB' \parallel BC' \quad \text{and} \quad A'B \parallel B'C,$$

then AA' $\|$ CC'.

Outline of proof. For convenience, a scalar map such as $x' = \mu x$, $y' = \mu y$ will be denoted simply by μ. Suppose scalar map μ takes A into B. Then from AB' $\|$ BC' and the trace, μ also takes B' into C'. If also scalar map ρ takes A' into B', then, from A'B $\|$ B'C and the trace, ρ also takes B into C. Thus scalar map $\rho\mu$ takes A into C and scalar map $\mu\rho$ takes A' into C'. Since $\mu\rho = \rho\mu$, we must have AA' $\|$ CC'. □

We have seen that

$$(\Sigma,\sigma) \text{ is a field plane} \Longrightarrow (\Sigma,\sigma) \text{ is an affine plane.}$$

The converse is, of course, not true. In a later chapter of our proposed text, we will show what can be done to an affine plane to make it into a field plane. We will start with an affine plane (Σ,σ) and impose on Σ an algebraic structure with an addition and a multiplication to form the basis of an analytic geometry. The adjunction of the vector theorem as an axiom will allow us to show that the system $(\Sigma,+,\cdot)$ is an Abelian group under addition and that multiplication is right distributive over addition. We will then adjoin the scalar theorem as an axiom, and the resulting multiplication will be associative and both left and right distributive over addition. Finally, we will adjoin the Pappus theorem as an axiom, the resulting multiplication will be commutative, and our algebraic system $(\Sigma,+,\cdot)$ will at last be a field. Allowing the Pappus statement to be universal, that is, no longer requiring the point O to be the origin of a given coordinate system, implies a universal scalar statement in any affine plane. This universal scalar statement is called the *affine Desargues theorem*. In any affine plane, the affine Desargues theorem implies the vector statement. In summary, a *Pappian plane*, that is, an affine plane satisfying the universal Pappus statement, is a field plane. In other words, *it takes only one additional axiom, the universal Pappus axiom, to make an affine plane into a field plane.* For more details on the above, the reader may consult Artin [1], or Bruck [3] for a longer explanation, or Blumenthal [2] for a textbook explanation.

7. *Ordered field planes.*

We introduce *positive* elements in the standard way for a field and define an *order relation* < in terms of the positive elements. We then have the *law of trichotomy*, which states that, for every field element x, exactly one of the following holds:

$$x > 0, \quad x = 0, \quad x < 0 \; (-x > 0).$$

If $A(a,b)$ and $C(c,d)$ are distinct points, then we say that point $B(u,v)$ lies *between* A and C if and only if there is a field element λ, $0 < \lambda < 1$, such that

$$u = \lambda a+(1-\lambda)c, \qquad v = \lambda b+(1-\lambda)d. \tag{9}$$

We introduce *open half-planes* $Ax+By+C > 0$ and $Ax+By+C < 0$ and show the *generalized law of trichotomy*: every point (x,y) of the plane satisfies exactly one of the relations

$$Ax + By + C > 0, \quad Ax + By + C = 0, \quad Ax + By + C < 0.$$

An open half-plane such as π: $Ax+By+C>0$ is a convex set. For if $A(a,b)$ and $C(c,d)$ are in π, then the point $B(u,v)$ defined by (9) is also in π since

$$\lambda Aa + \lambda Bb + \lambda C > 0 \;\text{ and }\; (1-\lambda)Ac + (1-\lambda)Bd + (1-\lambda)C > 0 \Longrightarrow Au + Bv + C > 0.$$

An important theorem for ordered field planes is

HILBERT'S SEPARATION THEOREM. Let k be any line. Then there are open half-planes α and γ such that

(i) every point of the plane lies in exactly one of k,α,γ;

(ii) each of k,α,γ is a convex set;

(iii) if $A \in \alpha$ and $C \in \gamma$, then there is a point $B \in k$ which lies between A and C.

Proof. If k is the line $Ax+By+C = 0$, then we claim that the open half-planes

$$\alpha:\ Ax+By+C > 0 \quad \text{and} \quad \gamma:\ Ax+By+C < 0$$

satisfy the theorem. It is clear that (i) follows from the generalized law of trichotomy. For (ii), convexity has already been discussed above insofar as it applies to the half-planes α and γ; and an even more general statement is true for lines, as we now show. If $A(a,b)$ and $C(c,d)$ are distinct points of line k, then, for *any* field element λ, the point $B(u,v)$ defined by (9) is easily shown to lie on k (and it lies between A and C if $0 < \lambda < 1$). Finally, to prove (iii), suppose $A(a,b) \in \alpha$ and $C(c,d) \in \gamma$. Then

$$Aa + Bb + C > 0 \quad \text{and} \quad Ac + Bd + C < 0,$$

from which we easily get

$$0 < Aa + Bb + C < A(a-c) + B(b-d).$$

Hence, if we set

$$1 - \lambda = \frac{Aa + Bb + C}{A(a-c) + B(b-d)},$$

then $0 < 1-\lambda < 1$ and $0 < \lambda < 1$, and it is easily verified that the point $B(u,v)$ defined by (9) now lies on k between A and C. □

We end this section with the

CHARACTERIZATION THEOREM. If A and C are distinct points, then for any point B on line AC, exactly one of the following statements is true: A is between B and C, B = A, B is between A and C, B = C, C is between A and B.

Outline of proof. If the distinct points are $A(a,b)$ and $C(c,d)$, and if $B(u,v)$ is defined by (9), the given statements correspond to the mutually exclusive cases $1 < \lambda$, $\lambda = 1$, $0 < \lambda < 1$, $\lambda = 0$, $\lambda < 0$.

8. *Ordered planes*.

Let Σ be a set of elements called *points*, and ω a ternary relation on Σ. If the ordered triple $(A,B,C) \in \omega$, then we will say that the points are *in the order* ABC and denote this by ωABC. If A and B are distinct points of Σ, then the *line* AB is the set of all points $P \in \Sigma$ such that

$$\omega PAB \quad \text{or} \quad P = A \quad \text{or} \quad \omega APB \quad \text{or} \quad P = B \quad \text{or} \quad \omega ABP.$$

(Note that every line has at least two distinct points by definition, since "line AB" implies $A \neq B$.) The pair (Σ,ω) is then called an *ordered plane* if the following six axioms are satisfied:

O_1. If ωABC, then A,B,C are all distinct.

O_2. If ωABC, then not ωBCA.

O_3. If A $\neq$ B, then there is at least one point C such that ωABC.

O_4. There are three noncollinear points.

O_5. If C and D are distinct points of line AB, then A is on line CD.

O_6. (The Pasch axiom) If A,B,C are noncollinear points, if D is a point such that ωADB, and if there is a line $k \neq$ AB such that D $\in k$ and C $\notin k$, then there is a point E such E $\in k$ and ωAEC or ωBEC.

O_1-O_5 are essentially Forder's axioms [5], but we have replaced his axiom O_6 by the Pasch axiom. The Pasch axiom implies Forder's O_6, which in turn implies the Pasch axiom in an affine plane. Without a parallel postulate or the Pasch axiom, we could be in a higher-dimensional space. Hilbert's separation axiom is equivalent to the Pasch axiom when given O_1-O_5, but we will only prove half of that statement here, as shown in the following

THEOREM 0. Hilbert's separation axiom and O_1-O_5 imply the Pasch axiom.

Proof. Line k separates the plane into two open half-planes α and β. With D on k, A and B are in different half-planes, say A $\in \alpha$ and B $\in \beta$. Since C $\notin k$, then C $\in \alpha$ or C $\in \beta$. If C $\in \alpha$, then E $\in k$ and ωBEC; and if C $\in \beta$, then E $\in k$ and ωAEC.

THEOREM 1. In a field plane whose field is ordered, ωABC implies that A,B,C are all distinct.

Proof. It is understood here that A $\neq$ C, so we need only prove that A $\neq$ B and B $\neq$ C. If the coordinates of A and C are (a,b) and (c,d), respectively, then the coordinates (u,v) of B are given by (9) for some λ with $0 < \lambda < 1$. If A = B, then

$$(1-\lambda)a = (1-\lambda)c \quad \text{and} \quad (1-\lambda)b = (1-\lambda)d,$$

so $a = c$, $b = d$, and A = C, a contradiction. So A $\neq$ B, and similarly B $\neq$ C.

THEOREM 2. In a field plane whose field is ordered, if ωABC then not ωBCA.

Proof. Using the points A,B,C of Theorem 1 and assuming ωBCA as well, there is a field element μ such that $0 < \mu < 1$ and

$$c = \mu\{\lambda a+(1-\lambda)c\} + (1-\mu)a, \quad d = \mu\{\lambda b+(1-\lambda)d\} + (1-\mu)b.$$

From field operations, we then get

$$(1-\mu+\mu\lambda)c = (1-\mu+\mu\lambda)a \quad \text{and} \quad (1-\mu+\mu\lambda)d = (1-\mu+\mu\lambda)b.$$

If $1-\mu+\mu\lambda \neq 0$, then $a = c$ and $b = d$, contrary to A $\neq$ C. Hence $1-\mu+\mu\lambda = 0$, a contradiction since $1-\mu > 0$ and $\mu\lambda > 0$.

THEOREM 3. If A $\neq$ B in a field plane whose field is ordered, then there is a point C such that ωABC.

Proof. Let $A(a,b)$ and $B(c,d)$ be distinct points. Since $2 = 1+1 \neq 0$ in an ordered field, for the point $C(2c-a, 2d-b)$ we will have ωABC since

$$c = \tfrac{1}{2}a + \tfrac{1}{2}(2c-a), \qquad d = \tfrac{1}{2}b + \tfrac{1}{2}(2d-b),$$

and $0 < \frac{1}{2} < 1$.

THEOREM 4. In a field plane whose field is ordered, there are three noncollinear points.

Proof. A field plane is an affine plane.

THEOREM 5. In a field plane whose field is ordered, if C and D are distinct points of line AB, then A is on line CD.

Proof. By Lemma 5 of section 4, line AB = line CD.

9. *What else?*

We do not wish to leave the impression that we have succumbed to Dieudonné's dictum that all geometry must be done analytically. There will be much more in the proposed text, with plenty of synthetic geometry, as well as sections showing relations with number theory, combinatorics, coding theory, design of experiments, computers, topology, complex variables, theory of equations, logic, set theory, and Jordan content. Here we just chose one thread common to finite and ordered geometries. There are also nonfield affine planes, nonfield affine ordered planes, and nonaffine ordered planes.

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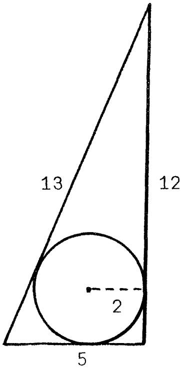
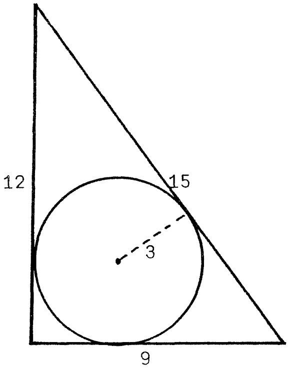
* * *

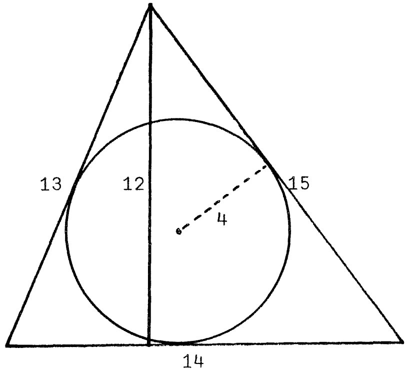
AN HERONIAN ODDITY

LEON BANKOFF

Triangles whose sides and areas are rational are known as *Heronian triangles.* This definition implies that each altitude must be rational. By multiplication with suitable factors, the sides and at least one altitude can be made integral. One of the best-known Heronian triangles is formed by the juxtaposition of two Pythagorean triangles, one primitive, with sides 5, 12, 13, and the other with sides triple those of the primitive 3-4-5 Pythagorean triangle. By placing these triangles so that the sides 5 and 9 are aligned and the sides 12 coincide, we obtain the 13-14-15 Heronian triangle (see figures).







One of the delightful surprises that contribute so much to the joy of mathematics is the revelation that the 13-14-15 triangle has one of its altitudes equal to 12, thus extending the sequence of consecutive integers. Although there exists an infinitude of Heronian triangles whose sides are consecutive integers, the 13-14-15 triangle is the only one having an altitude and its three sides measured by consecutive integers.

A serendipitous discovery, perhaps never before observed and published, is the astonishing but by no means world-shaking coincidence that the inradii of the 5-12-13, the 9-12-15, and the 13-14-15 triangles are also measured by consecutive integers: 2, 3, and 4, respectively. No other Heronian triangle can claim that distinction.

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* * *

NOTES ON NOTATION: VI

LEROY F. MEYERS

What is the value of $\lim\limits_{p\to\infty} \sin(\pi p)$?

There are two reasonable answers, depending on which numbers the letter p may represent. If p must be an integer, then the limit is 0, but if p is unrestricted, then the limit does not exist.

How can a reader tell which interpretation is meant? Sometimes an explicit statement is made, such as "for integral p"; most of the time, however, the reader must guess. Often, certain letters, such as those from i through n, are intended to represent integers, and certain other letters, such as t through z, are intended to represent arbitrary real (or complex) numbers, even if no explicit statement is made. (The choice of the letter p in the example above was deliberate, to avoid preconceptions.) But note the contrary-to-convention $\lim\limits_{n\to 0}$ in *Crux* problem 273 [1977: 226 and 1978: 87].

One way to avoid ambiguity is to state the intended interpretations explicitly in the symbol for the limit:

$$\lim_{p\to\infty,\, p\in N} \sin(\pi p) = 0,$$

but

$$\lim_{p\to\infty,\, p\in R} \sin(\pi p)$$

does not exist. This may be the best procedure if a single discussion uses several different conditions on the dummy letter in the limit symbol. However, if the only ambiguity is between integer and real number, then the repetition of the conditions is awkward. I propose (and often use) the following convention. If the letter p is to represent real numbers without restriction, then I keep the standard notation: $p\to\infty$; but if it is to represent only integers, then I use $p\uparrow$ and read it as " p goes up".

A related matter is that of the notation used to indicate one-sided limits. Here are some common notations for the limit of the function f from the left at a:

$$\lim_{x\to a,\, x<a} f(x), \quad \lim_{x\to a^-} f(x), \quad f(a^-), \quad f(a-0), \quad f_-(a), \quad \lim_{x\nearrow a} f(x), \quad \lim_{x\uparrow a} f(x).$$

The first of these notations is explicit, but may be awkward to use repeatedly. The fourth notation is highly confusing, since $f(a-0)$ should equal $f(a)$, independently of any limit. The second and third notations seem to imply the existence of a number a^- which is different from a (and does -2^- mean $(-2)^-$ or $-(2^-)$?). The fifth notation is compact, but may conflict with other uses of the same symbol or with subscripts. The only reason that I use the sixth rather than the seventh notation is that the vertical arrow reminds me of approach through integers, whereas here, unless $a = \infty$, the approach must be through arbitrary real numbers in some left neighborhood of a. In fact, I generally write $x \nearrow \infty$ rather than $x \to \infty$ for the approach to ∞ through real numbers since the approach must be one-sided. If the function f is named, then there are additional possibilities which use no dummy letter:

$$\lim_{a-} f, \qquad \lim_{\nearrow a} f, \qquad \lim_{\uparrow a} f.$$

Similar notations may be used for limits from the right and for two-sided limits. In particular, if f is a sequence (with $f(n)$ as term numbered n, n being an integer), then a compact notation for

$$\lim_{n\uparrow} f(n) \qquad \text{is} \qquad \lim_{\uparrow} f.$$

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* * *

THE PUZZLE CORNER

Puzzle No. 20: Rebus (11)

$$x - D = 0$$

In chemistry, it sometimes comes to pass,
A solid or a liquid or a gas
Will mix completely with another
In REBUS, scientific brother!

Puzzle No. 21: Heteronym (6)

To show a second (arc, not time),
Affix two primes above a one;
It takes a FINE, if this is done.
Don't make the second prime too PRIME.

ALAN WAYNE
Holiday, Florida

* * *

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before February 1, 1983, although solutions received after that date will also be considered until the time when a solution is published.

761. *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Solve the following independent *base eight* alphametics, each of which has a unique solution and is doubly-true in German.

$$\text{(a)}\quad \begin{array}{r} \text{EINS} \\ \text{EINS} \\ \text{EINS} \\ \underline{\text{EINS}} \\ \text{VIER} \end{array} \qquad\qquad \text{(b)}\quad \begin{array}{r} \text{EINS} \\ \text{EINS} \\ \underline{\text{ZWEI}} \\ \text{VIER} \end{array}$$

762. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

ABC is a triangle with area K and sides a,b,c in the usual order. The internal bisectors of angles A,B,C meet the opposite sides in D,E,F, respectively, and the area of triangle DEF is K'.

(a) Prove that

$$\frac{3abc}{4(a^3+b^3+c^3)} \le \frac{K'}{K} \le \frac{1}{4}.$$

(b) If $a = 5$ and $K'/K = 5/24$, determine b and c, given that they are integers.

763. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Given are n points in general position in space (i.e., no four in a plane). By a *tetrahedration* of their convex hull is meant a partitioning of the convex hull into nonoverlapping tetrahedra each having four of the given points as vertices. Show that the number of edges in every tetrahedration is independent of the number of vertices of the convex hull.

(This problem was inspired by Crux 502 [1981: 22].)

764. *Proposed by Kent Boklan, student, Massachusetts Institute of Technology.*

Find all positive integer pairs $\{m,n\}$ such that

$$\frac{1}{m} + \frac{1}{n} = \frac{q}{p},$$

where p and q are consecutive primes with $p > q$.

765. *Proposed by K.P. Shum and R.F. Turner-Smith, The Chinese University of Hong Kong.*

If n is a given positive integer, find all solutions $\theta \in [0,2\pi)$ of the equation

$$\cos^n\theta + \sin^n\theta = 1.$$

(The trivial case $n = 2$ may be omitted.)

766. *Proposed by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.*

Let ABC be an equilateral triangle with center O. Prove that, if P is a variable point on a fixed circle with center O, then the triangle whose sides have lengths PA, PB, PC has a constant area.

767. *Proposed by H. Kestelman, University College, London, England.*

Let $z_1, z_2, \ldots, z_k$ be complex numbers such that

$$z_1^s + z_2^s + \ldots + z_k^s = 0$$

for all $s = 1,2,\ldots,k$. Must all the z_r be 0?

768. *Proposed by Jack Garfunkel, Flushing, N.Y.; and George Tsintsifas, Thessaloniki, Greece.*

If A,B,C are the angles of a triangle, show that

$$\frac{4}{9}\Sigma \sin B \sin C \le \Pi \cos\frac{B-C}{2} \le \frac{2}{3}\Sigma \cos A,$$

where the sums and product are cyclic over A,B,C.

769. *Proposed by Loren C. Larson, St. Olaf College, Northfield, Minnesota.*

For each positive integer n, let $S_n = x^n+y^n+z^n$, where x,y,z are real numbers. Given that $S_1 = 0$, express S_n as a polynomial in S_2 and S_3.

770. *Proposed by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.*

Let P be an interior point of triangle ABC. Prove that

$$PA\cdot BC + PB\cdot CA > PC\cdot AB.$$

* * *

THE PUZZLE CORNER

Answer to Puzzle No. 18 [1982: 163]: Homeomorphisms, homomorphisms.

Answer to Puzzle No. 19 [1982: 163]: Centimeter; time, center.

* * *

SOLUTIONS

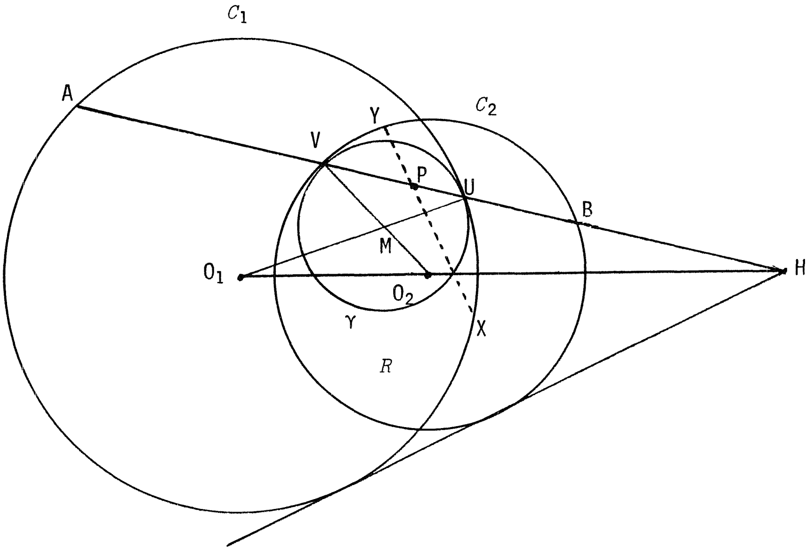
No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

656. [1981: 180] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

P is an interior point of a convex region R bounded by the arcs of two intersecting circles C_1 and C_2. Construct through P a "chord" UV of R, with U on C_1 and V on C_2, such that PU·PV is a minimum.

I. *Solution by Jordi Dou, Barcelona, Spain.*

Let O_1 and O_2 be the centers of C_1 and C_2, respectively, and let H be their homothetic center (intersection of a common tangent and the line of centers), as shown in the figure. If HP meets the boundary of R in U (on C_1) and V (on C_2), then UV is the required chord of R. (If C_1 and C_2 are congruent circles, then H is at infinity, so then we take UV through P parallel to O_1O_2.)



For it is easy to see that the circle γ with center $M = O_1U \cap O_2V$ and radius MU also goes through V and is bitangent to C_1 at U and to C_2 at V. Hence, if XY is *any* chord of R through P, with X on C_1 and Y on C_2, we have

$$PX \cdot PY \geq PU \cdot PV = \text{power of P with respect to } \gamma.$$

II. *Comment extracted from the proposer's solution.*

For *all* lines through P that meet C_1 in U an A, and meet C_2 in V and B, we have

$$PU \cdot PA = \text{constant} \quad \text{and} \quad PV \cdot PB = \text{constant},$$

and so $PU \cdot PA \cdot PV \cdot PB$ = constant. Hence, for the line HP of solution I (see figure), where $PU \cdot PV$ is a minimum, the product $PA \cdot PB$ is a maximum.

Also solved by HIPPOLYTE CHARLES, Waterloo, Québec; DAN SOKOLOWSKY, California State University at Los Angeles; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

Editor's comment.

One more solution, not recorded above, was received. It was a very long analytic outpouring, spiced with calculus, in which it is not clear what, if anything, has been proved.

* * *

657. [1981: 203] *Proposed by Ngo Tan, student, J.F. Kennedy H.S., Bronx, N.Y.*

A quadrilateral ABCD is circumscribed about a circle γ and

$$\gamma \cap AB,BC,CD,DA = E,F,G,H, \text{ respectively.}$$

Let $AC \cap BD = I$ and

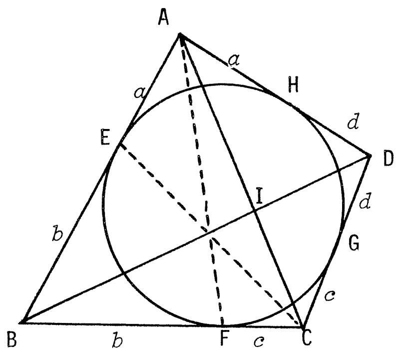
$$AH = AE = a, \quad BE = BF = b, \quad CF = CG = c, \quad DG = DH = d.$$

Prove that

$$\frac{IA}{IC} = \frac{a}{b} \quad \text{and} \quad \frac{IB}{ID} = \frac{b}{d}.$$

Solution by J.T. Groenman, Arnhem, The Netherlands (revised by the editor).

From Brianchon's Theorem applied to the degenerate circumscribed hexagon AEBFCD, we conclude that AF, BD, and CE are concurrent. Now, from Ceva's Theorem applied to triangle ABC, we have



$$\frac{AE}{EB} \cdot \frac{BF}{FC} \cdot \frac{CI}{IA} = 1,$$

or

$$\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{IC}{IA} = 1,$$

from which $IA/IC = a/c$. The other result, $IB/ID = b/d$, is proved in the same way.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; HIPPOLYTE CHARLES, Waterloo, Québec; GEORGE DEDAJ, student, Bronx High School of Science, Bronx, N.Y.; JORDI DOU, Barcelona, Spain; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, California State University at Los Angeles; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

* * *

658, [1981: 203] *Proposed by Charles W. Trigg, San Diego, California.*

In the decimal system, do there exist consecutive squares that have the same square digit sum?

Solution by Bob Prielipp, University of Wisconsin-Oshkosh.

It is well known that in the decimal system a number is congruent to the sum of its digits modulo 9. Hence a necessary condition for n^2 and $(n+1)^2$ to have the same digit sum (in symbols, $\Sigma n^2 = \Sigma(n+1)^2$) is that

$$(n+1)^2 \equiv n^2, \text{ so } 2n+1 \equiv 0 \text{ and } n \equiv 4 \pmod 9.$$

Moreover, if $n \equiv 4 \pmod 9$, then $n^2 \equiv 7 \pmod 9$, so if

$$\Sigma n^2 = \Sigma(n+1)^2 = \text{a square},$$

then that square must be one of

$$16,\ 25,\ 169,\ 196,\ 484,\ 529,\ \ldots\ .$$

If for a satisfactory n we have $\Sigma n^2 > 25$, then n^2 must have more than $[169/9] = 18$ digits, so we restrict our search to numbers $n \equiv 4 \pmod 9$ such that

$$\Sigma n^2 = \Sigma(n+1)^2 = 16 \text{ or } 25.$$

Even for $n < 1000$, there are 40 solutions. These are tabulated below.

n	$n+1$	n^2	$(n+1)^2$	Σn^2	n	$n+1$	n^2	$(n+1)^2$	Σn^2
13	14	169	196	16	274	275	75076	75625	25
22	23	484	529	16	283	284	80089	80656	25
58	59	3364	3481	16	301	302	90601	91204	16
76	77	5776	5929	25	391	392	152881	153664	25
103	104	10609	10816	16	418	419	174724	175561	25
130	131	16900	17161	16	427	428	182329	183184	25
139	140	19321	19600	16	454	455	206116	207025	16
157	158	24649	24964	25	463	464	214369	215296	25
193	194	37249	37636	25	472	473	222784	223729	25
202	203	40804	41209	16	481	482	231361	232324	16
229	230	52441	52900	16	508	509	258064	259081	25
247	248	61009	61504	16	553	554	305809	306916	25
256	257	65536	66049	25	598	599	357604	358801	25

n	$n+1$	n^2	$(n+1)^2$	Σn^2	n	$n+1$	n^2	$(n+1)^2$	Σn^2
643	644	413449	414736	25	769	770	591361	592900	25
661	662	436921	438244	25	778	779	605284	606841	25
679	680	461041	462400	16	796	797	633616	635209	25
688	689	473344	474721	25	850	851	722500	724201	16
724	725	524176	525625	25	868	869	753424	755161	25
742	743	550564	552049	25	904	905	817216	819025	25
760	761	577600	579121	25	949	950	900601	902500	16

Also solved by E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; CLAYTON W. DODGE, University of Maine at Orono; J.T. GROENMAN, Arnhem, The Netherlands; FRIEND H. KIERSTEAD, JR., Cuyahoga Falls, Ohio; ANDY LIU, University of Alberta; DANIEL PLOTNICK, Rockville Center, N.Y.; STANLEY RABINOWITZ, Digital Equipment Corp; Merrimack, New Hampshire; DONVAL R. SIMPSON, Fairbanks, Alaska; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Kierstead reported that there are 328 solutions for which $n^2 < 2^{31}$. Three solvers misinterpreted the problem. Dodge and Simpson gave some solutions for which $\Sigma n^2 = \Sigma(n+1)^2$, but did not require that in every case this common value be itself a square. Plotnick interpreted the proposal's "square digit sum" as "sum of the squares of the digits" (s.s.d.) and gave 155 computer-generated solutions in which

$$\text{s.s.d. } n^2 = \text{s.s.d. } (n+1)^2, \quad n^2 < 10^9.$$

Our proposer (who, as everyone knows by now, is a Count of Digit Delvers) will be glad to know of the following curiosities, which Plotnick found by scrutinizing his computer print-out. The following pairs of consecutive squares are permutations of the same digits (and hence have the same s.s.d.):

$13^2 = 169$	$157^2 = 24649$	$913^2 = 833569$
$14^2 = 196$	$158^2 = 24964$	$914^2 = 835396$
$4513^2 = 20367169$	$14647^2 = 214534609$	$19201^2 = 368678401$
$4514^2 = 20376196$	$14648^2 = 214563904$	$19202^2 = 368716804$
$19291^2 = 372142681$	$19813^2 = 392554969$	$27778^2 = 771617284$
$19292^2 = 372181264$	$19814^2 = 392594596$	$27779^2 = 771672841$

Plotnick also found the following sets of three consecutive squares with the same s.s.d.:

$$\left\{\begin{aligned} 6744^2 &= 45481536 \\ 6745^2 &= 45495025 \\ 6746^2 &= 45508516 \end{aligned}\right. \qquad \text{all with s.s.d. } 192,$$

$$\begin{cases} 13404^2 = 179667216 \\ 13405^2 = 179694025 \\ 13406^2 = 179720836 \end{cases} \quad \text{all with s.s.d. } 293,$$

$$\begin{cases} 13711^2 = 187991521 \\ 13712^2 = 188018944 \\ 13713^2 = 188046369 \end{cases} \quad \text{all with s.s.d. } 307.$$

* * *

659. [1981: 204] *Proposed by Leon Bankoff, Los Angeles, California.*

If the line joining the incenter I and the circumcenter O of a triangle ABC is parallel to side BC, it is known from an earlier problem in this journal (Crux 388 [1979: 201]) that

$$s^2 = \frac{(2R-r)^2(R+r)}{R-r}.$$

In the same configuration, $\cos B + \cos C = 1$, and the internal bisector of angle A is perpendicular to the line joining I to the orthocenter H (*Mathematics Magazine*, Problem 758, 43 (November 1970) 285-286).

Prove the following additional properties, which were listed but not proved in the last-mentioned reference:

(a) If the internal bisector of angle A meets the circumcircle in P, show that $AI/AP = \cos A$.

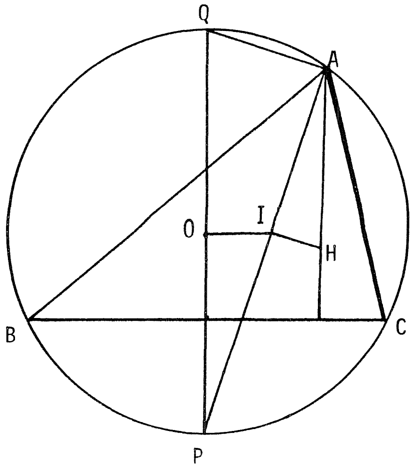
(b) The circumcircle of triangle AIH is equal to the incircle of triangle ABC.

(c) $AI \cdot IP = 2Rr = \sqrt{R \cdot AI \cdot BI \cdot CI}$.

(d) $\sin^2\frac{B}{2} + \sin^2\frac{C}{2} = \frac{1}{2}$; $\cos^2\frac{B}{2} + \cos^2\frac{C}{2} = \frac{3}{2}$.

(e) $\tan^2\frac{A}{2} = \frac{R-r}{R+r}$.

Solution by the proposer.



First we note that $\cos B + \cos C = 1$ implies that angles B and C are both acute; and $\Sigma \cos A = 1 + r/R$ gives $\cos A = r/R > 0$, so angle A also is acute. We will use the result $AH = 2R\cos A$, which is easily established for acute-angled triangles (or see [1, p. 5]).

(a) If Q is diametrically opposite P, as shown in the figure, then triangles AIH and PAQ are similar, so

$$\frac{AI}{AP} = \frac{AH}{PQ} = \frac{2R\cos A}{2R} = \cos A.$$

(b) This part follows from the fact that the circumradius of triangle AIH is

$$\tfrac{1}{2}\mathrm{AH} = R\cos\mathrm{A} = r.$$

(c) For a proof of the familiar results used here, see [1, p. 15]. We have

$$\mathrm{AI}\cdot\mathrm{IP} = R^2 - \mathrm{OI}^2 = 2Rr$$

and

$$\mathrm{AI}\cdot\mathrm{BI}\cdot\mathrm{CI} = 64R^3\Pi\sin^2\frac{\mathrm{A}}{2};$$

hence

$$\sqrt{R\cdot\mathrm{AI}\cdot\mathrm{BI}\cdot\mathrm{CI}} = 2R\left(4R\Pi\sin\frac{\mathrm{A}}{2}\right) = 2Rr.$$

(d) Here we have

$$\sin^2\frac{\mathrm{B}}{2} + \sin^2\frac{\mathrm{C}}{2} = 1 - \frac{1}{2}(\cos\mathrm{B} + \cos\mathrm{C}) = \frac{1}{2},$$

and

$$\cos^2\frac{\mathrm{B}}{2} + \cos^2\frac{\mathrm{C}}{2} = \frac{3}{2}$$

follows immediately.

(e) Finally, we have

$$\tan^2\frac{\mathrm{A}}{2} = \frac{1-\cos\mathrm{A}}{1+\cos\mathrm{A}} = \frac{R - R\cos\mathrm{A}}{R + R\cos\mathrm{A}} = \frac{R-r}{R+r}.$$

Also solved by HIPPOLYTE CHARLES, Waterloo, Québec; G.C. GIRI, Midnapore College, West Bengal, India; J.T. GROENMAN, Arnhem, The Netherlands; V.N. MURTY, Pennsylvania State University, Capitol Campus; and KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India.

REFERENCE

1. C.V. Durell and A. Robson, *Advanced Trigonometry*, G. Bell and Sons, London, 1959.

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660. [1981: 204, 274] (Corrected) *Proposed by Leon Bankoff, Los Angeles, California.*

Show that, in a triangle ABC with semiperimeter s, the line joining the circumcenter and the incenter is parallel to BC if and only if

$$\mathrm{DL} + \mathrm{EM} + \mathrm{FN} = s\tan\frac{\mathrm{A}}{2},$$

where L,M,N bisect the arcs BC,CA,AB, respectively, of the circumcircle and D,E,F bisect the sides BC,CA,AB, respectively, of the triangle.

I. *Solution by G.C. Giri, Midnapore College, West Bengal, India.*

We will use results mentioned in the preceding proposal 659 or in its proof. We have

$$DL = OL - OD = R - R\cos A,$$

with similar results for EM and FN. Hence

$$DL + EM + FN = R(3 - \Sigma\cos A) = 2R - r,$$

since $\cos B + \cos C = 1$ and $R\cos A = r$. But

$$2R - r = s\sqrt{\frac{R-r}{R+r}} = s\tan\frac{A}{2},$$

so

$$DL + EM + FN = s\tan\frac{A}{2}.$$

II. *Comment by Kesiraju Satyanarayana, Gagan Mahal Colony, Hyderabad, India.*

If r_1 is the radius of the excircle touching segment BC at an interior point, then we have

$$r_1 = s\tan\frac{A}{2} = 2R - r,$$

so $R = (r+r_1)/2$, that is, R is the arithmetic mean of r and r_1.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; J.T. GROENMAN, Arnhem, The Netherlands; V.N. MURTY, Pennsylvania State University, Capitol Campus; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; and the proposer.

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661. [1981: 204] *Antrag von Alan Wayne, Holiday, Florida.*

Zu finden: die einzige entsprechende Lösung der jeder der folgenden unabhängigen Additionen, welche die Sechs als Basis ihrer Aufzählung haben.

$$\text{(a)}\quad \begin{array}{r} \text{EINS} \\ \underline{\text{EINS}}, \\ \text{ZWEI} \end{array} \qquad\qquad \text{(b)}\quad \begin{array}{r} \text{ZWEI} \\ \underline{\text{ZWEI}}. \\ \text{VIER} \end{array}$$

Solution by Charles W. Trigg, San Diego, California.

(a) Operating in base six, I is even and E = 1 or 2. If E = 2, then S < 3, so either I or S = 2, a duplication. Consequently, E = 1, S > 2, and N = 0 or 3. The possible values of the letters are tabulated below, a star indicating that a duplication has occurred. The table shows that the unique reconstruction of the sum is

$$\begin{array}{r} 1405 \\ \underline{1405}. \\ 3214 \end{array}$$

E	S	I	N	W	Z
1	3	0	*		
	4	2	3	5	*
	4	2	0	*	
	5	4	3	*	
	5	4	0	2	3

(b) Here Z = 1 or 2, so I ≠ 1. If I = 2, then R = 4, Z = 1, V = 3, E = 0, and we have the duplication W = 4. Hence I > 2 and E is odd. Indeed, 2E + 1 = E + 10 (in base six, remember?), so E = 5 and I is odd, which means that I = 3. Now R = 0, and W = 1, Z = 2, V = 4. THus the unique reconstruction is

$$\begin{array}{r} 2153 \\ \underline{2153}. \\ 4350 \end{array}$$

Also solved by E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; CLAYTON W. DODGE, University of Maine at Orono; J.A.H. HUNTER, Toronto, Ontario; ALLAN WM. JOHNSON JR., Washington, D.C.; J.A. McCALLUM, Medicine Hat, Alberta; LEROY F. MEYERS, The Ohio State University; DANIEL PLOTNICK, Rockville Center, N.Y.; DONVAL R. SIMPSON, Fairbanks, Alaska; KENNETH M. WILKE, Topeka, Kansas; DAVID ZAGORSKI, student, Massachusetts Institute of Technology; and the proposer. One incorrect solution was received.

Editor's comment.

For a related problem, see Crux 761 in this issue, which takes us one step further in mastering the addition table in German.

*   *   *

662. [1981: 204] *Proposed by Kaidy Tan, Fukien Teachers' University, Foochow, Fukien, China.*

An isosceles triangle has vertex A and base BC. Through a point F on AB, a perpendicular to AB is drawn to meet AC in E and BC produced in D. With square brackets denoting area, prove synthetically that

$$[AFE] = 2[CDE] \iff AF = CD.$$

Solution by E.C. Buissant des Amorie, Amstelveen, The Netherlands.

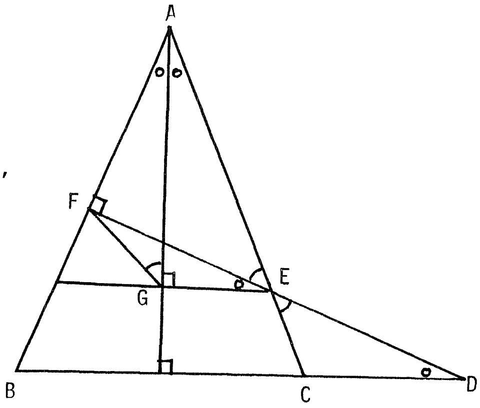
If a line through E parallel to BC meets the altitude issued from A in G, then the points A,F,G,E are concyclic, and it is easy to verify that certain angles are properly designated as congruent in the figure on the next page, as a consequence of which

$$\text{triangles CDE and FAG are (inversely) similar.} \tag{1}$$

Since clearly [AFE] = 2[AFG], we have, from (1),

$$[AFE] = 2[CDE] \iff [AFG] = [CDE] \iff AF = CD.$$

Also solved by JORDI DOU, Barcelona, Spain; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, California State University at Los Angeles; GEORGE TSINTSIFAS, Thessaloniki, Greece; DAVID ZAGORSKI, student, Massachusetts Institute of Technology; and the proposer.



Editor's comment.

The proposal asked for a synthetic solution, and the one we gave above was by far the best. Five of the other solutions used "synthetic" trigonometry.

*   *   *

663. [1981: 205] *Proposed by Allan Wm. Johnson Jr., Washington, D.C.*

Prove that every fifth-order pandiagonal magic square can be written in the form

E+F-I	A	B+G-I	H	C+D-I
B	C+H-I	D+E-I	A+F-I	G
A+D-I	F+G-I	I	B+C-I	E+H-I
C	B+E-I	A+H-I	D+G-I	F
G+H-I	D	C+F-I	E	A+B-I

.

Comment by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.

Benson and Jacoby [1] have shown that every fifth-order pandiagonal magic square can be written in the form

$A+a$	$B+b$	$C+c$	$D+d$	$E+e$
$C+d$	$D+e$	$E+a$	$A+b$	$B+c$
$E+b$	$A+c$	$B+d$	$C+e$	$D+a$
$B+e$	$C+a$	$D+b$	$E+c$	$A+d$
$D+c$	$E+d$	$A+e$	$B+a$	$C+b$

.

It is easily seen that the substitutions

A $\to$ *B+b*	D $\to$ *E+d*	G $\to$ *B+c*
B $\to$ *C+d*	E $\to$ *B+a*	H $\to$ *D+d*
C $\to$ *B+e*	F $\to$ *A+d*	I $\to$ *B+d*

transform the proposer's basic square into Benson and Jacoby's basic square.

Also solved by the proposer.

Editor's comment.

The proposer's solution and his ensuing comments contain much interesting information about fifth-order pandiagonal magic squares which we have insufficient room to publish. Interested readers can obtain this information by writing directly to the proposer at 524 S. Court House Road, Apt. 301, Arlington, Virginia 22204, U.S.A.

REFERENCE

1. William H. Benson and Oswald Jacoby, *New Recreations With Magic Squares*, Doves, New York, 1976, p. 126.

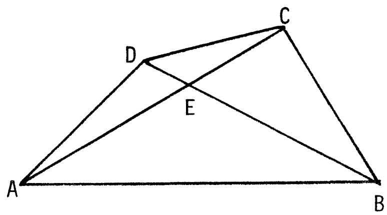
* * *

664. [1981: 205] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

An isosceles trapezoid ABCD, with parallel bases AB and DC, is inscribed in a circle of diameter AB. Prove that AC > (AB + DC)/2.

Solution by M.S. Klamkin, University of Alberta.

Let ABCD be any convex quadrilateral with diagonals AC and BD intersecting at E, as shown in the figure. From



$$AE + EB > AB$$

and

$$DE + EC > DC,$$

we obtain the known elementary result

$$AC + BD > AB + DC.$$

If AC is the longer diagonal (or either one if they are equal), we therefore have

$$2AC \geq AC + BD > AB + DC,$$

and AC > (AB + DC)/2 follows. The isosceles trapezoid of the proposal is a very, very special case of this more general result.

Also solved by W.J. BLUNDON, Memorial University of Newfoundland; E.C. BUISSANT DES AMORIE, Amstelveen, The Netherlands; S.C. CHAN, Singapore; GEORGE

DEDAJ, student, The Bronx High School of Science, Bronx, N.Y.; CLAYTON W. DODGE, University of Maine of Orono; ROLAND H. EDDY, Memorial University of Newfoundland; MILTON P. EISNER, Mount Vernon College, Washington, D.C.; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; LEROY F. MEYERS, The Ohio State University; DANIEL PLOTNICK, Rockville Center, N.Y.; STANLEY RABINOWITZ, Digital Equipment Corp., Merrimack, New Hampshire; KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, California State University at Los Angeles; ROBERT A. STUMP, Hopewell, Virginia; ROBERT TRANQUILLE, Collège de Maisonneuve, Montréal, Québec; DAVID ZAGORSKI, student, Massachusetts Institute of Technology; and the proposer. Also, one unsigned solution from Georgia Southern College, Statesboro, Georgia.

Editor's comment.

Nearly all solvers proved only the very special case of the proposal, several by trigonometry.

*　　　　*　　　　*

665. [1981: 205] *Proposed by Jack Garfunkel, Queens College, Flushing, N.Y.*

If A,B,C,D are the interior angles of a convex quadrilateral ABCD, prove that

$$\sqrt{2}\,\Sigma\cos\frac{A+B}{4} \le \Sigma\cot\frac{A}{2}$$

(where the four-term sum on each side is cyclic over A,B,C,D), with equality if and only if ABCD is a rectangle.

Solution by E.C. Buissant des Amorie, Amstelveen, The Netherlands.

Using repeatedly the transformation of the sum of two cosines into a product, and noting that $\Sigma A/8 = \pi/4$, we have

$$\sqrt{2}\,\Sigma\cos\frac{A+B}{4} = \sqrt{2}\cdot 2\cos\frac{\Sigma A}{8}\left(\cos\frac{A+B-C-D}{8} + \cos\frac{A-B-C+D}{8}\right)$$
$$= 4\cos\frac{A-C}{8}\cos\frac{B-D}{8},$$

from which

$$\sqrt{2}\,\Sigma\cos\frac{A+B}{4} \le 4, \tag{1}$$

with equality just when A = C and B = D, that is, just when ABCD is a parallelogram.

The half-angles involved in the right member of the proposed inequality are all acute, so we can use the convexity of the cotangent function for acute angles, thus:

$$\Sigma\cot\frac{A}{2} \ge 2\cot\frac{A+B}{4} + 2\cot\frac{C+D}{4} \ge 4\cot\frac{\Sigma A}{8} = 4, \tag{2}$$

with equality just when A = B = C = D, that is, just when ABCD is a rectangle.

Finally, the proposed inequality follows from (1) and (2), with equality just when ABCD is a rectangle.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; M.S. KLAMKIN, University of Alberta; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

Editor's comment.

One more solution, not recorded above, was received. It was by calculus, with very extensive calculations involving first and second partial derivatives (for three independent variables), and it fell short of being a complete solution. We have frequently said in recent months that calculus should usually be avoided in proving triangle inequalities. The same caution applies to most geometric inequalities, including the present one. After long but straightforward calculations, our solver showed that a sufficient condition for a certain function ϕ to attain its maximum was that ABCD be a rectangle. But he did not show that the condition was also necessary.

* * *

666, [1981: 205] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

The symmedians issued from vertices A,B,C of triangle ABC meet the opposite sides in D,E,F, respectively. Through D,E,F, lines d,e,f are drawn perpendicular to BC,CA,AB, respectively. Prove that d,e,f are concurrent if and only if triangle ABC is isosceles.

Solution by Stanley Rabinowitz, Digital Equipment Corp., Merrimack, New Hampshire.

It is known that the segments into which a side of a triangle is divided by the corresponding symmedian are proportional to the squares of the adjacent sides of the triangle [1]. Thus, if the side lengths are a,b,c in the usual order, we have

$$\mathrm{BD} = \frac{ac^2}{b^2+c^2} \qquad \text{and} \qquad \mathrm{DC} = \frac{ab^2}{b^2+c^2},$$

with similar expressions for the other segments. Furthermore, it is easy to show (or see [2]) that if D,E,F are arbitrary points on BC,CA,AB, respectively, then d,e,f are concurrent if and only if

$$\mathrm{AF}^2 + \mathrm{BD}^2 + \mathrm{CE}^2 = \mathrm{FB}^2 + \mathrm{DC}^2 + \mathrm{EA}^2. \tag{1}$$

With the expressions for the segments found above, the necessary and sufficient condition (1) becomes

$$\left(\frac{cb^2}{a^2+b^2}\right)^2 + \left(\frac{ac^2}{b^2+c^2}\right)^2 + \left(\frac{ba^2}{c^2+a^2}\right)^2 = \left(\frac{ca^2}{a^2+b^2}\right)^2 + \left(\frac{ab^2}{b^2+c^2}\right)^2 + \left(\frac{bc^2}{c^2+a^2}\right)^2,$$

an equation which is easily (but tediously) shown to be equivalent to

$$(b^2-c^2)(c^2-a^2)(a^2-b^2)(a^2+b^2+c^2) = 0. \tag{2}$$

It follows from (2) that the required necessary and sufficient condition is that $b = c$ or $c = a$ or $a = b$, that is, that triangle ABC be isosceles.

Also solved by KESIRAJU SATYANARAYANA, Gagan Mahal Colony, Hyderabad, India; DAN SOKOLOWSKY, California State University at Los Angeles; GEORGE TSINTSIFAS, Thessaloniki, Greece; and the proposer.

REFERENCES

1. Nathan Altshiller Court, *College Geometry*, Barnes & Noble, New York, 1952, p. 248.

2. Leslie H. Miller, *College Geometry*, Appleton-Century-Crofts, New York, 1957, p. 180.

* * *

THE OLYMPIAD CORNER: 37

M.S. KLAMKIN

The Twenty-third International Mathematical Olympiad (I.M.O.) was held this year in Budapest and Ceglēd, Hungary, on July 7-14, 1982. Teams from thirty countries took part in the contest. This was a record number of countries, but not a record number of participants because this year, for the first time, the team size was reduced from eight to four for each country, perhaps in anticipation of the large turnout. The reduced team size will probably be maintained in the future since the number of participating countries is unlikely to decrease.

The standard of difficulty for this year's competition was back to "normal", for there were only three perfect papers, compared to twenty-six in 1981. Also, continuing last year's break with tradition (see last year's report [1981: 220]), the six problems of the competition were assigned equal weights of 7 points each, for a maximum possible score of 42. The three perfect papers were achieved by

Bruno Haible (West Germany),
Grigorij Perelman (Soviet Union),
Le Tu Quoc Thang (Vietnam).

The results of the competition (see table) are announced officially only for individual team members. However, team standings are always compiled unofficially by adding up the individual scores. In this unofficial team ranking, the top five teams (and their total scores) were

West Germany	145	East Germany	136	tie
Soviet Union	137	United States	136	tie
		Vietnam	133.	

Country	Contestant No. 1	2	3	4	Total max 168	Prizes 1st	2nd	3rd.	Total Prizes	Rank
Algeria	11	10	2		23	-	-	-	-	
Australia	20	23	10	13	66	-	-	1	1	
Austria	11	11	38	22	82	1	-	1	2	
Belgium	7	2	22	19	50	-	-	1	1	
Brazil	24	10	19	13	66	-	-	1	1	
Bulgaria	26	29	26	27	108	-	-	4	4	9
Canada	14	12	23	29	78	-	-	2	2	
Colombia	3	9	18	4	34	-	-	-	-	
Cuba	17	7	17	3	44	-	-	-	-	
Czechoslovakia	29	21	31	34	115	-	2	2	4	7
East Germany	37	40	27	32	136	2	1	1	4	3
Finland	16	35	28	34	113	-	2	1	3	8
France	38	17	14	20	89	1	-	-	1	
Great Britain	23	23	28	29	103	-	-	4	4	10
Greece	14	19	9	13	55	-	-	-	-	
Hungary	21	36	33	35	125	-	3	1	4	6
Israel	22	18	17	18	75	-	-	1	1	
Kuwait	2	1	1	0	4	-	-	-	-	
Mongolia	21	12	13	10	56	-	-	1	1	
Netherlands	17	22	34	19	92	-	1	1	2	
Poland	30	23	16	27	96	-	1	2	3	
Romania	26	14	26	33	99	-	1	2	3	
Soviet Union	37	42	30	28	137	2	1	1	4	2
Sweden	23	15	11	25	74	-	-	2	2	
Tunisia	7	8	1	3	19	-	-	-	-	
United States	40	35	29	32	136	1	2	1	4	3
Venezuela	11	10	1	1	23	-	-	-	-	
Vietnam	42	30	32	29	133	1	2	1	4	5
West Germany	42	35	31	37	145	2	2	-	4	1
Yugoslavia	30	20	18	30	98	-	2	-	2	
						10	20	31	61	

The problems of this year's competition are given below. Solutions to these problems (along with those of the Eleventh U.S.A. Mathematical Olympiad given earlier in this column [1982: 165]) will appear later this year in a booklet, *Olympiads for 1982*, to be compiled by Samuel S. Greitzer and obtainable from

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Executive Director
M.A.A. Committee on H.S. Contests
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XXIII INTERNATIONAL MATHEMATICAL OLYMPIAD

July 9, 1982
Time: 4½ hours

1. The function $f(n)$ is defined for all positive integers n and takes on non-negative integer values. Also, for all m,n,

$$f(m+n) - f(m) - f(n) = 0 \text{ or } 1;$$
$$f(2) = 0,\ f(3) > 0, \text{ and } f(9999) = 3333.$$

Determine $f(1982)$.

2. A scalene triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3 (a_i is the side opposite A_i). For all $i = 1,2,3$, M_i is the midpoint of side a_i, T_i is the point where the incircle touches side a_i, and the reflection of T_i in the interior bisector of A_i yields the point S_i. Prove that the lines M_1S_1, M_2S_2, and M_3S_3 are concurrent.

3. Consider the infinite sequences $\{x_n\}$ of positive real numbers with the following properties:

$$x_0 = 1 \text{ and, for all } i \geq 0,\quad x_{i+1} \leq x_i.$$

(a) Prove that, for every such sequence, there is an $n \geq 1$ such that

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \ldots + \frac{x_{n-1}^2}{x_n} \geq 3.999.$$

(b) Find such a sequence for which

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \ldots + \frac{x_{n-1}^2}{x_n} < 4$$

for all n.

July 10, 1982
Time: 4½ hours

4. Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers (x,y), then it has at least three such solutions.

Show that the equation has no solution in integers when $n = 2891$.

5. The diagonals AC and CE of the regular hexagon ABCDEF are divided by the interior points M and N, respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Determine r if B, M, and N are collinear.

6. Let S be a square with sides of length 100 and let L be a path within S which does not meet itself and which is composed of linear segments A_0A_1, A_1A_2, ..., $A_{n-1}A_n$ with $A_n \neq A_0$. Suppose that for every point P of the boundary of S there is a point of L at a distance from P not greater than $\frac{1}{2}$. Prove that there are two points X and Y in L such that the distance between X and Y is not greater than 1 and the length of that part of L which lies between X and Y is not smaller than 198.

*

In a later column, I shall give some of the problems that were proposed but unused at this year's I.M.O..

The Canadian team, led by Professors G.J. Butler, University of Alberta and E.J. Barbeau, University of Toronto, consisted of

Charles Timar	(Third Prize Award),
Alistair Rucklidge	(Third Prize Award),
Todd Cardno,	
Edward Hatton.	

The United States team, led by the author and Professor Andy Liu, University of Alberta, consisted of

Noam Elkies	(First Prize Award),
Brian Hunt	(Second Prize Award),
Washington Taylor IV	(Second Prize Award),
Douglas Jungreis	(Third Prize Award).

The 1983 I.M.O. is scheduled to be held in Paris, France. For information about past national and international olympiads, see Olympiad Corner: 8 [1979: 220-227].

Editor's Note. All communications about this column should be sent to Professor M.S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

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