Crux

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Journal title history:

- The first 32 issues, from Vol. 1, No. 1 (March 1975) to Vol. 4, No.2 (February 1978) were published under the name *EUREKA*.
- Issues from Vol. 4, No. 3 (March 1978) to Vol. 22, No. 8 (December 1996) were published under the name *Crux Mathematicorum*.
- Issues from Vol 23., No. 1 (February 1997) to Vol. 37, No. 8 (December 2011) were published under the name *Crux Mathematicorum with Mathematical Mayhem*.
- Issues since Vol. 38, No. 1 (January 2012) are published under the name *Crux Mathematicorum*.

ISSN 0700 - 558X

E U R E K A

Vol. 3, No. 7

August - September 1977

Sponsored by

Carleton-Ottawa Mathematics Association Mathématique d'Ottawa-Carleton
A Chapter of the Ontario Association for Mathematics Education

Publié par le Collège Algonquin

EUREKA *is published monthly (except July and August). The following yearly subscription rates are in Canadian or U.S. dollars. Canada and USA: $6.00; elsewhere: $7.00. Bound copies of combined Volumes 1 and 2: $10.00. Back issues: $1.00 each. Make cheques or money orders payable to Carleton-Ottawa Mathematics Association.*

All communications about the content of the magazine (articles, problems, solutions, book reviews, etc.) should be sent to the editor: Léo Sauvé, Mathematics Department, Algonquin College, 281 Echo Drive, Ottawa, Ont., K1S 1N3.

All changes of address and inquiries about subscriptions and back issues should be sent to the Secretary-Treasurer of COMA: F.G.B. Maskell, Algonquin College, 200 Lees Ave., Ottawa, Ont., K1S 0C5.

Typist-compositor: Nancy Makila.

* * *

CONTENTS

EXPRESSING ONE AS A SUM OF DISTINCT RECIPROCALS

Comments and a Bibliography

E.J. BARBEAU

University College, University of Toronto

My interest in the expression of one as a sum of distinct reciprocals arose out of the consideration of a 1950 Putnam problem concerning a certain number-theoretic function $D(n)$. This was defined as follows:

$$D(1) = 0,$$
$$D(p) = 1 \text{ for every prime } p,$$
$$D(mn) = mD(n) + nD(m) \text{ for every pair } m, n \text{ of natural numbers.}$$

One had to investigate the behaviour of this function under iteration (see [8] for details). The numbers n for which $D(n) = n$ are precisely those of the form p^p for prime p. Are there any numbers n such that $D(n) \neq n$ while $D(D(n)) = n$?

It is not too hard to verify that if $D(D(n)) = n$ while $D(n) \neq n$, then both n and $D(n)$ must be squarefree and their greatest common divisor must be 1. Thus, for distinct primes p_i and q_j,

$$n = p_1 p_2 \cdots p_r \quad \text{and} \quad D(n) = q_1 q_2 \cdots q_s .$$

Using the fact that

$$D(n) = n \sum_{i=1}^{r} \frac{1}{p_i} \quad \text{and} \quad n = D(D(n)) = D(n) \sum_{j=1}^{s} \frac{1}{q_j},$$

we see that the existence of such n would give the following representation of 1:

$$1 = \left(\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_r}\right)\left(\frac{1}{q_1} + \frac{1}{q_2} + \ldots + \frac{1}{q_s}\right).$$

I do not know whether such a representation exists.

Such a representation would mean, in particular, that 1 can be expressed as a finite sum of reciprocals of distinct positive integers, each the product of exactly two primes. With the aid of a pocket calculator, I was able to show that 1 is the sum of the reciprocals of the following 101 numbers of the required type:

6, 10, 14, 15, 21, 22, 26, 33, 34, 35, 38, 39, 51, 55, 57, 62,
65, 69, 77, 82, 85, 86, 91, 93, 115, 118, 119, 123, 129, 133,
143, 155, 161, 203, 209, 213, 217, 221, 287, 295, 299, 323, 355,
371, 407, 413, 415, 481, 493, 497, 527, 553, 559, 565, 611, 622,
933, 955, 989, 1081, 1139, 1141, 1195, 1219, 1315, 1337, 1555,

1673, 1841, 1883, 2077, 2177, 2353, 2483, 2771, 2933, 3349, 3437, 4043, 4063, 4163, 4187, 4609, 4727, 4997, 5401, 6107, 7961, 8507, 11293, 13199, 14257, 21251, 34835, 76637, 81335, 211471, 257779, 578261, 699481, 1838171.

(See [20] for an account of the approach used.) What is the shortest such representation?

R.L. Graham informs me, by letter, that he and Erdős have shown the following: if q is squarefree, then p/q can always be written as a finite sum of reciprocals of squarefree integers each having exactly m distinct prime factors, for $m \geq 3$. Also, there are many cases for which m can be taken as 2. This work has not yet been published.

Turning to another question, we might ask whether it is always possible to express 1 as the sum of finitely many reciprocals of integers drawn from a given infinite arithmetic progression. For the shortest representation of 1 as a sum of odd reciprocals, we can take the numbers in each of the following sets:

(3, 5, 7, 9, 11, 15, 21, 135, 10395),
(3, 5, 7, 9, 11, 15, 21, 165, 693),
(3, 5, 7, 9, 11, 15, 21, 231, 315),
(3, 5, 7, 9, 11, 15, 33, 45, 385),
(3, 5, 7, 9, 11, 15, 35, 45, 231).

(These were supplied by Harry Nelson and Peter Montgomery; for a simple derivation of a slightly longer representation, see [18].) The shortest representation using numbers congruent to 2 modulo 3 uses the integers

(2, 5, 8, 11, 20, 41, 110, 1640).

It has in fact been shown by Van Albada and Van Lint [10] that every positive integer can be expressed as the finite sum of reciprocals of numbers chosen from any given positive infinite arithmetic progression. One might ask for information on the length of the shortest such representation.

Paul Erdős and R.L. Graham are preparing a survey paper which will include material on this topic. In the meantime, it seems useful to record the following bibliography.

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Further bibliographic information might be sent to the author or to Dr. R.L. Graham, Bell Laboratories, 600 Mountain Avenue, Murray Hill, NJ 07974.

* * *

THE VOLUME OF THE REGULAR TETRAHEDRON

CHARLES W. TRIGG,
Professor Emeritus, Los Angeles City College

Early in the study of volumes, it is learned that the volume of any pyramid is equal to one-third the product of its base and its *altitude*. The simplest pyramid is the regular tetrahedron. Can the volume of a regular tetrahedron be computed *without* using or knowing its altitude? Seeking an answer to this question reveals many interesting relationships among various simple solids. Following are three procedures that provide affirmative answers to the question.

1. *From a Cube.*

As shown in Figure 1, each of the six edges of a regular tetrahedron inscribed in a cube is a diagonal of a face of the cube. The opposide edges of the tetrahedron are nonparallel diagonals. The faces of the tetrahedron cut off four trirectangular tetrahedrons from the cube. Each of these four tetrahedrons has a volume of (1/3)(1/2) or 1/6 that of the cube. Consequently, the volume of the residual regular tetrahedron is one-third the volume of the cube [2].

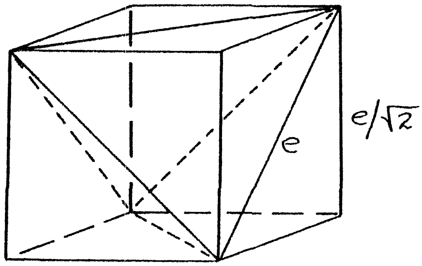

Figure 1

If an edge of the tetrahedron is e, then an edge of the cube is $e/\sqrt{2}$. Therefore the volume of the tetrahedron is

$$\frac{1}{3}\left(\frac{e}{\sqrt{2}}\right)^3 = \frac{\sqrt{2}}{12}e^3.$$

2. *From a Regular Octahedron.*

A regular octahedron is composed of two congruent square pyramids, A-BCDE and F-BCDE (see Figure 2). Hence the volume of the regular octahedron [3] is

$$V = 2\cdot\frac{1}{3}\cdot AO\cdot BCDE = 2\cdot\frac{1}{3}\cdot\frac{e}{\sqrt{2}}\cdot e^2 = \frac{\sqrt{2}}{3}e^3.$$

A plane through the midpoints of three edges of a regular tetrahedron issuing from one vertex will cut off a smaller tetrahedron with a volume $(\frac{1}{2})^3$ that of the

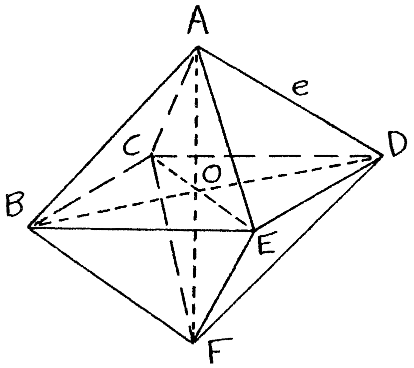

Figure 2

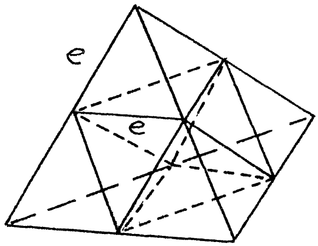

Figure 3

larger similar one (since their edges are in the ratio 1:2). The four tetrahedrons formed by similarly cutting off each vertex of the original tetrahedron have a combined volume of 4(1/8) or ½ that of the large tetrahedron (see Figure 3). It follows that the residual regular octahedron has a volume four times that of the smaller tetrahedron with which it has three common edges [1]. Hence the volume of a regular tetrahedron with edge e is

$$\frac{1}{4} \cdot \frac{\sqrt{2}}{3} e^3 = \frac{\sqrt{2}}{12} e^3.$$

3. *From a Bisphenoid.*

In an isosceles tetrahedron the opposide edges are equal. A bisphenoid or double wedge is an isosceles tetrahedron with four equal edges. Hence its faces are congruent isosceles triangles. In Figure 4, the four equal edges have length x, and the other two edges length e.

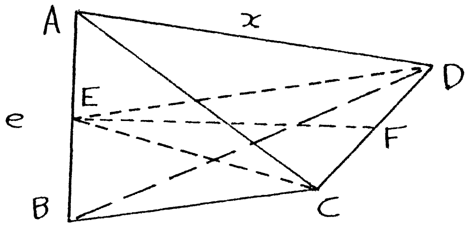

Figure 4

The plane through DC and E, the midpoint of AB, is perpendicular to AB and divides the bisphenoid into two congruent pyramids. Triangle DEC is isosceles. Its altitude EF, which bisects DC, is a *bimedian* of the bisphenoid. Now

$$|EF|^2 = |EC|^2 - |FC|^2 = |AC|^2 - |AE|^2 - |FC|^2 = x^2 - 2\left(\frac{e}{2}\right)^2.$$

It follows that the volume of the bisphenoid is

$$2 \cdot \frac{1}{3} \cdot \frac{1}{2} e \cdot \frac{1}{2} e \cdot \sqrt{x^2 - \tfrac{1}{2}e^2} = \frac{e^2}{6} \sqrt{x^2 - \tfrac{1}{2}e^2}.$$

When $x = e$, the bisphenoid becomes a regular tetrahedron with a volume of

$$\frac{e^2}{6}\sqrt{\tfrac{1}{2}e^2} = \frac{\sqrt{2}}{12}e^3.$$

Extensions.

1. Starting with its volume, show that the altitude of a regular tetrahedron of edge e is $e\sqrt{2/3}$.

2. Show that the volume of a regular tetrahedron in terms of an altitude h is $\frac{\sqrt{3}}{8}h^3$.

3. Find the volume of a regular tetrahedron in terms of a bimedian b (see Problem 245 [1977: 130]).

4. Show that the volume of a regular tetrahedron in terms of the area T of a face is $(2T/9)\sqrt[4]{12T^2}$.

5. Show that the volume of a regular tetrahedron in terms of its circumradius R is $8\sqrt{3}R^3/27$.

6. Show that the volume of a regular tetrahedron in terms of its inradius r is $8\sqrt{3}r^3$.

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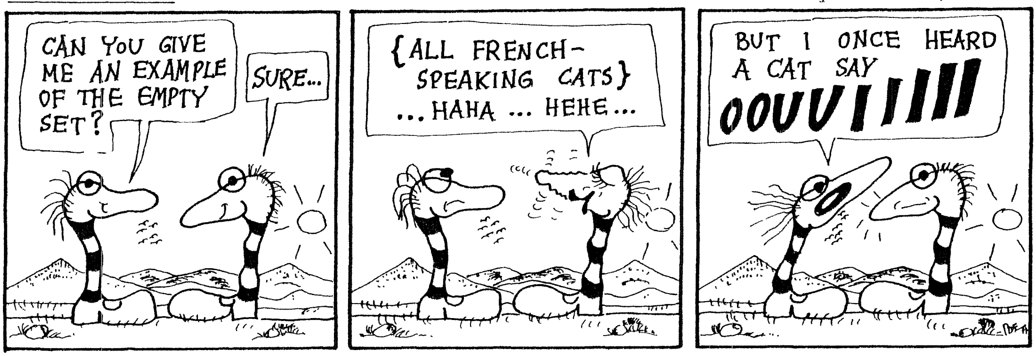
* * *

FOOTIES

ANDREJS DUNKELS,
University of Luleå, Sweden



Reminiscences

MY INTRODUCTION TO CIPHERING

R. ROBINSON ROWE, Naubinway, Michigan

Reluctantly[1], I was born on 26 August[2] 1896 at Spencer, Massachusetts, in the Baptist parsonage, but at age 2 moved with the family to a farm in Michigan near which, at age 6, I entered a small country school.[3]

The school was a one-room, one-teacher, eight-grade building, heated by a central potbellied wood stove, with a well and playground in front and a woodshed and privies in back. Inside, a teacher's desk faced 24 pupils' desks arranged in columns with aisles between them. The room was girded with blackboards on the front and rear walls, and more alternating with windows on the side walls.

The pupils, all farm children, were a varied lot. Early settlers had been Irish, Welsh and Cornish, but newcomers were mostly Dutch, and a majority of the pupils came from homes where Dutch was the family language, so they were handicapped in Readin' and 'Ritin', but not in 'Rithmetic.

The teachers (a new one each year) were Trainee graduates of Ypsilanti Normal School[4] and were very dedicated, patient and conscientious people. Since all used the routine I will describe, I presume it was taught there.

School hours were from 9 a.m. to 4 p.m., with a noon hour and midmorning and midafternoon 15-minute recesses. Always on Friday after the afternoon recess we had a recreational treat—a nature walk in the nearby woods, a spelling contest or spelldown, storytelling, a game of charades or, most exciting to me, a contest called "Ciphering".

When teacher announced "Ciphering", each pupil readied his slate and teacher chalked a problem in arithmetic on the front blackboard. As each pupil got an answer, he ran up and slammed his slate face down on teacher's desk in what grew into a single stack of slates.

When all the slates were stacked, teacher turned the stack over and graded them in the order of stacking. Every wrong answer was scored "zero". The first correct answer was scored 100, the next 95, then 90, and so on—down to zero if there were as many as 20 correct ones. Thus it paid to be quick as well as right.

---

[1]Coaxing modest me to write an autobiography, the editor said he would permit me to start with the word "Reluctantly"—and so I did.

[2]This is the August-September issue, so Happy Birthday, Mr. Rowe. (Editor)

[3]Hurd School, on Burton Road in Paris Township, Kent County; the site is now a shopping center in suburban Grand Rapids.

[4]Later upgraded to become Eastern Michigan University.

Each pupil then recorded his score on the back of his slate and cleaned his slate for the next problem. There was time for about 10 problems, then each pupil totalled his 10 scores and teacher would have a token prize for the winner.

Reminiscing, I am struck by the equities involved in this game of ciphering. One might think the 8th-graders had a tremendous advantage over the 1st-graders, but the pupils were assigned seats by grades from front to rear; younger pupils sat nearer the teacher's desk, with a shorter run with their slates. Then there were the Dutch, handicapped by language; but 2 + 2 = 4 in any country—and they were smart at arithmetic. Further, the teacher selected problems with increasing difficulty: the first might be simple addition, the tenth a square root. There was a chance for everyone to achieve a score, and a 1st-grader's 30 was as much of a brag as an 8th-grader's 90.

Numbers fascinated me and the competition was a challenge. I very soon found some ways to improve my score. Since teacher usually began with a problem like writing 12, 73, 49, 34, 96, 62, then drawing a line and saying "Add", while others were copying down the figures I would be mentally adding 12, 85, 134, 168, 264, 326 and at the word "Add" I would just write 326 on my slate and run up to her desk with it. For the more advanced problems, I was helped by the teaching of eight grades in one room and regular use of blackboards. Long division was not taught until the 5th grade, nor square root until the 7th but if one was fascinated by numbers, as I was, he could learn by watching the higher grades recite and work at the blackboard. Also, by the 4th grade we had to know the multiplication tables up to 9×9, but I had figured them up to 20×20 and memorized them. This helped me to score if a problem in long division had a divisor of, say, 19.

The square root problems were my nemesis for a long time; we lower-graders just sat those out. But squares were easy to compute, so I began memorizing those up to 50×50. I was in the 5th grade when, fortuitously, as the last problem of a ciphering, the teacher wrote "2025—square root". I scribbled "45" on my slate and ran up with it. My 100 on that problem was enough to win me the prize, which was an orange. In Michigan in 1907 an orange was a rare exotic; so I took it home, peeled it at suppertime, segmented it, and divided it with my folks.

My purpose in recalling *My Introduction to Ciphering* is to do a bit of preaching or moralizing. Arithmetic can be made interesting—as a competitive game—with equitable rules so that all can score. Learning multiplication tables is universally despised, but will appeal to many if it is a means of winning a game. Assuredly, slates are passé, but scratch pads of uniform size would serve as well. Neither are eight grades often taught in one room today, but abilities run the gamut in any grade and problems can be devised to give all a chance to score. It's an old game: Bell says that Gauss played it [1]. But none of my children, grandchildren nor great-grandchildren have had it in their classrooms. Why?

In conclusion, a few words on the semantics and derivation of "cipher". Long before there was a figure "0", *sifr* was an Arabic word for "nothing", from which derived two English words, "cipher" and "zero". Neither was so used in my grade school: the figure "0" was "naught". We never thought of "ciphering" as related to "nothing" or to "zero". Curious, I looked in a handy unabridged dictionary [2] under "cipher" for its verb definition and read:

> —v.i. 10. to use figures or numerals arithmetically.
>
> —v.t. 11. to calculate numerically; figure.

So, after all these years, "ciphering" is a good meaningful word—for an excellent teaching aid for early grades in mathematics.

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2. *The American College Dictionary*, Random House, New York, 1953.

* * *

CHOICE SELECTIONS FROM THE LIBRARY OF E.P.B. UMBUGIO

R.S. JOHNSON, Montreal, Quebec

The following books from the library of E.P.B. Umbugio were unaccountably left out of previously published lists [1977: 123-125].

1. Extracting Roots from the Primal Jungle
2. Evaluation of π to e Decimal Places
3. Borromean Rings for Today's Fiancée
4. Trisecting an Angle with Straightedge and Compass

Contents	*Page*
Stating the Problem	1
Explaining the Method	2
Conclusion (see footnote)	753
Approval of Censor Board	754
Obituary of Author	755

5. A Cute Triangle and Other Love Stories
6. The Triangular Life Style of Blaise Pascal
7. The Anatomical Origin of Digital Computers

8. Measuring Angles in Equilateral Triangles (A Short Course)
9. Travels in Saudi Arabia, by Leonhard Euler
10. A Treasury of Marginal Notes, by Pierre Fermat
11. A List of Exceptions to Goldbach's Conjecture (Abridged)
12. How to Throw a Baseball on a Bernoulli Lemniscate
13. Adam, Eve, Apples and Me, by Sir Isaac Newton
14. Integral Solutions to $x^{n+3} + y^{n+3} = z^{n+3}$ (Doomsday Press)
15. The Sexagesimal System (Adults Only)
16. The Slide Rule, by George H. (Babe) Ruth
17. A Meeting of Minds, by Trigg and Euclid
18. All About Nothing, by Zero Mostel
19. Circling the Square, by Jack Dempsey and Joe Louis
20. The Superior Intellect of E.P.B. Umbugio, by E.P.B. Umbugio

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LETTER TO THE EDITOR

Dear Editor:

Since I am a rather slow reader, it was only recently that I read "Popular Misconceptions" by Léo Sauvé in the October 1975 EUREKA. On page 70 a proof is given that 1 is the largest positive integer. It states that if the largest positive integer $N \neq 1$, then $N+1 > N$, so N is not the largest. By elimination, then 1 is largest. One weakness in this argument is that application of the argument to 1 produces $1+1>1$, so the process contained in the argument produces an integer greater than 1. One is reminded of the preacher who wrote in the margin of a sermon, "Shout and pound on lectern;argument weak here." Since I have no quarrel with the result, only with its proof, I quote Problem 8.4.10 on page 270 in H. Eves, *An Introduction to the Foundations and Fundamental Concepts of Mathematics*, Revised Edition (Holt, Rinehart & Winston, 1965), where a proof appears with the above weakness removed:

"Consider the following two arguments:

I. *THEOREM. Of all triangles inscribed in a circle, the equilateral is the greatest.*

(1) If ABC is a nonequilateral triangle inscribed in a circle, so that AB $\neq$ AC, say, construct triangle XBC, where X is the intersection of the perpendicular bisector of BC with arc BAC. (2) Then triangle XBC > triangle ABC. (3) Hence, if we have a nonequilateral triangle inscribed in a circle, we can always construct a greater inscribed triangle. (4) Therefore, of all triangles inscribed in a circle, the equilateral is the greatest.

II. *THEOREM. Of all natural numbers,* 1 *is the greatest.*

(1) If m is a natural number other than 1, construct the natural number m^2. (2) Then $m^2 > m$. (3) Hence, if we have a natural number other than 1, we can always construct a greater natural number. (4) Therefore, of all natural numbers, 1 is the greatest.

Now the conclusion in argument I is true, and that in argument II is false. But the two arguments are formally identical. What, then, is wrong?"

If the editor will overcome his natural modesty[1] and not edit it out of this letter, perhaps we should add the following:

III. *THEOREM. Of all mathematics journal editors, L. Sauvé is the greatest.*
(1) Given any mathematics journal editor other than L. Sauvé, consider his predecessor. (2) Since things were always better in the good old days, the predecessor is greater than the incumbent. Etc.[2]

CLAYTON W. DODGE,
University of Maine at Orono.

* * *

DR. RICHARD J. SEMPLE

Dr. Richard J. Semple, Associate Professor of Mathematics at Carleton University, died suddenly last spring. He was 47.

Dick, as he was known to his friends, was born in Toronto on January 21, 1930, and attended public and secondary schools in Toronto. He obtained his B.A. and M.A. degrees from the University of Toronto and in 1951 went to Princeton University to do further graduate studies in mathematics. After obtaining his A.M. degree at Princeton in 1952, Dick was awarded a Charlotte Elizabeth Procter Fellowship for 1952-53 and started on his doctorate. He completed his Ph.D. in 1957 with a thesis titled "Cohomology with unit coefficients in local fields."

In 1954, Dick began his long association with Carleton University. From 1954-55 he was a lecturer in the Department of Mathematics and in 1955 he was promoted to an Assistant Professorship. In 1960 Dick left Carleton University for the University of Waterloo, where he was an Associate Professor till 1963. That year he returned to Carleton University where he held an Associate Professorship till his untimely death.

Dick was a member of both the Canadian Mathematical Congress and the American Mathematical Society . He was also a member of C.O.M.A., and was involved in the founding of EUREKA.

Dick was a teacher who communicated his love of mathematics very effectively to his students. He will be greatly missed by his friends, students and colleagues. He leaves a wife, Ann, and a son, Jimmy.

KENNETH S. WILLIAMS
Carleton University

In memory of Dick Semple, Carleton University has approved the establishment of a memorial fund to be used to award an annual prize for an outstanding student enrolled in an honours mathematics program, and entering third or fourth year at Carleton.

If you wish to contribute to this fund, please send your contribution to the Development Office, Carleton University, Colonel By Drive, Ottawa, Ontario K1S 5B6. Cheques should be made payable to Carleton University (Richard J. Semple Memorial Award). All gifts (by Canadians) are deductible for Income Tax purposes under Department of National Revenue Registration No. 0051912-20-10. An official receipt will be sent.

* * *

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[1] The editor is a modest fellow with a great deal to be modest about. (Editor's wife)
[2] Clearly Theorem III $\Longleftrightarrow$ Theorem II. (Editor)

PROBLEMS--PROBLÈMES

Problem proposals and solutions should be sent to the editor, whose address appears on the front page of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before December 1, 1977, although solutions received after that date will also be considered until the time when a solution is published.

261. *Proposé par Alan Wayne, Pasco-Hernando Community College, New Port Richey, Floride.*

Identifier les chiffres de l'addition décimale que voici:

UN + DEUX + DEUX + DEUX + DEUX = NEUF

262. *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

In *Challenging Problems in Algebra* 2, by Alfred S. Posamentier and Charles T. Salkind, Macmillan, New York, 1970, p. 14, occurs the following problem (originally proposed in a New York City Junior Contest on April 9, 1965):

Find the real values of x *such that* $3^{2x^2-7x+3} = 4^{x^2-x-6}$.

In their solution, the authors "prove" that 3 is the only solution. Show this to be incorrect, and find all solutions of this equation.

263. *Proposed by Sahib Ram Mandan, Indian Institute of Technology, Kharagpur, India.*

Ten friends, identified by the digits 0, 1,...,9, form a lunch club. Each day *four* of them meet and have lunch together. Describe minimal sets of lunches $ijkl$ such that

(a) every two of the friends lunch together an equal number of times;

(b) every three of them lunch together just once;

(c) every four of them lunch together just once.

264*. *Proposed by Gilbert W. Kessler, Canarsie H.S., Brooklyn, N.Y.*

Find a formula that gives the number of digits in the nth Fibonacci number explicitly in terms of n.

265. *Proposed by David Wheeler, Concordia University, Montreal, Quebec.*

A game involves tossing a coin n times. What is the probability that two heads will turn up in succession somewhere in the sequence of throws?

266*. *Proposed by Daniel Rokhsar, Susan Wagner H.S., Staten Island, N.Y.*

Let d_n be the first digit in the decimal representation of $n!$, so that

$$d_0 = 1,\ d_1 = 1,\ d_2 = 2,\ d_3 = 6,\ d_4 = 2, \ldots .$$

Find expressions for d_n and $\Sigma_{i=0}^{n} d_i$.

267. *Proposed by John Veness, Cremorne, N.S.W., Australia.*

Some products, like $56 = 7 \cdot 8$ and $17820 = 36 \cdot 495$, exhibit consecutive digits without repetition. Find more (if possible, all) such products $c = a \cdot b$ which exhibit without repetition four, five,..., ten consecutive digits.

268. *Proposed by Gali Salvatore, Ottawa, Ont.*

Show that in ΔABC, with $a \ge b \ge c$, the sides are in arithmetic progression if and only if

$$2\cot\frac{B}{2} = 3(\tan\frac{C}{2} + \tan\frac{A}{2}).$$

269. *Proposed by Kenneth M. Wilke, Topeka, Kansas.*

Let $<\sqrt{10}>$ denote the fractional part of $\sqrt{10}$. Prove that for any positive integer n there exists an integer I_n such that

$$(<\sqrt{10}>)^n = \sqrt{I_n + 1} - \sqrt{I_n}.$$

270. *Proposed by Dan Sokolowsky, Antioch College, Yellow Springs, Ohio.*

Call a *chord* of a triangle a segment with endpoints on the sides. Show that for every acute-angled triangle there is a unique point P through which pass three *equal* chords each of which is bisected by P.

* * *

WHO WAS TINCA TINCA?

The Gauss Bicentennial issue of EUREKA (April 1977) contained an article under the mysterious name *Tinca Tinca* [1977: 101-102]. I had invited readers to send me their guesses as to the true identity of Tinca Tinca. Only two readers did so, and both were wrong. So who is Tinca Tinca?

Tinca Tinca is the scientific name of a fish called in Latin *tinca*, in English *tench*, in French *tanche*, in German *Schleie*, in Latvian *linis*, in Swahili..., wait! *Linis*? LINIS? Of course! It must be VIKTORS LINIS, University of Ottawa, one of the earliest and continuing supporters of EUREKA (one of its founding members, in fact). Dr. Linis, to whom is due the original idea of having a Gauss Bicentennial issue of EUREKA, had in that same issue another article under his own name, indeed on the very same page as the *Tinca Tinca* article. So how could anyone miss it?

* * *

S O L U T I O N S

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

154. [1976: 110, 159, 197, 225; 1977: 20, 108] *Proposed by Kenneth S. Williams, Carleton University, Ottawa, Ont.*

Let p_n denote the nth prime, so that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, etc. Prove or disprove that the following method finds p_{n+1} given $p_1, p_2, \ldots, p_n$.

In a row list the integers from 1 to $p_n - 1$. Corresponding to each r $(1 \le r \le p_n - 1)$ in this list, say $r = p_1^{a_1} \ldots p_{n-1}^{a_{n-1}}$, put $p_2^{a_1} \ldots p_n^{a_{n-1}}$ in a second row. Let ℓ be the smallest odd integer not appearing in the second row. The claim is that $\ell = p_{n+1}$.

Example. Given $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, $p_6 = 13$.

1	2	3	4	5	6	7	8	9	10	11	12
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
1	3	5	9	7	15	11	27	25	21	13	45

We observe that $\ell = 17 = p_7$.

V. *Comment by the proposer.*

Let x be a real number greater than 2. Then Nelson's conjecture [1977: 108] is equivalent to the assertion that the interval $[x,\ x + 2\sqrt{x} + 1]$ always contains at least one prime. However, the best known result in this direction (obtained by deep methods) is the following:

Let c be a real number greater than 7/12. Then, whenever x is sufficiently large, the interval $[x,\ x + x^c]$ always contains a prime.

This result is due to Huxley [9]. I think it is therefore very unlikely that Nelson's conjecture is known to be true.

VI. *Comment by Leroy F. Meyers, The Ohio State University.*

I agree with the proposer [1977: 20-22] that this is a hard problem, for the reasons stated below. But first I proceed as the proposer did. Let the function f from the set N of positive integers into the set O of odd positive integers be defined so that

$$x = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k} \quad \Longrightarrow \quad f(x) = p_2^{a_1} p_3^{a_2} \ldots p_{k+1}^{a_k}.$$

(Note that neither x nor $f(x)$ is affected by the presence of a "phantom" prime with exponent 0. Hence $f(1) = f(2^0) = 3^0 = 1$.) The proposer shows that f is bijective and that the given problem is equivalent to the following, which he calls Conjecture 2.

Conjecture 2. *For all* $x \in N$, *if* $x \geq p_n$ *then* $f(x) \geq p_{n+1}$.

Now suppose Conjecture 2 holds. Let r (and p_r) be given, and let p_n be the largest prime not exceeding p_r^2, so that $p_{n+1} > p_r^2 \geq p_n$. Then, by the conjecture, we have $p_{r+1}^2 = f(p_r^2) \geq p_{n+1}$. Since equality cannot hold, we now have:

Conjecture 3. *For each positive integer* r *there is a prime* q *such that* $p_r^2 < q < p_{r+1}^2$.

Now this conjecture looks suspiciously like, and is an obvious consequence of, a well-known conjecture:

Conjecture 4. *For each positive integer* m *there is a prime* q *such that* $m^2 < q < (m+1)^2$.

In turn, this conjecture is a consequence of the further

Conjecture 5. *If* $x \geq 117$ *(where* x *is not necessarily an integer), then there is a prime* q *such that* $x < q < x + \sqrt{x}$. (There are exceptions for certain smaller numbers.)

Proof. If m is a positive integer and $m \geq \sqrt{117}$ then, by Conjecture 5, there is a prime q such that

$$m^2 < q < m^2 + m < m^2 + 2m + 1 = (m+1)^2,$$

as needed for Conjecture 4. Direct verification can be used if $m < \sqrt{117}$.

Because of the connection between Conjecture 3 and the well-known Conjectures 4 and 5, I wrote to Hans Riesel (author of [1], cited in EUREKA [1976: 18]) and spoke to Andrzej Schinzel when he visited Ohio State earlier this year. (Schinzel is probably the most prominent recent student of Sierpiński.) Both Riesel and Schinzel replied that, as far as each knows, Conjectures 4 and 5 are still open, and that the present problem, like Conjecture 3, seems to be of the same order of difficulty as the more well-known conjectures.

Thus the comment by Harry L. Nelson [1977: 108] needs confirmation, in the form of a proof or a reference to a published proof.

For further (older) information on Conjectures 4 and 5, as well as on related matters, see [2-8].

Editor's comment.

From a conversation I had with Harry L. Nelson some time ago, I understand he did not mean to imply that his statement in [1977: 108] was proven fact. Rather, it is a conjecture he was led to make on the basis of computer-generated data.

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3. W. Sierpiński, *O stu prostych, ale trudnych zagadnieniach arytmetyki* (One hundred elementary but difficult problems in arithmetic), bound together with *Pogranicza geometrii i arytmetyki* (On the border between geometry and arithmetic), with supplementary remarks by A. Mąkowski; Warsaw (1959). (I cannot give the page references, since I have not seen the book.)

4. W. Sierpiński, *Co wiemy, a czego nie wiemy o liczbach pierwszych* (What we know and what we don't know about prime numbers), Warsaw (1961), 35-37, 74.

5. V. Serpinskij [W. Sierpiński], *Sto prostykh, no odnovremenno i trudnykh voprosov arifmetiki*, bound together with *Na granice geometrii i arifmetiki*; a translation of [3] into Russian, with further supplementary remarks by V. A. Golubev; Moscow (1961), 15-16, 59, 71-72.

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7. W. Sierpiński, *Elementary theory of numbers*, translated from the Polish by A. Hulanicki; Warsaw (1964), 145-147.

8. D. Shanks, "On maximal gaps between successive primes", *Mathematics of Computation* 18 (1964), 646-651. (Riesel referred me to this.)

9. M.N. Huxley, *The Distribution of prime numbers*, Oxford (1962), 119.

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189. [1976: 194; 1977: 74] *Proposed by Kenneth S. Williams, Carleton University.*

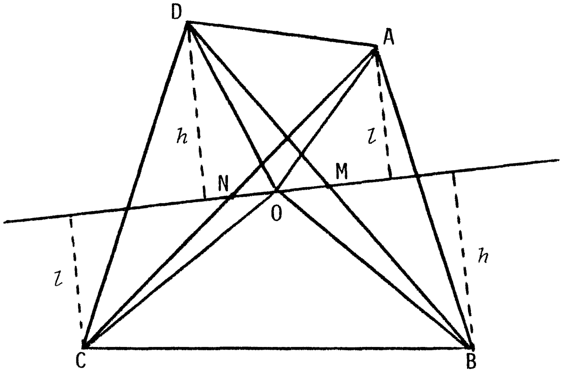
If a quadrilateral circumscribes an ellipse, prove that the line through the midpoints of its diagonals passes through the centre of the ellipse.

Editor's Comment.

In my earlier comment [1977: 75], I showed that this problem was a trivial consequence of a theorem of Léon Anne. Some readers have observed that in order to make the proof of this problem self-contained it would be helpful to have a proof of the theorem of Léon Anne, which seems to be little-known today. I give below the theorem and its proof exactly as it appears in [4], a reference (given in [1977: 76]) which is not easily accessible.

THÉORÈME. Dans tout quadrilatère, le lieu géométrique des points tels que la somme des triangles ayant ce point pour sommet et deux côtés opposés pour base est équivalente à celle des deux autres triangles, est la droite qui joint les milieux des diagonales. (Léon ANNE.)

Soit MN la droite qui joint les points milieux des diagonales; les perpendiculaires abaissées des sommets sur MN sont égales deux à deux; donc les triangles de même base, située sur MN, sont équivalents quand ils ont leur sommet en A et en C; il en est de même pour ceux qui ont le sommet en B et en D.



Or on a

AOB = AMB + AOM + BOM
DOC = DMC - COM - DOM

Mais

AOM = COM, BOM = DOM

donc

AOB + DOC = AMB + DMC (1)

De même

AOD = AMD + DOM - AOM
COB = CMB - BOM + COM

Donc

AOD + COB = AMD + CMB (2)

Or les triangles AMB et AMD des formules (1) et (2) ont même sommet A et des bases égales DM et BM; donc ils sont équivalents; il en est de même de DMC et de BMC.

Donc AOB + DOC = AOD + COB

Remarque. Le lieu qui répond directement à l'énoncé est compris dans le quadrilatère; au dehors, il y a changement de signe. (Journal de Mathématiques élémentaires de M. BOURGET, 1879, p. 29.)

Now I have a treat for those readers who would like to see how such a problem could be tackled today (and for those whose French is not up to snuff). In a practice session with the U.S. team for the International Mathematical Olympiad (which took place on 3-4 July 1977 in Belgrade), Murray S. Klamkin had assigned the proof of the theorem of Léon Anne. Two of the team members, Victor Milenkovic and Paul Weiss, each came up with equivalent elegant vector solutions which Klamkin submitted to me as follows:

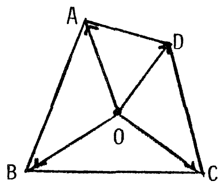
THEOREM (Léon Anne). Let ABCD *be a quadrilateral. Then the line joining the midpoints of the diagonals* AC *and* BD *is the locus of the points* O *such that the sum of the areas of* Δs OAD, OBC *is equal to the sum of the areas of* Δs OAB, OCD.

Proof. Let $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ denote vectors from any permissible point O to the vertices A, B, C, D (see adjoining figure). Then, by hypothesis,

$$\vec{a}\times\vec{b}+\vec{c}\times\vec{d}=\vec{b}\times\vec{c}+\vec{d}\times\vec{a},$$

whence

$$\frac{\vec{a}+\vec{c}}{2}\times\frac{\vec{b}+\vec{d}}{2}=\vec{0}.$$



Thus O must be on the line joining the midpoints of AC and BD. However, if these midpoints coincide (if ABCD is a parallelogram), then O can be any point within or on ABCD. Also, O can be outside if we use signed areas.

* * *

195. [1976: 220; 1977: 83] *Proposed by John Karam, Coop. Student, University of Waterloo.*

The following two problems are given together since they both appear to be related to the celebrated Birthday Problem, which says that if 23 persons are in a room the odds are better than 50% that two persons in the room have the same birthday.

(a) How many persons would have to be in a room for the odds to be better than 50% that three persons in the room have the same birthday?

(b) In the Quebec-based lottery *Loto Perfecta*, each entrant picks six distinct numbers from 1 to 36. If, at the draw, his six numbers come out in some order (*dans le désordre*) he wins a sum of money; if his numbers come out in order (*dans l'ordre*), he wins a larger sum of money. How many entries would there have to be for the odds to be better than 50% that two persons have picked the same numbers (i) *dans le désordre*, (ii) *dans l'ordre*?

Editor's comment.

R. Robinson Rowe wrote to say that he had discovered an error in the derivation of his formula (1) given in [1977: 84], and that therefore his answer $n=83$ for part (a) was not reliable.

Rowe's discovery had been anticipated by Donald E. Knuth, Stanford University, who wrote to say that the correct formula for part (a) is

$$q_n=\frac{n!}{N^n}\sum_{k=0}^{n}2^{-k}\binom{N}{n-k}\binom{n-k}{k}$$

and that the smallest n for which $q_n<\frac{1}{2}$ is $n=88$. (If desired, the upper limit in the sum can be replaced by $[\frac{1}{2}n]$ since all succeeding terms vanish.) Knuth said that this correct formula is due to Pinzka [4] and that it is quoted in [3]. The result

$n = 88$ is also mentioned in [2]. Pinzka also reported in [4] that the corresponding result for 4 and 5 coinciding birthdays is $n = 187$ and $n = 314$.

M.S. Klamkin wrote to say that Rowe's answers to part (b), $n = 1644$ and $n = 44093$, appear to be correct. He added that the Klamkin-Newman formula is also correct, even though it gives $n = 1750$ and $n = 46936$. I quote from his letter: "The reason they do not agree is that the two problems are different even though similar. In the Rowe problem, we have a fixed-size k and ask for the probability of a coincidence. In the Klamkin-Newman model, we sample one at a time without replacement until there has been a coincidence. See Feller [1]."

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3. ________, *The Art of Computer Programming*, Vol. 3 (Sorting and Searching), Addison-Wesley (1973), 542 (Exercise 4), 687.

4. C.F. Pinzka, "Remarks on some problems in the American Mathematical Monthly," *American Mathematical Monthly*, 67 (1960), 830.

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205. [1977: 10, 142] *Late solution: R.S. JOHNSON, Montreal, Quebec.*

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210. [1977: 10, 160] *Proposed by Murray S. Klamkin, University of Alberta.*

P, Q, R denote points on the sides BC, CA and AB, respectively, of a given triangle ABC. Determine all triangles ABC such that if

$$\frac{BP}{BC} = \frac{CQ}{CA} = \frac{AR}{AB} = k \qquad (\neq 0,\ 1/2,\ 1),$$

then PQR (in some order) is similar to ABC.

Editor's comment.

Solution I of this extremely interesting problem, a composite of those submitted by W.J. Blundon and Dan Sokolowsky, brought out hitherto unknown facts about the geometry of the triangle. Sokolowsky's solution contained more interesting information which should have been included but which instead, by an oversight, ended up on the cutting-room floor. Here it is:

The three triangles $P_iQ_iR_i$ were determined by

$$k_1 = \frac{a^2 - b^2}{2a^2 - b^2 - c^2}, \qquad k_2 = \frac{c^2 - a^2}{2c^2 - a^2 - b^2}, \qquad k_3 = \frac{b^2 - c^2}{2b^2 - c^2 - a^2}.$$

When $a > b > c$, k_1 and k_2 always exist, but k_3 does not exist if $2b^2 = c^2 + a^2$, which implies that $k_1 = 1/3$ and $k_2 = 2/3$. Thus, when $2b^2 = c^2 + a^2$ there are only two, not three, triangles $P_iQ_iR_i$ (indirectly) similar to ABC. Furthermore, since the ratio (see [1977: 160])

$$\frac{\Delta\,\mathrm{PQR}}{\Delta\,\mathrm{ABC}} = 1 - 3k + 3k^2$$

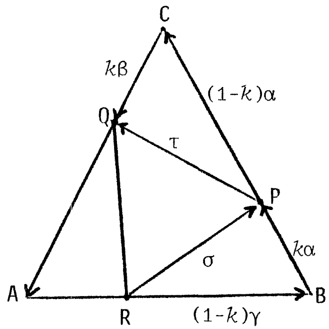
equals 1/3 for both $k = k_1 = 1/3$ and $k = k_2 = 2/3$, it follows that triangles $P_1Q_1R_1$ and $P_2Q_2R_2$ are equal in area and hence (since they are similar) they are *congruent*.

I hope some readers will try to discover other unsuspected special properties of scalene triangles ABC for which $2b^2 = c^2 + a^2$.

I give below a new solution for the direct similarity case. Although, like solution II [1977: 163], it uses complex numbers, the new solution is more self-contained in that it does not assume knowledge of the determinantal condition used in solution II, and it also brings right out in the open the rôle played by the restrictions $k \neq 0, \frac{1}{2}, 1$.

III. *Solution by Sam LaMacchia, Yellow Springs, Ohio; and Dan Sokolowsky, Antioch College (jointly).*

Consider the vertices of ΔABC as points in the complex plane and let $\alpha = C - B$, $\beta = A - C$, $\gamma = B - A$; $\sigma = P - R$, $\tau = Q - P$ (see figure).



Then

$$\alpha^2 - \beta\gamma = \beta^2 - \gamma\alpha = \gamma^2 - \alpha\beta,$$

and we have the equivalent conditions

$$\alpha^2 = \beta\gamma \iff \beta^2 = \gamma\alpha \iff \gamma^2 = \alpha\beta, \qquad (1)$$

each of which implies each of

$$\alpha^3\beta = \gamma^3\beta, \quad \beta^3\gamma = \alpha^3\gamma, \quad \gamma^3\alpha = \beta^3\alpha,$$

and each of

$$|\alpha| = |\gamma|, \quad |\beta| = |\alpha|, \quad |\gamma| = |\beta|.$$

Thus any one of equalities (1) implies that ΔABC is equilateral.

Now suppose ΔPQR is directly similar to ΔABC; then

$$A \longleftrightarrow P \implies \frac{\beta}{\gamma} = \frac{\sigma}{\tau} = \frac{(1-k)\gamma + k\alpha}{(1-k)\alpha + k\beta} \implies \alpha^2 = \beta\gamma \quad \text{if } k \neq \tfrac{1}{2},$$

and similarly

$$B \longleftrightarrow P \implies \frac{\gamma}{\alpha} = \frac{\sigma}{\tau} \implies \alpha^2 = \beta\gamma \quad \text{if } k \neq 0,$$

$$C \longleftrightarrow P \implies \frac{\alpha}{\beta} = \frac{\sigma}{\tau} \implies \alpha^2 = \beta\gamma \quad \text{if } k \neq 1.$$

Hence, if ΔPQR is directly similar to ΔABC and $k \neq 0, \frac{1}{2}, 1$ then ΔABC is equilateral. Conversely, if ΔABC is equilateral, it is clear that ΔPQR is directly similar to ΔABC for all k.

* * *

215. [1977: 42, 167] *Late solution: CATHERINE A. CALLAGHAN, The Ohio State University.*

* * *

216. [1977: 42, 169] *Proposed by L.F. Meyers, The Ohio State University.*

For which positive integers n is it true that

$$\sum_{k=1}^{(n-1)^2} [\sqrt[3]{kn}] = \frac{(n-1)(3n^2-7n+6)}{4}\ ?$$

The brackets, as usual, denote the greatest integer function.

II. *Comments by David Stone, Georgia Southern College, Statesboro, Georgia.*

1. Adopting the notation used in Blundon's solution (with λ_n instead of λ), we have

$$S_n = \frac{(n-1)(3n^2-7n+6)}{4} + \frac{1}{2}\lambda_n,$$

where λ_n counts the number of terms in the sequence $1^3, 2^3, \ldots, (n-2)^3$ which are divisible by n. As Blundon shows, $\lambda_n = 0$ if and only if n is squarefree.

But in fact we can calculate λ_n for any n by using its prime-power decomposition. Let

$$n = \prod_{i=1}^{s} p_i^{e_i}$$

and let $\{x\}$ denote the smallest integer $\geq x$. Then $n|t^3$ if and only if t is a multiple of

$$a_n = \prod_{i=1}^{s} p_i^{\{e_i/3\}}.$$

Now $a_n|n$ and there are $\frac{n}{a_n} - 1$ such t among $1, 2, \ldots, n-2$. Thus

$$\lambda_n = \prod_{i=1}^{s} p_i^{e_i - \{e_i/3\}} - 1,$$

and so S_n can be calculated even when $\lambda_n \neq 0$.

The numbers λ_n have some interesting properties:

(i) If m and n are relatively prime, then

$$\begin{aligned}\lambda_{mn} &= (\lambda_m + 1)(\lambda_n + 1) - 1 \\ &= \lambda_m + \lambda_n + \lambda_m\lambda_n.\end{aligned}$$

(ii) Since $\lambda_4 = 1$, we have $\lambda_n = 1$ if and only if $n = 4m$ where m is squarefree and odd.

(iii) If m is squarefree, then $\lambda_{m^2} = m - 1$. In a limited sense, λ_n is determined by the "square part" of n: if $n = m^2r$, with m and r relatively prime and squarefree, then $\lambda_n = \lambda_{m^2} = m - 1$.

(iv) $\frac{1}{2}\lambda_n = 1$, that is, $\lambda_n = 2$, if and only if $n = 9m$, with m squarefree and relatively prime to 3.

2. *A related problem.* When I first wrote a computer program to compute some examples for this problem, I did not realize that, for example, $[\sqrt[3]{2\cdot 4}]$ would be calculated as $[1.999...9] = 1$. So the interesting results I obtained were actually about the related problem:

For which n does

$$\sum_{k=1}^{(n-1)^2} \langle\sqrt[3]{kn}\rangle = \frac{(n-1)(3n^2-7n+6)}{4},$$

where $\langle x\rangle$ denotes the greatest integer strictly less than x?

Since $\langle x\rangle = [x] - 1$ or $[x]$ according as x is or is not an integer, the sum considered here is less than S_n, losing 1 for each k such that kn is a cube. But since $kn = t^3$ if and only if $n|t^3$, there are λ_n such k among $1, 2, \ldots, (n-1)^2$. So we have

$$\sum_{k=1}^{(n-1)^2} \langle\sqrt[3]{kn}\rangle = S_n - \lambda_n = \frac{(n-1)(3n^2-7n+6)}{4} - \frac{1}{2}\lambda_n .$$

Thus the answer to my computer-generated problem is the same as that to the original question: n squarefree!

* * *

221. [1977: 65] *Proposed by Clayton W. Dodge, University of Maine at Orono.*

Solve this cryptarithm: $\text{CW}^2 = \text{TRI.GG}$, subject to the condition that, since he is unique (the only 1 of his kind), the solution should not contain the digit 1.

I. *Solution by Jack LeSage, Eastview Secondary School, Barrie, Ont.*

Since CW is a natural (number, that is), we must have G = 0; and avoiding 1 and repeated digits in CW and TRI separately leaves only the possibilities

$$CW = 23,\ 24,\ 25,\ 27,\ 28.$$

Of these, only $24^2 = 576$ provides five distinct digits for CW and TRI. The unique answer is $24^2 = 576.00$.

II. *Solution by Charles W. Trigg (himself, the* 1 *and only!), San Diego, California.*

Having gotten to the point where GG makes no cents, and $22 < CW < 29$, and $W \neq 5$ or 6, clearly the unique (i.e. the only 1) result sought is $24^2 = 576.00$.

Note that $C + W + T = 11$ (lucky, what?)

and $C + W - T = 1$ (unique?)

and $(C)(W) - T = 3 = (T + R + I + G + G)/(C + W)$.

What's in a name?

Also solved by STEVEN R. CONRAD, Benjamin N. Cardozo H.S., Bayside, N.Y.; ROBERT S. JOHNSON, Montreal, Que.; JUDY LYNCH, 6th grade, Bulloch Academy, Statesboro, Georgia; HARRY L. NELSON, Livermore, California; R. ROBINSON ROWE, Naubinway, Michigan; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Some solvers sent in only an answer (pardonable in such an easy problem), and some interpreted CW^2 as $(C)(W^2)$ which is not usual in cryptarithms.

This problem does nothing for mathematics, and it does not increase the already great reputation of its subject. It only serves to enhance the pages of EUREKA with the name of one of America's (Americans should realize this includes Canada) greatest problemists.

In other words, we're just name-dropping.

* * *

222. [1977: 65] *Proposed by Bruce McColl, St. Lawrence College, Kingston, Ontario.*

Prove that

$$\tan\frac{\pi}{11}\tan\frac{2\pi}{11}\tan\frac{3\pi}{11}\tan\frac{4\pi}{11}\tan\frac{5\pi}{11} = \sqrt{11}.$$

I. *Solution by W.J. Blundon, Memorial University of Newfoundland.*

The stated equation is equivalent to

$$\tan\frac{\pi}{11}\tan\frac{2\pi}{11}\tan\frac{3\pi}{11}\ldots\tan\frac{10\pi}{11} = -11. \qquad (1)$$

Let $x = \tan\frac{k\pi}{11}$, where k is any of the integers $1, 2, \ldots, 10$. Then

$$\cos\frac{2k\pi}{11} + i\ \sin\frac{2k\pi}{11} = \frac{1 - x^2 + 2xi}{1 + x^2} = \frac{(1 + xi)^2}{1 + x^2},$$

and de Moivre's Theorem gives

$$(1 + xi)^{22} = (1 + x^2)^{11}.$$

Equating real parts, we have

$$x^{20} - 110x^{18} + \ldots + 121 = 0.$$

Hence the square of the required product (1) is 121. Taking only the negative square root, since exactly five of the tangents in (1) are negative, we have the required result.

II. *Solution by Leon Bankoff, Los Angeles, California.*

We will use the well-known formula (see [1975: 31])

$$\frac{\tan n\theta}{\tan\theta} = \frac{\binom{n}{1} - \binom{n}{3}\tan^2\theta + \binom{n}{5}\tan^4\theta - \ldots}{1 - \binom{n}{2}\tan^2\theta + \binom{n}{4}\tan^4\theta - \ldots}.$$

Substitution of $n = 11$, $t = \tan^2\theta$, and $\theta = \frac{k\pi}{11}$ where k is any of the integers 1, 2,..., 5, makes the left member vanish and yields

$$t^5 - 55t^4 + 330t^3 - 462t^2 + 165t - 11 = 0.$$

The product of the roots is

$$\prod_{k=1}^{5} \tan^2\frac{k\pi}{11} = 11, \qquad \text{whence} \qquad \prod_{k=1}^{5} \tan\frac{k\pi}{11} = \sqrt{11}.$$

Also solved by LEON BANKOFF, Los Angeles, California (three additional solutions); STEVEN R. CONRAD, Benjamin N. Cardozo H.S., Bayside, N.Y.; J. WALTER LYNCH, Georgia Southern College, Statesboro, Georgia; F.G.B. MASKELL, Algonquin College, Ottawa; LEROY F. MEYERS, The Ohio State University; BOB PRIELIPP, The University of Wisconsin-Oshkosh; R. ROBINSON ROWE, Naubinway, Michigan; KENNETH M. WILKE, Topeka, Kansas; and the proposer.

Editor's comment.

Lynch and Meyers derived the more general formulas

$$\prod_{k=1}^{m} \sin\frac{k\pi}{2m+1} = \frac{\sqrt{2m+1}}{2^m}, \qquad \prod_{k=1}^{m} \cos\frac{k\pi}{2m+1} = \frac{1}{2^m}, \qquad \prod_{k=1}^{m} \tan\frac{k\pi}{2m+1} = \sqrt{2m+1}.$$

Wilke observed that derivations of these formulas can be found in Nagell [2]; Rowe mentioned that they go back at least to Gauss [1]; and Conrad, Lynch and Prielipp reported that their derivations can also be found in [3].

REFERENCES

1. Carl Friedrich Gauss, *Disquisitiones Arithmeticae*, Art. 337.

2. Trygve Nagell, *Introduction to Number Theory*, Chelsea, New York (1964), 173-174.

3. D.O. Shklarsky, N.N. Chentzov, I.M. Yaglom, *The USSR Olympiad Problem Book*, W.H. Freeman and Co., San Francisco (1962), 53, 334-339 (Problem 232).

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223. [1977: 65] *Proposed by Steven R. Conrad, Benjamin N. Cardozo H.S., Bayside, N.Y.*

Without using any table which lists Pythagorean triples, find the smallest integer which can represent the area of two noncongruent primitive Pythagorean triangles.

I. *Solution by R. Robinson Rowe, Naubinway, Michigan.*

The area of a Pythagorean triangle with legs a, b and hypotenuse c is $A = \frac{1}{2}ab$. All legs can be generated by $a, b = 2mn, m^2 - n^2$, so that

$$A = mn(m-n)(m+n).$$

For primitive triangles, m and n must be relatively prime and of opposite parity. The small range that needs to be investigated is shown in the following table.

m	n	$m-n$	$m+n$	A		
2	1	1	3	6		
3	2	1	5	30		
4	1	3	5	60		
4	3	1	7	84		
5	2	3	7	210	a,b,c = 20,21,29	These two triangles are the required set.
5	4	1	9	180		
6	1	5	7	210	a,b,c = 12,35,37	
6	5	1	11	330		
7	2	5	9	630		
7	4	3	11	924		
7	6	1	13	546		
8	1	7	9	504		

Obviously now, as m increases, the least areas A occur for $n = 1$ or $m - n = 1$, and all will exceed 210.

The next such set, with $A = 2730$, is

$$a, b, c = 60, 91, 109 \quad \text{and} \quad 28, 195, 197.$$

II. *Comment by Charles W. Trigg, San Diego, California.*

The result in this problem is given in Sierpiński [2]. He further shows algebraically that right-angled triangles with the same area and same hypotenuse must be congruent. (Geometrically, the proof is trivial.) He also states that the smallest area common to three primitive Pythagorean triangles is $A = 13{,}123{,}110$; these triangles are

$$(4485,\ 5852,\ 7373), \qquad (1380,\ 19019,\ 19069), \qquad (3059,\ 8580,\ 9109).$$

Then he proceeds to prove the following theorem of Fermat: *For each natural number n, there exist n Pythagorean triangles with different hypotenuses and the same area.*

III. *Comment by H.L. Ridge, University of Toronto.*

A related problem would be to find out if a given positive integer can represent the area of at least one primitive Pythagorean triangle. For a discussion of this problem, see Ore [1].

Also solved by CLAYTON W. DODGE, University of Maine at Orono; ROBERT S. JOHNSON, Montreal, Quebec; H.L. RIDGE, University of Toronto (solution as well); CHARLES W. TRIGG, San Diego, California (solution as well); KENNETH M. WILKE, Topeka, Kansas; and the proposer. One incorrect solution was received.

REFERENCES

1. Oystein Ore, *Invitation to Number Theory*, Random House and The L.W. Singer Co. (1967), 57-59.

2. W. Sierpiński, *Pythagorean Triangles*, Yeshiva University, New York (1962), 9, 36.

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224. [1977: 65] (Corrected) *Proposed by M.S. Klamkin, University of Alberta.*

Let P be an interior point of a given n-dimensional simplex of vertices $A_1, A_2, \ldots, A_{n+1}$. Let P_i ($i = 1,2,\ldots,n+1$) denote points on A_iP such that $A_iP_i/A_iP = 1/n_i$. Finally, let V_i denote the volume of the simplex cut off from the given simplex by a hyperplane through P_i parallel to the face of the given simplex opposite A_i. Determine the minimum value of ΣV_i and the location of the corresponding point P.

Solution by the proposer.

Let r_i denote the distances from the point P to the corresponding faces of content F_i of the given simplex. Then if h_i denotes the altitude of the given simplex from A_i and V the volume of the given simplex,

$$\frac{V_i}{V} = \left(\frac{h_i - r_i}{n_i h_i}\right)^n = n_i^{-n}\left(1 - \frac{r_iF_i}{nV}\right)^n .$$

By Hölder's inequality,

$$\left\{\sum n_i^{-n}\left(1-\frac{r_iF_i}{nV}\right)^n\right\}^{1/n}\cdot\left\{\sum n_i^{n/(n-1)}\right\}^{(n-1)/n}\geq\sum\left(1-\frac{r_iF_i}{nV}\right)=n.$$

Consequently,

$$\sum V_i\geq n^n\left\{\sum n_i^{n/(n-1)}\right\}^{1-n}\cdot V$$

with equality if and only if

$$\frac{r_iF_i}{nV}=1-\lambda n_i^{n/(n-1)}$$

where

$$\lambda=\frac{n}{\sum n_i^{n/(n-1)}}.$$

If n_i = constant, then equality holds if and only if P is the centroid of the given simplex.

Note. This problem extends a result of S.I. Zetel', corresponding to the special case $n=2$, $n_i=2$ (see [1]).

Editor's comment.

The original proposal had

$$A_iP_i/P_iP=1/n_i \quad\text{instead of}\quad A_iP_i/A_iP=1/n_i.$$

But it is clear from the proposer's solution that the latter was intended.

REFERENCE

1. O. Bottema, R.Ž. Djordjević, R.R. Janić, D.S. Mitrinović, P.M. Vasić, *Geometric Inequalities*, Wolters-Noordhoff, Groningen, Netherlands (1968), 128 (14.30).

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225. [1977: 65] *Proposed by Dan Sokolowsky, Antioch College, Yellow Springs, Ohio.*

C is a point on the diameter AB of a circle. A chord through C, perpendicular to AB, meets the circle at D. Two chords through B meet CD at T_1, T_2 and arc AD at U_1, U_2 respectively. It is known from Problem 220 that there are circles C_1, C_2 tangent to CD at T_1, T_2 and to arc AD at U_1, U_2 respectively. Prove that the radical axis of C_1 and C_2 passes through B.

I. *Solution by Leon Bankoff, Los Angeles, California.*

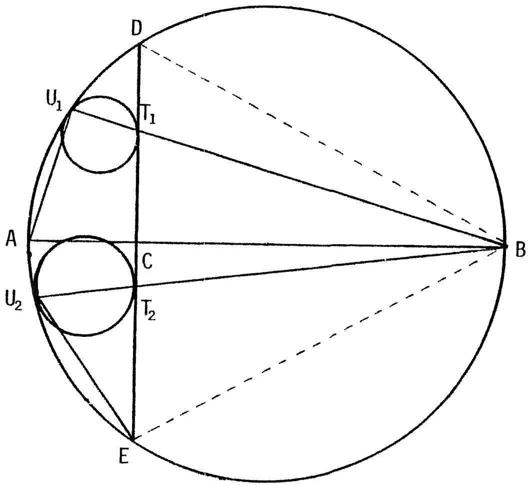
Since the circles C_1 and C_2 are self-inverse with respect to the circle of inversion centered at B with radius of inversion equal to BD, all tangents from B to the circles C_1 and C_2 are equal. Hence B lies on the radical axis of C_1 and C_2.

II. *Solution by Sahib Ram Mandan, Indian Institute of Technology, Kharagpur, India.*

Since $\angle AU_1T_1 = \angle ACT_1 = 90°$, ACT_1U_1 is a cyclic quadrilateral (see figure) and

$$BT_1 \cdot BU_1 = BC \cdot BA = BD^2,$$

and similarly $BT_2 \cdot BU_2 = BD^2$. Thus the powers of B with respect to circles C_1 and C_2 are equal, and so B lies on the radical axis of these two circles.



III. *Solution by Clayton W. Dodge, University of Maine at Orono.*

Since angles BU_2E and BET_2 are equal, as they intercept equal arcs BE and BD (see figure), it follows that $\Delta BU_2E \sim \Delta BET_2$ and so

$$\frac{BU_2}{BE} = \frac{BE}{BT_2}, \quad \text{that is,} \quad BT_2 \cdot BU_2 = BE^2,$$

and similarly $BT_1 \cdot BU_1 = BE^2$. The desired conclusion now follows as in solution II.

Also solved by the proposer.

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226. [1977: 66] *Proposed by David L. Silverman, West Los Angeles, California.*

The positive integers are divided into two disjoint sets A and B. A positive integer is an A-number if and only if it is the sum of two different A-numbers or of two different B-numbers. Find A.

I. *Solution by R. Robinson Rowe, Naubinway, Michigan; and the proposer (independently).*

Evidently 1 and 2 are in B, whence 3 is in A. It follows that 4 is in B, and 5 and 6 in A. Next, 7 is in B, 8 and 9 in A and 10 in B. Now we have 11, 12, 13 all in A and this, together with the fact that 3 is in A, enables us to show by a simple induction that all succeeding integers are in A.

Therefore A is the set of all positive integers except 1, 2, 4, 7, 10.

II. *Comment by Clayton W. Dodge, University of Maine at Orono.*

If the procedure is applied to the set Z of all integers, the solution is not unique. We could have, for example,

$$A = \text{all even integers}, \quad B = \text{all odd integers} \tag{1}$$

or

$$A = Z, \quad B = \emptyset. \tag{2}$$

Also solved by CLAYTON W. DODGE, University of Maine at Orono (solution as well); SAHIB RAM MANDAN, Indian Institute of Technology, Kharagpur, India; H.L. RIDGE, University of Toronto; and KENNETH M. WILKE, Topeka, Kansas (two solutions). An answer was supplied by HARRY L. NELSON, Livermore, California.

Editor's comment.

Some solvers ignored or did not see the word *positive* in the first line of the proposal, and proved that (1) was the unique answer. It apparently did not occur to them that in (2) the sets A and B *are* disjoint.

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THE 1977 INTERNATIONAL MATHEMATICAL OLYMPIAD

The following is a news release from The Mathematical Association of America.

U.S. MATHLETES WINNERS IN
INTERNATIONAL MATHEMATICAL OLYMPIAD

A team of eight U.S. high school students won the first prize in the 19th International Mathematical Olympiad (I.M.O.) held in Belgrade, Yugoslavia on July 3-4. Two U.S. team members, Randall Dougherty of Fairfax, Virginia, and Michael Larsen of Lexington, Massachusetts, won individual first prizes with perfect scores on the two-day I.M.O. examination. Three U.S. teammates, Peter Shor of Mill Valley, California, Mark Kleiman of New York City,[1] and Victor Milenkovic of Glencoe, Illinois, won individual second prizes. Team member James Propp of Great Neck, New York, won a third prize.

The I.M.O. brought together teams of high school students from 21 nations[2] for a spirited competition based on an examination requiring both broad knowledge and great mathematical ingenuity. The U.S. students topped teams from the U.S.S.R. (second place), Great Britain and Hungary (tied for third) and The Netherlands (fifth place).

The United States has competed in the I.M.O. only since 1974. This year's U.S. team is the first to win top honors, although the Americans have never finished below third place.

The U.S. team in the I.M.O. is chosen on the basis of performance in the

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[1]Mark Kleiman is a EUREKA subscriber and contributor. (Editor)

[2]There was no team from Canada. Why not? (Editor)

U.S.A. Mathematical Olympiad which was held this year on May 3. The team was honored in Washington on June 7 at the Sixth U.S.A. Mathematical Olympiad Awards Ceremony and prepared for the I.M.O. at a training session held at the U.S. Military Academy, June 8-29.

The Mathematical Olympiad activities are sponsored by four national societies in the mathematical sciences. Financial support is provided by IBM, the Army Research Office, and the Office of Naval Research.

The Members of the U.S. team were:
Randall Dougherty, Fairfax, Virginia
Ronald Kaminsky, Albany, New York
Mark Kleiman, Staten Island, New York
Michael Larsen, Lexington, Massachusetts
Victor Milenkovic, Glencoe, Illinois
James Propp, Great Neck, New York
Peter Shor, Mill Valley, California
Paul Weiss, Brooklyn, New York

A 32-page pamphlet on the 1977 Olympiads has been compiled by Samuel L. Greitzer. It contains, along with complete solutions, all problems proposed at the 6th U.S.A. Mathematical Olympiad and at the 19th International Mathematical Olympiad.

Further information about the Olympiads can be obtained from

Samuel L. Greitzer,
Mathematics Department,
Rutgers University,
New Brunswick, N.J. 08903.

For information about the Annual High School Mathematics Examination, and for copies of the abovementioned pamphlet (at 50¢ per copy), write to

Dr. Walter E. Mientka, Executive Director,
MAA Committee on High School Contests,
917 Oldfather Hall,
University of Nebraska,
Lincoln, Nebraska 68588.

To whet your appetite, you will find below the last problem (No. 6) proposed at the 1977 International Olympiad, together with a solution by U.S. team member Peter Shor which Murray S. Klamkin, who sent it to the editor, thinks is superior to the "official" solution. Get a copy of the Olympiad pamphlet, then compare and see if you agree with Professor Klamkin.

6. *Let $f(n)$ be a function defined on the set of all positive integers and taking on all its values in the same set. Prove that if*

$$f(n+1) > f(f(n))$$

for each positive integer n, then $f(n) = n$ for each n.

Solution by Peter Shor (U.S.A. Mathematical Olympiad Team, 1977*).*

Define a sequence g_n by

$$g_0 = 2; \qquad g_{n+1} = f(g_n - 1), \quad n = 0, 1, 2, \ldots .$$

This sequence is defined for g_{n+1} unless $g_n = 1$. From the hypothesis,

$$f(2) = f(g_0) > f(g_1) > f(g_2) > \ldots > f(g_n).$$

Since $f(2)$ is finite, the g_n sequence must terminate. This implies that for some g_n (the last one), $g_n = 1$ and correspondingly $f(g_{n-1} - 1) = 1$. Hence there exists a positive integer x such that $f(x) = 1$. If $x > 1$, then

$$1 = f(x) > f\big(f(x-1)\big),$$

which is impossible since f takes on only positive integral values, and so $x = 1$.

Define a new function

$$F(n) \equiv f(n+1) - 1, \qquad n = 1, 2, 3, \ldots, \tag{$*$}$$

so that $F(n)$ is also a positive integer. It then follows by substitution in

$$f(n+2) - 1 > f\big(f(n+1)\big) - 1$$

that

$$F(n+1) > F\big(F(n)\big), \qquad n = 1, 2, 3, \ldots .$$

Thus we have exactly the same conditions on $F(n)$ as were given for $f(n)$.

We complete the proof by induction. Assume that $f(n) = n$ for $n = 1, 2, \ldots, k$. We have already proven the case $n = 1$. It then follows that $F(n) = n$ for $n = 1, 2, \ldots, k$. Then, from $(*)$, $f(n) = n$ for $n = k+1$. Finally, by induction $f(n) = n$ for all positive integral n.

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LETTER TO THE EDITOR

Dear Editor:

I have recently subscribed to EUREKA, purchased Volumes 1 and 2, and spent a very enjoyable time thumbing through all the past issues. I look forward to future issues and wish the EUREKA adventure huge success.

Thank you for helping to make Mathematics more entertaining.

RICHARD A. GIBBS,
Fort Lewis College,
Durango, Colorado.

Editor's comment. The first edition of the bound combined EUREKA Vol. 1-2 (1975-1976) is rapidly being snapped up. Get your copy now. See the front page of this issue for details.

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