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GENERAL INFORMATION

Crux Mathematicorum is a problem-solving journal at the senior secondary and university undergraduate levels for those who practise or teach mathematics. Its purpose is primarily educational, but it serves also those who read it for professional, cultural or recreational reasons.

Problem proposals, solutions and short notes intended for publication should be sent to the Editor:

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THE OLYMPIAD CORNER
No. 106
R.E. WOODROW

All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

We begin this month with the *Canadian Mathematics Olympiad* for 1989, which we reproduce with the permission of the Canadian Mathematical Society. My thanks to Andy Liu of The University of Alberta, Edmonton, for forwarding copies of the exam. We will discuss official solutions in a later issue, but we welcome any particularly "nice" solutions or generalizations that you may find.

CANADIAN MATHEMATICS OLYMPIAD
April 5, 1989
Time: 3 hours

1. The integers $1,2,\ldots,n$ are placed in order so that each value is either strictly bigger than all the preceding values or is strictly smaller than all preceding values. In how many ways can this be done?

2. Let ABC be a right–angled triangle of area 1. Let A', B', and C' be the points obtained by reflecting A, B, C, respectively, in their opposite sides. Find the area of $\Delta A'B'C'$.

3. Define $\{a_n\}_{n=1}^{\infty}$ as follows: $a_1 = 1989^{1989}$, and a_n for $n > 1$ is the sum of the digits of a_{n-1}. What is the value of a_5?

4. There are 5 monkeys and 5 ladders and at the top of each ladder there is a banana. A number of ropes connect the ladders; each rope connects two ladders. No two ropes are attached to the same rung of the same ladder. Each monkey starts at the bottom of a different ladder. The monkeys climb up the ladders but each time they encounter a rope they climb along it to the other end of the rope and then continue climbing upwards. Show that, no matter how many ropes there are, each monkey gets a banana.

5. Given the numbers $1,2,2^2,\ldots,2^{n-1}$. For a specific permutation $\sigma = x_1,x_2,\ldots,x_n$ of these numbers we define

$$S_1(\sigma) = x_1, \quad S_2(\sigma) = x_1 + x_2, \quad S_3(\sigma) = x_1 + x_2 + x_3, \ \dots$$

and $Q(\sigma) = S_1(\sigma)S_2(\sigma)\dots S_n(\sigma)$. Evaluate $\sum 1/Q(\sigma)$ where the sum is taken over all possible permutations.

* * *

The next set of problems we pose are from the eighteenth annual *United States of America Mathematical Olympiad*, written April 25, 1989. These problems are copyrighted by the Committee on the American Mathematics Competitions of the Mathematical Association of America and may not be reproduced without permission. Solutions, and additional copies of the problems, may be obtained for a nominal fee from Professor Walter E. Mientka, C.A.M.C. Executive Director, 917 Oldfather Hall, University of Nebraska, Lincoln, NE, U.S.A., 68588–0322. As always we welcome your original "nice" solutions and generalizations.

18th U.S.A. MATHEMATICAL OLYMPIAD – 1989

Time: 3 1/2 hours

1. For each positive integer n, let

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

$$T_n = S_1 + S_2 + S_3 + \cdots + S_n,$$

$$U_n = \frac{T_1}{2} + \frac{T_2}{3} + \frac{T_3}{4} + \cdots + \frac{T_n}{n+1}.$$

Find, with proof, integers $0 < a,b,c,d < 1000000$ such that $T_{1988} = aS_{1989} - b$ and $U_{1988} = cS_{1989} - d$.

2. The 20 members of a local tennis club have scheduled exactly 14 two–person games among themselves, with each member playing in at least one game. Prove that within this schedule there must be a set of 6 games with 12 distinct players.

3. Let

$$P(z) = z^n + c_1z^{n-1} + c_2z^{n-2} + \cdots + c_n$$

be a polynomial in the complex variable z, with real coefficients c_k. Suppose that $|P(i)| < 1$. Prove that there exist real numbers a and b such that

$$P(a + bi) = 0$$

and

$$(a^2 + b^2 + 1)^2 < 4b^2 + 1.$$

4. Let ABC be an acute-angled triangle whose side lengths satisfy the inequalities $AB < AC < BC$. If point I is the center of the inscribed circle of triangle ABC and point O is the center of the circumscribed circle, prove that the line IO intersects line segments AB and BC.

5. Let u and v be real numbers such that

$$(u + u^2 + u^3 + \cdots + u^8) + 10u^9 = (v + v^2 + v^3 + \cdots + v^{10}) + 10v^{11} = 8.$$

Determine, with proof, which of the two numbers, u or v, is larger.

*   *   *

The third set of problems was forwarded by Professor Francisco Bellot, Valladolid, Spain. They are the problems proposed for the 3rd Ibero–American Mathematical Olympiad, held in Lima, Peru, April 24 to May 1, 1988.

3RD IBERO–AMERICAN OLYMPIAD

First Day

Time: 4 1/2 hours

1. (Proposed by Cuba). The angles of a triangle are in arithmetical progression. The altitudes of the triangle are also in arithmetical progression. Show that the triangle is equilateral.

2. (Proposed by Brazil). Let a, b, c, d, p and q be natural numbers different from zero such that

$$ad - bc = 1 \quad \text{and} \quad \frac{a}{b} > \frac{p}{q} > \frac{c}{d}.$$

Show that

(i) $q \geq b + d$;

(ii) if $q = b + d$ then $p = a + c$.

3. (Proposed by Cuba). Show that, of all triangles whose vertices are at distances 3, 5 and 7 from a given point P, the triangle with maximum perimeter has P as incenter.

Second Day

Time: 4 1/2 hours

4. (Proposed by Brazil). Let ABC be a triangle with sides a, b, c. Each side is divided in n equal parts. Let S be the sum of the squares of distances from each vertex to each one of the points of subdivision of the

opposite side (excepting the vertices). Show that

$$\frac{S}{a^2 + b^2 + c^2}$$

is rational.

5. (Proposed by Venezuela). We consider expressions of the form

$$x + yt + zt^2,$$

where $x, y, z \in \mathbb{Q}$ and $t^2 = 2$. Show that, if $x + yt + zt^2 \neq 0$, then there exist $u, v, w \in \mathbb{Q}$ such that

$$(x + yt + zt^2)(u + vt + wt^2) = 1.$$

6. (Proposed by Brazil). We consider all sets of n non–zero natural numbers, no three of which are in arithmetical progression. Show that in one of these sets, the sum of the reciprocals of its elements is maximal.

* * *

We now turn to solutions received to problems from the December 1987 number of the Corner. These problems were proposed but not used for the 1987 I.M.O. in Havana, Cuba.

Belgium 1. [1987: 308]

Twenty–eight random draws are made from the set

{1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, J, K, L, U, X, Y, Z}

containing 20 elements. What is the probability that the sequence

CUBAJULY1987

occurs in that order in the chosen sequence?

Solutions by George Evagelopoulos, Law student, Athens, Greece, and by Curtis Cooper, Central Missouri State University, U.S.A.

Let A_i be the set of choices w which contain the given substring starting at position i. Notice that for $i < j$ we have $A_i \cap A_j \neq \phi$ iff $1 \leq i \leq 5$ and $i + 12 \leq j \leq 17$. Also whenever $A_i \cap A_j \neq \phi$ we have $|A_i \cap A_j| = 20^4$. Thus the probability that the substring CUBAJULY1987 occurs is

$$\frac{1}{20^{28}}\left(\sum_{i=1}^{17}|A_i| - \sum_{i=1}^{5}\sum_{j=12+i}^{17}|A_i \cap A_j|\right) = \frac{17\cdot 20^{16} - 15\cdot 20^4}{20^{28}}.$$

Bulgaria 2. [1987: 309]

Let $P_1, P_2, \ldots, P_n$ be n points in the plane. What is the smallest number of segments P_iP_j to join and how should they be selected so that whenever 4 points are chosen a triangle is formed by three of them (with edges amongst the selected segments)?

Solution by Edward T.H. Wang and Zun Shan, Wilfrid Laurier University, Waterloo, Ontario.

We solve the problem in terms of a graph on $n \geq 4$ vertices to avoid unnecessary complication arising from possible collinearity of three or more points. Thus, let $G = (V,E)$ denote a graph with the described property, where V and E denote the set of all vertices and edges, respectively. If vertices u and v are adjacent, we write $\overline{uv} \in E$. We show that

$$|E| \geq \binom{n-1}{2}$$

with equality if and only if G consists of an isolated point and K_{n-1}, the complete graph on $n-1$ vertices.

If $G = K_n$, then $|E| = \binom{n}{2}$. If $G \neq K_n$, then there exist distinct $u, v \in V$ such that $\overline{uv} \notin E$. Let $H = G\backslash\{u,v\}$ denote the induced subgraph obtained from G by deleting u and v. If there are distinct vertices x and y in H such that $\overline{xy} \notin E$, then clearly there is no triangle among the four points u, v, x, y. Hence H is a copy of K_{n-2}. Let x denote an arbitrary vertex in H. If $\overline{xu} \notin E$ and $\overline{xv} \notin E$, then there is no triangle among the four points u, v, x and y where y denotes any vertex in H different from x. Hence $\overline{xu} \in E$ or $\overline{xv} \in E$ and it follows

$$|E| \geq \binom{n-2}{2} + (n-2) = \binom{n-1}{2} .$$

If equality holds, then for any vertex x in H exactly one of $\overline{xu} \in E$ and $\overline{xv} \in E$ is true. But if there are two distinct vertices x, y in H such that $\overline{xu} \in E$ and $\overline{yv} \in E$ then there is no triangle among the points u, v, x and y. Therefore, in this case either $\overline{xu} \in E$ for all vertices x in H or $\overline{xv} \in E$ for all x in H. That is, G consists of an isolated point and K_{n-1}. Conversely, it is obvious that such a graph has $\binom{n-1}{2}$ edges and satisfies the described property.

<u>Finland 1</u>. [1987: 309]

Let A be an infinite set of integers such that every $a \in A$ is the product of at most 1987 prime numbers. Prove that there is an infinite set $B \subset A$ and a number p such that the greatest common divisor of any two numbers in B is p.

Solution by George Evagelopoulos, Law student, Athens, Greece.

Replace 1987 by n and use induction on n. Then the assertion is true if $n = 1$: the numbers are all prime and the gcd of any pair is 1. Assume the assertion holds for all $k < n$ and assume that A is an infinite set of numbers which are products of at most n primes. There are two cases.

***Case* 1.** If there is a fixed integer $q > 1$ for which there is an infinite subset $A_0 \subset A$ such that q divides a for each $a \in A_0$, then the set $A' = \{a/q\colon\ a \in A_0\}$ is an infinite set of numbers, each a product of at most $n - 1$ prime numbers. Thus there is r and an infinite subset $B' \subset A'$ such that $\gcd(c,d) = r$ for all c, d in B'. But then also $\gcd(qc,qd) = qr$, and $B = \{qc\colon\ c \in B'\}$ has the desired property.

***Case* 2.** Assume then that for every $q > 1$ the set of $a \in A$ such that q divides a is finite. Select $a_1 \in A$. There are only a finite number of $a \in A$ such that $\gcd(a,a_1) > 1$. Delete all these from A to form A_1. Select a_2 in A_1. Again there are only finitely many $a \in A_1$ such that $\gcd(a,a_2) > 1$. Delete these to form A_2. Notice that for each $a \in A_2$, $\gcd(a,a_i) = 1$ for $i = 1, 2$. Continuing in this way we select an infinite sequence $a_1,a_2,\ldots$ such that $\gcd(a_i,a_j) = 1$ for $i \neq j$, completing case 2, and the result follows by induction.

<u>France 1</u>. [1987: 309]

For each whole number $k > 0$ let $a^k_{n_k}\cdots a^k_0$ ($a^k_{n_k} \neq 0$) denote the decimal representation of $(1987)^k$. (Thus $n_0 = 0$ and $a^0_{n_0} = 1$ since $(1987)^0 = 1$. Also $n_1 = 3$, $a^1_3a^1_2a^1_1a^1_0 = 1987$, etc.) Form the infinite decimal

$$x = 0.1\ 1987\ a^2_{n_2}\cdots a^2_0\ \cdots\ a^k_{n_k}\cdots a^k_0\ \cdots\ .$$

Show that x is irrational.

Solution by George Evagelopoulos, Law student, Athens, Greece.

First note that $(1987)^4$ ends in a 1, i.e. $(1987)^4 \equiv 1 \bmod 10$. We show by induction on $n \geq 1$ that

$$A_n\colon\quad (1987)^{4\cdot 10^{n-1}} \equiv 1 \bmod 10^n,$$

for which the truth of A_1 is just the observation above. So assume A_n holds. Then for some $k \in \mathbb{N}$ we have

$$(1987)^{4\cdot 10^{n-1}} = 1 + 10^n k,$$

so

$$\begin{aligned}(1987)^{4\cdot 10^n} &= (1987^{4\cdot 10^{n-1}})^{10} = (1 + 10^n k)^{10}\\ &= 1 + \binom{10}{1}10^n k + \binom{10}{2}(10^n k)^2 + \cdots + \binom{10}{10}(10^n k)^{10}\\ &= 1 + 10^{n+1}l\end{aligned}$$

for some integer l. Thus A_{n+1} holds, and we have that, for all $n \geq 1$,

$$(1987)^{4\cdot 10^{n-1}} \equiv 1 \bmod 10^n.$$

But this means that the decimal representation of $(1987)^{4\cdot 10^{n-1}}$ ends with

$$\underbrace{0\ .\ .\ .\ 0}_{n\text{-}1\text{ zeros}}\,1\ .$$

Therefore the decimal expansion of x cannot be periodic, and x is irrational.

U.S.S.R. 1. [1987: 309]

The positive quantities α, β and γ are such that $\alpha + \beta + \gamma < \pi$. Prove that a triangle can be formed from segments of length $\sin \alpha$, $\sin \beta$, and $\sin \gamma$ such that the area of the triangle does not exceed

$$\frac{\sin 2\alpha + \sin 2\beta + \sin 2\gamma}{8}\ .$$

Comment by Edward T.H. Wang, Wilfrid Laurier University and by Murray S. Klamkin, Mathematics Department, The University of Alberta, Edmonton.

There is probably some misprint or error due to translation since the conditions given do not guarantee that $\sin \alpha$, $\sin \beta$ and $\sin \gamma$ can form the three sides of a triangle. We must have the sum of any two of $\sin \alpha$, $\sin \beta$ and $\sin \gamma$ larger than the third. For a counterexample take $\alpha = 90^\circ$, $\beta = \gamma = 30^\circ$.

Correction and solution by Murray S. Klamkin, Mathematics Department, The University of Alberta, Edmonton.

A *sufficient* condition on α, β, and γ that ensures satisfaction of the necessary condition cited above, is that the sum of any two of α, β and γ is greater than the third one. This follows from

$$\begin{aligned}\sin \alpha + \sin \beta &= 2 \sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\\ &> 2 \sin \tfrac{\gamma}{2} \cos\left(\frac{\alpha-\beta}{2}\right)\\ &> 2 \sin \tfrac{\gamma}{2} \cos \tfrac{\gamma}{2} = \sin \gamma\end{aligned}$$

(using $0 < \gamma < \alpha + \beta < \pi$ and $\beta - \gamma < \alpha < \gamma + \beta$, i.e. $|\alpha - \beta| < \gamma$), etc. From Heron's formula, 16 times the square of the area of the triangle is

$$2 \sum \sin^2\beta \sin^2\gamma - \sum \sin^4\alpha$$

(where the summations here and subsequently are symmetric over α, β and γ). Thus we have to show that

$$8 \sum \sin^2\beta \sin^2\gamma - 4 \sum \sin^4\alpha \le \left(\sum \sin 2\alpha\right)^2. \tag{1}$$

Using the double angle formula, (1) becomes

$$2\sum(1-\cos 2\beta)(1-\cos 2\gamma)-\sum(1-\cos 2\alpha)^2$$
$$\le \sum \sin^2 2\alpha + 2\sum \sin 2\beta \sin 2\gamma.$$

After expanding out and combining, this becomes

$$\sum \cos 2(\alpha+\beta) \le \sum \cos 2\alpha\,. \qquad (2)$$

Then using

$$\cos 2\alpha - \cos 2(\beta+\gamma) = 2\sin(\beta+\gamma-\alpha)\sin(\alpha+\beta+\gamma), \text{ etc.},$$

(2) becomes

$$[\sin(\beta+\gamma-\alpha)+\sin(\gamma+\alpha-\beta)+\sin(\alpha+\beta-\gamma)]\sin(\alpha+\beta+\gamma) \ge 0.$$

This is trivially true under the strengthened hypothesis given above, as $\beta+\gamma-\alpha > 0$, etc. (In fact the conditions give strict inequality.) However it is also true under the weaker assumption that $\sin\alpha$, $\sin\beta$, $\sin\gamma$ are the sides of a triangle. To show this we use the known identity

$$\sin 2x + \sin 2y + \sin 2z - \sin 2(x+y+z) = 4\sin(y+z)\sin(z+x)\sin(x+y)$$

to give

$$\sin(\beta+\gamma-\alpha)+\sin(\gamma+\alpha-\beta)+\sin(\alpha+\beta-\gamma)$$
$$= \sin(\alpha+\beta+\gamma) + 4\sin\alpha\sin\beta\sin\gamma > 0.$$

It is to be noted that there is a connection here with a tetrahedron $PABC$ where $PA = PB = PC = 1/2$ and $\angle BPC = 2\alpha$, $\angle CPA = 2\beta$, $\angle APB = 2\gamma$. For then $\alpha+\beta+\gamma < \pi$, the sum of any two of the angles α, β, γ is greater than the third angle, and ABC is a triangle of sides $\sin\alpha$, $\sin\beta$, $\sin\gamma$. If $\alpha+\beta+\gamma = \pi$, then the tetrahedron is degenerate and lies in a plane. For this case the upper bound on the area is achieved.

U.S.S.R. 2. [1987: 309–310]

For each natural number $k \ge 2$ the sequence $a_n(k)$ is generated according to the rule

$$a_0 = k, \quad a_n = \tau(a_{n-1}), \qquad n = 1,2,3,\ldots,$$

where $\tau(a)$ is the number of positive integral divisors of a. Find all k for which the sequence $a_n(k)$ does not contain squares of whole numbers.

Solution by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.

We claim that the sequence $\{a_n\}_{n=0}^{\infty}$ where $a_n = a_n(k)$ does not contain squares of whole numbers if and only if k is prime. The sufficiency is clear since if $a_0 = k$ is a prime, then $a_1 = \tau(a_0) = 2$ and thus $a_n = 2$ for all $n \ge 1$. To see the necessity, suppose $a_0 \ge 2$ is a composite number and that the sequence $\{a_n\}_{n=0}^{\infty}$ does not contain any square. Let $a_0 = p_1^{\alpha_1}p_2^{\alpha_2}...p_m^{\alpha_m}$ be the prime power decomposition of a_0, where $\alpha_i \geq 1$, $i = 1,2,...,m$ and $p_1,p_2,...,p_m$ are distinct primes. Then

$$a_1 = \tau(a_0) = (\alpha_1 + 1)(\alpha_2 + 1)...(\alpha_m + 1).$$

Since a_0 is not a square, α_i is odd for at least one i and thus a_1 is even. Clearly $a_1 > 2$, since if $m > 1$ then $a_1 > \alpha_1 + 1 \geq 2$ while if $m = 1$ then $\alpha_1 > 1$ implies $a_1 = \alpha_1 + 1 > 2$. Hence a_1 is composite. Furthermore since

$$\alpha_i + 1 \leq (1 + 1)^{\alpha_i} \leq p_i^{\alpha_i}$$

with equality if and only if $\alpha_i = 1$ and $p_i = 2$, we have $a_1 < a_0$. Continuing in this manner, we obtain a strictly decreasing sequence of positive integers $a_0 > a_1 > a_2 > \dots$, which is obviously impossible.

[*Editor's note*: This problem was also solved by George Evagelopoulos, Law student, Athens, Greece.]

U.S.S.R. 3. [1987: 310]

Find the largest value of the expression

$$(a + b)^4 + (a + c)^4 + (a + d)^4 + (b + c)^4 + (b + d)^4 + (c + d)^4$$

where a, b, c and d are real numbers satisfying

$$a^2 + b^2 + c^2 + d^2 \leq 1.$$

Solutions by George Evagelopoulos, Law student, Athens, Greece; Murray S. Klamkin, Mathematics Department, The University of Alberta, Edmonton; and by Zun Shan and Edward T.H. Wang, Wilfrid Laurier University, Waterloo.

Let S denote the given sum and let $\sum$ denote summation over all the six selections of two elements of a, b, c, d. Then

$$\begin{aligned} S = \sum (a + b)^4 &\leq \sum (a + b)^4 + \sum (a - b)^4 \\ &= 2 \sum (a^4 + 6a^2b^2 + b^4) \\ &= 6(a^4 + b^4 + c^4 + d^4) + 6 \sum 2a^2b^2 \\ &= 6(a^2 + b^2 + c^2 + d^2)^2. \end{aligned}$$

From this homogeneous inequality and $a^2 + b^2 + c^2 + d^2 \leq 1$ we deduce $S \leq 6$ with equality if and only if $a = b = c = d$ and $a^2 + b^2 + c^2 + d^2 = 1$, i.e. if and only if $a = b = c = d = 1/2$.

Generalization and problems by Murray S. Klamkin, Mathematics Department, The University of Alberta, Edmonton.

More generally, consider the analogous inequality with n variables

$$n \sum_{1 \le i < j \le n} (a_i + a_j)^4 \le 8(n-1)(a_1^2 + a_2^2 + \cdots + a_n^2)^2 \qquad (I_n)$$

with equality when all the a_i's are equal. We assume here and subsequently that the variables are *non–negative.* The case $n = 4$ has already been taken care of. We show that the case $n = 3$ is also valid. First rewrite I_3 as

$$10(x^4 + y^4 + z^4) + 14(x^2y^2 + x^2z^2 + y^2z^2) - 12(x^3y + xy^3 + x^3z + xz^3 + y^3z + yz^3) \ge 0. \qquad (*)$$

Now let

$$\begin{aligned} L &= (x-y)^4 + (y-z)^4 + (z-x)^4 + (x^2+y^2)(x-y)^2 \\ &\qquad + (y^2+z^2)(y-z)^2 + (x^2+z^2)(x-z)^2 \\ &= 4(x^4 + y^4 + z^4) + 8(x^2y^2 + x^2z^2 + y^2z^2) \\ &\qquad - 6(x^3y + xy^3 + x^3z + xz^3 + y^3z + yz^3). \end{aligned}$$

Thus the left hand side of (*) is

$$\begin{aligned} &2L + 2(x^4 + y^4 + z^4) - 2(x^2y^2 + x^2z^2 + y^2z^2) \\ &\qquad = 2L + (x^2-y^2)^2 + (x^2-z^2)^2 + (y^2-z^2)^2 \ge 0. \end{aligned}$$

The complementary (going the other way) inequality for $n = 3$,

$$(x+y)^4 + (y+z)^4 + (z+x)^4 \ge 2(x^2+y^2+z^2)^2,$$

is also valid since it can be rewritten as

$$4x^3y + 2x^2y^2 + 4xy^3 + 4x^3z + 2x^2z^2 + 4xz^3 + 4y^3z + 2y^2z^2 + 4yz^3 \ge 0,$$

with equality for two of x, y, z equaling zero. (Here we use that x, y, $z \ge 0$.) The complementary inequality to I_4 is

$$\sum_{1 \le i < j \le 4} (a_i + a_j)^4 \ge 3(a_1^2 + a_2^2 + a_3^2 + a_4^2)^2$$

with equality if three of the variables equal zero. This is valid since it can be rewritten as

$$4 \sum_{1 \le i < j \le 4} a_i a_j (a_i^2 + a_j^2) \ge 0.$$

(Again we use that the a_i are non–negative.)

By letting $a_1 = 1$, $a_2 = 2$ and $a_3 = a_4 = \ldots = a_8 = 0$ we see that I_8 is invalid. However it is valid if we reverse the inequality sign, since it is equivalent to

$$\sum a_i a_j (a_i - a_j)^2 \ge 0.$$

(All the $a_i \ge 0$.)

As *open problems*, determine the least k and the maximum l such that

$$k(a_1^2 + a_2^2 + \cdots + a_n^2)^2 \geq \sum_{1 \leq i < j \leq n} (a_i + a_j)^4 \geq l(a_1^2 + a_2^2 + \cdots + a_n^2)^2$$

for all non-negative a_i. Also determine the maximum l if the a_i's are only restricted to being real. The latter problem seems rather difficult.

* * *

This brings us to the end of solutions submitted for problems from the 1987 numbers of *Crux*. We next will take up problems from January 1988. There are still some very good problems from 1987 to which you can submit solutions. Take up the challenge. Please collect and send in the 1989 Olympiads so we may share them!

* * *

MINI-REVIEWS

by

ANDY LIU

THE NEW MATHEMATICAL LIBRARY SERIES OF THE MATHEMATICAL ASSOCIATION OF AMERICA

This paperback series is written by professional mathematicians with the high school student in mind. The books cover topics which are not usually included in the high school curriculum, but are nevertheless not too far removed from classroom mathematics. Each volume contains numerous exercises with answers.

Volume 1. *Numbers: Rational and Irrational*, by I. Niven, 1961. (140 pp.)

The book begins with a review of elementary number theory and the basic properties of rational numbers. It then goes on to prove that irrational numbers exist, with explicit examples. Algebraic and transcendental numbers are then introduced. There is a discussion of the impossibility of the three classical problems in geometric construction, as well as the problem of approximating irrational numbers by rational numbers.

Volume 2. *What is Calculus About*, by W.W. Sawyer, 1961. (118 pp.)

The book uses a practical example to introduce the reader to the subject of calculus. From the consideration of distance, velocity and acceleration as functions of time, the concepts and techniques of differential calculus emerge. There is a brief discussion of integral calculus towards the end of the book.

Volume 3. ***An Introduction to Inequalities*, by E. Beckenbach and R. Bellman, 1961. (133 pp.)**

The heart of this book lies in the fourth chapter where classical inequalities are discussed. These include the Arithmetic–Mean–Geometric–Mean Inequality, Cauchy's Inequality, Hölder's Inequality, the Triangle Inequality and Minkowski's Inequality. There is a brief discussion of basic properties of inequalities and absolute values in the earlier chapters. The classical inequalities are later applied to maxima and minima problems.

Volume 4. ***Geometric Inequalities*, by N.D. Kazarinoff, 1961. (132 pp.)**

The book covers the Arithmetic–Mean–Geometric–Mean Inequality and the Isoperimetric Theorem. It emphasizes the method of reflection in solving maxima and minima problems.

Volume 5. ***The Contest Problem Book I*, by C.T. Salkind, 1961. (112 pp.)**

The book contains reprints of the American High School Mathematics Examinations from 1950 to 1960. Each paper consists of 50 multiple–choice questions except for 1960 where there are only 40 questions. An answer key is provided, followed by complete solutions.

Volume 6. ***The Lore of Large Numbers*, by P.J. Davis, 1961. (165 pp.)**

The book presents a fascinating account of the notations and techniques in computation and approximation. The large numbers serve as a binding theme in the discussion. It starts with fairly basic material and a historical background, which later leads to glimpses of more advanced mathematics.

Volume 7. ***Uses of Infinity*, by L. Zippin, 1962. (151 pp.)**

Starting from an account of the popular notion of infinity, the book leads the reader onto mathematical treatments of sequences and series, limit and convergence, irrational numbers and their approximation, as well as countability and cardinal numbers. The golden ratio is featured as a detailed example.

Volume 8. ***Geometric Transformations I*, by I.M. Yaglom, 1962. (133 pp.)**

The book discusses Euclidean geometry from the transformation point of view. Only distance–preserving transformations or isometries are considered, and these are classified into translations, rotations, reflections and glide reflections.

Volume 9. ***Continued Fractions*, by C.D. Olds, 1963. (162 pp.)**

The book begins with a definition of continued fractions and shows that finite continued fractions are equivalent to rational numbers. These continued fractions are used to solve linear Diophantine equations. To represent irrational numbers, infinite continued fractions are introduced. It is then shown that periodic continued fractions are equivalent to quadratic irrationals. These continued fractions are then used to solve Pell's equation.

Volume 10. ***Graphs and their Uses*, by O. Ore, 1963. (131 pp.)**

This is an excellent introduction to the theory of graphs, covering connected graphs, trees, directed graphs, planar graphs, map colouring and matchings.

Volume 11. ***Hungarian Problem Book I*, edited by G. Hajós, G. Neukomm and J. Surányi, 1963. (111 pp.)**

Hungary has probably the longest and strongest tradition in mathematical contests for high school students. This volume collects the papers from 1894 to 1905 inclusive, with detailed solutions.

Volume 12. ***Hungarian Problem Book II*, edited by G. Hajós, G. Neukomm and J. Surányi, 1963. (120 pp.)**

This volume collects the papers of the Hungarian contests from 1906 to 1928. Contest activities were interrupted in 1919, 1920 and 1921 as an aftermath of World War I.

Volume 13. ***Episodes from the Early History of Mathematics*, by A. Aaboe, 1964. (133 pp.)**

The book consists of four chapters, one on Babylonian mathematics and three on Greek mathematics, the latter centred around Euclid, Archimedes and Ptolemy respectively.

Volume 14. ***Groups and their Graphs*, by I. Grossman, 1964. (195 pp.)**

The book introduces the reader to the abstract algebraic concept of groups via many concrete examples. The graphs of the groups come into the picture when the groups are defined by generators and relations. There is a discussion of the result that there are seventeen essentially different wallpaper patterns.

Volume 15. ***Mathematics of Choice*, by I. Niven, 1965. (195 pp.)**

As its subtitle "How to Count without Counting" suggests, this book deals with counting techniques. Starting from the basic ideas of permutations, combinations and the Binomial Theorem, it leads the reader into more sophisticated topics such as the Principle of Inclusion–Exclusion, generating functions, and recurrence relations along with many applications. There is a brief discussion of mathematical induction and the Pigeonhole Principle.

Volume 16. ***From Pythagoras to Einstein*, by K.O. Friedrichs, 1965. (88 pp.)**

The book discusses Pythagoras' Theorem and the concept of vectors in various mathematical and mechanical settings, leading eventually to the roles they play in the theory of relativity. Unlike other titles in this series, there are no exercises.

Volume 17. ***The MAA Problem Book II*, by C.T. Salkind, 1966. (112 pp.)**

The book contains reprints of the American High School Mathematics Examinations from 1961 to 1965. Each paper consists of 40 multiple–choice questions. An answer key is provided, followed by complete solutions.

Volume 18. ***First Concepts of Topology*, by W.G. Chinn and N.E. Steenrod, 1966. (160 pp.)**

The first of two chapters in this book introduces the reader to point–set topology, leading to an important result in analysis, the Bolzano–Weierstrass Theorem. The second chapter involves concepts of algebraic topology from which the fundamental theorem of algebra is deduced.

Volume 19. ***Geometry Revisited*, by H.S.M. Coxeter and S. Greitzer, 1967. (193 pp.)**

This is an excellent review of Euclidean geometry, dealing with points and lines connected with a triangle, some properties of circles, collinearity and concurrence and transformations. There is also an introduction to inversive geometry and one to projective geometry.

Volume 20. ***Invitation to Number Theory*, by O. Ore, 1967. (129 pp.)**

This is an excellent introduction to number theory, covering concepts such as divisibility, primes, greatest common divisors, congruence, Diophantine equations and numeration systems.

Volume 21. ***Geometric Transformations II*, by I.M. Yaglom, 1968. (189 pp.)**

The book adds similarity transformations to isometries and gives many applications of the transformation approach to geometric problems.

Volume 22. ***Elementary Cryptanalysis*, by A. Sinkov, 1968. (222 pp.)**

The book gives a mathematical approach to the popular topic of secret codes. There are discussions of various systems of codes, mostly based on modular arithmetic. Methods of cracking these codes without the knowledge of the key are also given.

Volume 23. ***Ingenuity in Mathematics*, by R. Honsberger, 1970. (206 pp.)**

The book consists of nineteen essays from elementary combinatorics, number theory and geometry. Topics include Sylvester's problem, the Isoperimetric problem, the Theorem of Barbier, probability and π, the Farey series, complementary sequences, abundant numbers, squaring the square and the construction problems of Mascheroni and Steiner.

Volume 24. ***Geometric Transformations III*, by I.M. Yaglom, 1973. (237 pp.)**

The book takes the reader beyond Euclidean geometry to affine and projective geometries, again discussed from the transformational point of view. There is also a supplement on hyperbolic geometric.

Volume 25. ***The Contest Problem Book III*, by C.T. Salkind and J.M. Earl, 1973. (186 pp.)**

The book contains reprints of the American High School Mathematics Examinations from 1966 to 1972. Each of the first two papers consists of 40 multiple–choice questions, the number dropping to 35 for subsequent papers. An answer key is provided, followed by complete solutions.

Volume 26. ***Mathematical Methods in Science*, by G. Pólya, 1977. (234 pp.)**

A distinguished mathematician and scientist illustrates the applications of various mathematical concepts such as measurement, successive approximation, vectors and differential equations in astronomy, statics and dynamics.

Volume 27. ***International Mathematical Olympiads 1959–1977*, by S. Greitzer, 1978. (204 pp.)**

The International Mathematical Olympiad began in 1959 in Romania as an all East–European affair. It has since grown to truly international proportions. This book contains the papers of the first nineteen Olympiads and their solutions.

Volume 28. ***The Mathematics of Games and Gambling*, by E. Packel, 1981. (141 pp.)**

The book treats the subject of probability in a gambling setting, discussing various games of dice and cards such as backgammon, craps, poker and bridge. Basic concepts such as permutations, combinations, the binomial distribution and mathematical expectation are covered. There is also a chapter on elementary game theory.

Volume 29. ***The Contest Problem Book IV*, by R.A. Artino, A.M. Gaglione and N. Shell, 1982. (184 pp.)**

The book contains reprints of the American High School Mathematics

Examinations from 1973 to 1982. Each paper consists of 30 multiple–choice questions except for 1973 where there are 35 questions. An answer key is provided, followed by complete solutions.

Volume 30. *The Role of Mathematics in Science*, by M.M. Schiffer and L. Bowden, 1984. (207 pp.)

The book contains seven chapters, dealing with the beginnings of mechanics, growth functions, the role of mathematics in optics, mathematics with matrices – transformations, Einstein's space–time transformation problem, relativistic addition of velocities and energy.

Volume 31. *International Mathematical Olympiads 1978–1985*, by M.S. Klamkin, 1986. (150 pp.)

This book contains the International Mathematical Olympiad papers from 1978 to 1985, except for 1980 when it was not held. In addition to complete solutions, there are also forty supplementary problems. Professor Klamkin, of the University of Alberta, is the acknowledged authority on problem–solving in the world.

Volume 32. *Riddles of the Sphinx*, by Martin Gardner, 1987. (184 pp.)

This is an anthology of Martin Gardner's contributions to Isaac Asimov's *Science Fiction Magazine.* There are thirty–six puzzles in science fictional settings. Often, when the "first" answers are given, further questions arise, to be answered in a "second" section. "Third" answers and "fourth" answers pursue the matter even further.

Volume 33. *U.S.A. Mathematical Olympiads 1972–1986*, by M.S. Klamkin, 1988. (127 pp.)

This book contains the first fifteen U.S.A. Mathematical Olympiad papers. The solutions are grouped under the topics of Algebra, Number Theory, Plane Geometry, Solid Geometry, Geometric Inequalities, Inequalities, Combinatorics and Probability.

Address of the M.A.A.: 1829 Eighteenth Street N.W., Washington, D.C. 20036.

* * *

PROBLEMS

Problem proposals and solutions should be sent to the editor , whose address appears on the inside front cover of this issue. Proposals should, whenever possible, be accompanied by a solution, references, and other insights which are likely to be of help to the editor. An asterisk (*) after a number indicates a problem submitted without a solution.

Original problems are particularly sought. But other interesting problems may also be acceptable provided they are not too well known and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted by somebody else without his or her permission.

To facilitate their consideration, your solutions, typewritten or neatly handwritten on signed, separate sheets, should preferably be mailed to the editor before **January 1, 1990,** *although solutions received after that date will also be considered until the time when a solution is published.*

1400. [1988: 302] **(Corrected)** *Proposed by Robert E. Shafer, Berkeley, California.*

In a recent issue of the *American Mathematical Monthly* (June–July 1988, page 551), G. Klambauer showed that if $x^s e^{-x} = y^s e^{-y}$ (x, y, $s > 0$, $x \neq y$) then $x + y > 2s$. Show that if $x^s e^{-x} = y^s e^{-y}$ where $x \neq y$ and x, y, $s > 0$ then $xy(x + y) < 2s^3$.

1451. *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Find positive integers a and b such that $a \leq 2000 \leq b$ and 2, $a(a + 1)$, $b(b + 1)$ are in arithmetic progression.

1452. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let x_1,x_2,x_3 be positive reals satisfying $x_1 + x_2 + x_3 = 1$, and consider the inequality

$$(1 - x_1)(1 - x_2)(1 - x_3) \geq c_r(x_1x_2x_3)^r. \qquad (1)$$

For each real r, find the greatest constant c_r such that (1) holds for all choices of the x_i, or prove that no such constant c_r exists.

1453. *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

Triangle ABC moves in such a way that AB passes through a fixed point P and AC through a fixed point Q. Prove that throughout the motion BC is tangent to a fixed circle.

1454. *Proposed by Marcin E. Kuczma, Warsaw, Poland.*

Given a convex pentagon of area S, let $S_1,\ldots,S_5$ denote the areas of the five triangles cut off by the diagonals (each triangle is spanned by three consecutive vertices of the pentagon). Prove that the sum of some four of the S_i's exceeds S.

1455. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $A'B'C'$ be a triangle inscribed in a triangle ABC, so that $A' \in BC$, $B' \in CA$, $C' \in AB$. Suppose that

$$\frac{BA'}{A'C} = \frac{CB'}{B'A} = \frac{AC'}{C'B} \neq 1$$

and that $\Delta A'B'C'$ is similar to ΔABC. Prove that the triangles are equilateral.

1456. *Proposed by Murray S. Klamkin, University of Alberta.*

(a) Find a pair of integers (a,b) such that

$$x^{13} - 233x - 144 \quad \text{and} \quad x^{15} + ax + b$$

have a common (nonconstant) polynomial factor.

(b)* Is the solution unique?

1457. *Proposed by Colin Springer, student, University of Waterloo.*

In ΔABC, the sides are a, b, c, the perimeter is p and the circumradius is R. Show that

$$R^2 p \geq \frac{a^2 b^2}{a + b - c} .$$

Under what conditions does equality hold?

1458. *Proposed by Edward T.H. Wang, Wilfrid Laurier University.*

(a) Characterize all natural numbers n for which there exists a group with exactly two elements of order n.

(b) Characterize all natural numbers n for which there exists a group with exactly three elements of order n.

1459. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

ABC is an isosceles triangle in which $AB = AC$ and $\angle A < 120^\circ$. Let D be the point on side BC such that $BD = 2DC$, and let E be the point on segment AD such that $\angle BED = 2\angle DEC$. Prove that $\angle BED = \angle BAC$.

1460. *Proposed by Miha'ly Bencze, Brasov, Romania.*

P is an interior point of a convex n-gon $A_1A_2\cdots A_n$. For each $i = 1,\ldots,n$ let $R_i = \overline{PA_i}$ and w_i be the length of the bisector of $\angle P$ in ΔA_iPA_{i+1} $(A_{n+1} \equiv A_1)$. Also let $c_1,\ldots,c_n$ be positive real numbers. Prove that

$$2 \cos \frac{\pi}{n} \sum_{i=1}^{n} c_i^2 \geq \sum_{i=1}^{n} c_i c_{i+1} w_i \left(\frac{1}{R_i} + \frac{1}{R_{i+1}}\right)$$

$(R_{n+1} \equiv R_1)$.

* * *

SOLUTIONS

No problem is ever permanently closed. The editor will always be pleased to consider for publication new solutions or new insights on past problems.

1338. [1988: 110] *Proposed by Jean Doyen, Université Libre de Bruxelles, Brussels, Belgium; J. Chris Fisher, University of Regina, Regina, Saskatchewan; and Günter Kist, Technische Universität, Munich, Federal Republic of Germany.*

In a theoretical version of the Canadian lottery "Lotto 6–49", a ticket consists of six distinct integers chosen from 1 to 49 (inclusive). A *t–prize* is awarded for any ticket having t or more numbers in common with a designated "winning" ticket. Denote by $f(t)$ the smallest number of tickets required to be certain of winning a t–prize. Clearly $f(1) = 8$ and $f(6) = \binom{49}{6}$. Show that $f(2) \le 19$. Can you do better?

I. *Solution by Kee–Wai Lau, Hong Kong.*

We can only show that $f(2) \le 19$.

Denote by $F(k)$ the smallest number of tickets required in order to include all combinations of the k positive integers $i, i+1, \dots, i+k-1$ taken two at a time. Clearly $F(k)$ is independent of i. By considering the four tickets

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6\,, \\ 1 & 2 & 7 & 8 & 9 & 10\,, \\ 3 & 4 & 7 & 8 & 9 & 10\,, \\ 5 & 6 & 7 & 8 & 9 & 10\,, \end{array}$$

we see that $F(10) \le 4$. By considering the three tickets

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6\,, \\ 1 & 2 & 3 & 7 & 8 & 9\,, \\ 4 & 5 & 6 & 7 & 8 & 9\,, \end{array}$$

we see that $F(9) \le 3$.

Now divide the integers from 1 to 49 into five groups, namely from 1 to 10, from 11 to 20, from 21 to 30, from 31 to 40, and from 41 to 49 (all inclusive). By the pigeon–hole principle at least one of these five groups has two or more numbers in common with the "winning" ticket. Hence

$$f(2) \le 4F(10) + F(9) \le 19\ .$$

II. *Solution by Sam Maltby, student, Calgary.* (Adapted by the editor.)

We say that a set of numbers is *saturated* by a group of tickets if those tickets contain every pair of numbers from the set. If we divide the 49 numbers into five sets, one of the sets must contain a pair from the winning ticket. Thus any group of tickets which saturates each of these five sets will guarantee a 2–prize. Let $g(n)$ be the least number of tickets needed to saturate a set of n numbers; then

$$f(2) \le g(n_1) + g(n_2) + g(n_3) + g(n_4) + g(n_5)$$

for every five positive integers n_i such that

$$n_1 + n_2 + n_3 + n_4 + n_5 = 49.$$

We will show that the minimum value of $g(n_1) + \cdots + g(n_5)$ is 19.

Each number in a set of n numbers must appear on a ticket with every other number in the set. Since there are five other spaces on each ticket, the number must appear on at least $\lceil (n-1)/5 \rceil$ tickets ($\lceil x \rceil$ denotes the least integer greater than or equal to x). Since there are n numbers, there must be at least $n \cdot \lceil (n-1)/5 \rceil$ spaces on the tickets, and so

$$g(n) \ge \left\lceil \frac{n}{6} \cdot \left\lceil \frac{n-1}{5} \right\rceil \right\rceil . \tag{1}$$

Let

$$h(n) = n - g(n) ,$$

so that

$$\sum_{i=1}^{5} h(n_i) = 49 - \sum_{i=1}^{5} g(n_i)$$

if $n_1 + \cdots + n_5 = 49$. If

$$\sum_{i=1}^{5} g(n_i) < 19 ,$$

then

$$\sum_{i=1}^{5} h(n_i) > 30 , \tag{2}$$

so $h(n_i) \ge 7$ for at least one n_i.

We claim that $h(n) \ge 7$ only if $n = 14$, and then $h(14) = 7$. From (1),

$$g(n) \ge \frac{n(n-1)}{30} ,$$

and so

$$h(n) \le n - \frac{n(n-1)}{30} .$$

It follows that $h(n) \ge 7$ only if $10 \le n \le 21$. Further, from (1) we get the following table:

n	10	11	12	13	14	15	16	17	18	19	20	21
$g(n)$	≥ 4	≥ 4	≥ 6	≥ 7	≥ 7	≥ 8	≥ 8	≥ 12	≥ 12	≥ 13	≥ 14	≥ 14

(3)

Thus $h(n) < 7$ unless $n = 11$, 14, 15, 16, or 21. However, for $n = 11$ for example, if $g(11) = 4$ there would be 4 tickets, with 24 spaces, saturating the set $\{1,2,\ldots,11\}$. This means that at least nine of the numbers must appear the minimum two times in the tickets. Without loss of generality, assume 1 appears only twice, in the tickets $\{1,2,3,4,5,6\}$ and $\{1,7,8,9,10,11\}$. Also without loss of generality 2 appears only twice. Then its second ticket must be $\{2,7,8,9,10,11\}$. But then the numbers 7 to 11 must each appear at least three times, so $g(11) \geq 5$. Using this thinking it can similarly be shown that $g(15) \geq 9$, $g(16) \geq 10$, and $g(21) \geq 15$. Thus $h(n) \geq 7$ only if $n = 14$, as claimed. Finally, $h(14) = 7$ follows from $g(14) = 7$, which is shown by the tickets

$\{1,2,3,4,5,6\}$, $\{1,2,7,8,9,10\}$, $\{1,2,11,12,13,14\}$,
$\{3,4,7,8,11,12\}$, $\{3,4,9,10,13,14\}$, $\{5,6,7,8,13,14\}$, $\{5,6,9,10,11,12\}$.

From (1) it follows that $g(n) \geq 1$ for $n \leq 6$ and $g(n) \geq 3$ for $7 \leq n \leq 9$. Thus $h(n) \geq 5$ implies $n \geq 6$ and $h(n) \geq 6$ implies $n \geq 9$. Now suppose (2) holds. If exactly one $n_i = 14$, say $n_1 = 14$, then $h(n_1) = 7$ so that by (2) $h(n_i) = 6$ for $i = 2,\ldots,5$. Thus

$$\sum_{i=1}^{5} n_i \geq 14 + 4\cdot 9 = 50 ,$$

contradicting $n_1 + \cdots + n_5 = 49$. With exactly two $n_i = 14$, say $n_1 = n_2 = 14$, we need $h(n_3) = h(n_4) = 6$ and $h(n_5) = 5$, say. Thus

$$\sum_{i=1}^{5} n_i \geq 2\cdot 14 + 2\cdot 9 + 6 = 52.$$

With exactly three $n_i = 14$, say $n_1 = n_2 = n_3 = 14$, we need either $h(n_4) = 6$ (say) or $h(n_4) = h(n_5) = 5$, both of which are similarly impossible. Since at most three of the n_i's can be 14, we have shown that

$$\sum_{i=1}^{5} g(n_i) \geq 19 . \tag{4}$$

We obtain equality in (4), and thus $f(2) \leq 19$, when (n_1,n_2,n_3,n_4,n_5) is one of

$(14,14,9,6,6)$, $(14,10,10,9,6)$, $(14,9,9,9,8)$,
$(13,9,9,9,9)$, $(12,10,9,9,9)$, $(10,10,10,10,9)$.

This follows from (3) and the following groups of tickets:

n	$g(n)$	tickets for saturation
6	1	{1,2,3,4,5,6}
8,9	3	{1,2,3,4,5,6} , {1,2,3,7,8,9} , {4,5,6,7,8,9}
10	4	{1,2,3,4,5,6} , {1,2,7,8,9,10} , {3,4,7,8,9,10} , {5,6,7,8,9,10}
12	6	{1,2,3,4,5,6} , {1,2,3,7,8,9} , {1,2,3,10,11,12} , {4,5,6,7,8,9} , {4,5,6,10,11,12} , {7,8,9,10,11,12}
13,14	7	shown above

Also solved by CURTIS COOPER, Central Missouri State University; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; PETER WATSON–HURTHIG, Columbia College, Burnaby, B.C.; and the proposers.

The proposers mention the known results

$$15 \le f(2) \le 19 \quad [1] ,$$
$$84 \le f(3) \le 175 \quad [2] ,$$
$$f(4) \le 13120 ,$$
$$f(5) \le 285384 ,$$

with the references

[1] *H. Hanani, D. Ornstein, and V.T. Sós, On the lottery problem,* Magyar Tud. Akad. Mat. Kutató Int. Közl. *9 (1964) 155–158.*

[2] *F. Sterboul, Le problem du loto,* Coll. Math. Discrètes: Codes et Hypergraphes, *Bruxelles, 1978. Cahiers C.E.R.O. 20, 3–4 (1978) 443–449.*

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1339. [1988: 110] *Proposed by Weixuan Li, Changsha Railway Institute, Changsha, Hunan, China, and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Ontario.*

Let a, b, m, n denote positive real numbers such that $a \le b$ and $m \le n$. Show that

$$(b^m - a^m)^n \le (b^n - a^n)^m$$

and determine all cases when equality holds.

I. *Solution by J.L. Brenner, Palo Alto, California.*

Equality holds if and only if $a = b$ or $m = n$.

Set $x = b/a$ and $k = m/n$; then $x \ge 1$ and $k \ge 1$. Set $y = x^m - 1$.

The proposed inequality is then equivalent to

$$y^k \le (y+1)^k - 1\ , \qquad y \ge 0. \tag{1}$$

This is trivial if $y = 0$. If $k > 1$, the right member obviously increases faster than the left member for all $y > 0$. (Differentiate!) The inequality (1) is strict except when $k = 1$ or $y = 0$, as claimed.

II. *Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

We prove a more general inequality. Let $0 < m \le n$, $0 < u \le v$ and $0 \le t \le 1$. Then

$$(1 - t^u)^n \le (1 - t^v)^m \tag{1}$$

with equality if and only if $t = 0$, $t = 1$, or $m = n$ and $u = v$.

If $t \in \{0,1\}$ or $m = n$, $u = v$, (1) holds with equality. Otherwise, we have

Case (i): $u = v$. Then

$$0 < 1 - t^u = 1 - t^v < 1\ ,$$

and so

$$(1 - t^u)^n < (1 - t^v)^m.$$

Case (ii): $u < v$. Then

$$0 < 1 - t^u < 1 - t^v < 1$$

and thus also

$$(1 - t^u)^n < (1 - t^v)^m.$$

Done for (1)!

Putting in (1) $u = m$, $v = n$ and $t = a/b$ we get the original inequality (with equality if and only if $m = n$ or $a = b$).

Also solved by SEUNG–JIN BANG, Seoul, Korea; RICHARD I. HESS, Rancho Palos Verdes, California; MURRAY S. KLAMKIN, University of Alberta; KEE–WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; M.A. SELBY, University of Windsor; ROBERT E. SHAFER, Berkeley, California; COLIN SPRINGER, student, University of Waterloo; C. WILDHAGEN, Breda, The Netherlands; and the proposers.

The proposers remark that a special case of this problem appeared as problem 1188 in Mathematics Magazine *Vol. 57 no. 2 (1984) p. 109.*

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<u>1340</u>. [1988: 110] *Proposed by Jordi Dou, Barcelona, Spain.*

Given are lines a, b, m and a point P, all in the same plane. Find a line r through P such that the point $r \cap m$ is the midpoint of the segment joining $r \cap a$ and $r \cap b$.

Solution by H. Fukagawa, Aichi, Japan.

We use oblique coordinates.

Case (i): $a \cap b = O$ $(a \cap b \neq \phi)$.

We put a, b, and O as the x-axis, y-axis, and origin, respectively. Let

$$l : \frac{x}{x_1} + \frac{y}{y_1} = 1$$

be any line passing through $P(\alpha,\beta)$, where l cuts a in $A(x_1,0)$ and b in $B(0,y_1)$. Then

$$\frac{\alpha}{x_1} + \frac{\beta}{y_1} = 1,$$

and the midpoint of AB is $Q(x_1/2,y_1/2)$. The locus of $Q(x,y)$ is then the hyperbola

$$\frac{\alpha}{2x} + \frac{\beta}{2y} = 1.$$

Let the given line m meet this hyperbola in M. The desired line is then the line passing through P and M.

Case (ii): $a \cap b = \phi$.

In this case, the locus of the midpoint Q is the line L parallel to (and midway between) a and b. Let the given line m meet with L in M. The desired line is the line passing through P and M.

Also solved by J.T. GROENMAN, Arnhem, The Netherlands; L.J. HUT, Groningen, The Netherlands; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; COLIN SPRINGER, student, University of Waterloo; and the proposer. One other reader merely pointed out that the construction is not possible in special cases (which is not unusual in these problems).

Smeenk mentions that the Euclidean construction of the intersection of a hyperbola and a straight line (e.g. to find the point M in the above solution) may be considered well known.

* * *

1341. [1988: 140] *Proposed by Stanley Rabinowitz, Alliant Computer Systems Corp., Littleton, Massachusetts.*

An ellipse has center O and the ratio of the lengths of the axes is $2 + \sqrt{3}$. If P is a point on the ellipse, prove that the (acute) angle between the tangent to the ellipse at P and the radius vector PO is at least 30°.

I. *Solution by J.T. Groenman, Arnhem, The Netherlands.*

The parametric equation of the ellipse can be

$$x = (2+\sqrt{3})\cos t\ , \quad y = \sin t.$$

Thus the slope of the radius vector OP is

$$m_1 = \frac{\sin t}{(2+\sqrt{3})\cos t} = (2-\sqrt{3})\tan t$$

and the slope of the tangent line at P is

$$m_2 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-(2+\sqrt{3})\sin t} = -\frac{2-\sqrt{3}}{\tan t}\ .$$

If φ is the acute angle between these two lines,

$$\tan\varphi = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{(2-\sqrt{3})(\tan t + 1/\tan t)}{1-(2-\sqrt{3})^2}$$

$$= \frac{2-\sqrt{3}}{-6+4\sqrt{3}}\left(\tan t + \frac{1}{\tan t}\right)\ .$$

The minimum value of $\tan t + 1/\tan t$ is 2, when $\tan t = 1$. Thus

$$\tan\varphi \geq \frac{4-2\sqrt{3}}{-6+4\sqrt{3}} = \frac{4-2\sqrt{3}}{\sqrt{3}(4-2\sqrt{3})} = \frac{1}{\sqrt{3}}\ ,$$

so $\varphi \geq 30^\circ$.

II. *Generalization by Murray S. Klamkin, University of Alberta.*

We consider the problem of finding the maximum acute angle the normal to an n-dimensional ellipsoid at any point P can make with the radius vector from the center O of the ellipsoid to P. For $n = 2$ we will obtain the complement of the angle in the above problem.

For simplicity we first will treat the problem in E^3 for the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

If P has coordinates (h,k,l), the radius vector OP has direction numbers (h,k,l) while the normal has direction numbers

$$\left(\frac{h}{a^2}, \frac{k}{b^2}, \frac{l}{c^2}\right)\ .$$

Thus the acute angle ϕ between them is given by

$$\cos\phi = \frac{h^2/a^2 + k^2/b^2 + l^2/c^2}{\sqrt{h^2+k^2+l^2}\sqrt{h^2/a^4+k^2/b^4+l^2/c^4}}$$

$$= \frac{1}{\sqrt{h^2+k^2+l^2}\sqrt{h^2/a^4+k^2/b^4+l^2/c^4}}\ .$$

Hence we want to maximize

$$(h^2 + k^2 + l^2)\left(\frac{h^2}{a^4} + \frac{k^2}{b^4} + \frac{l^2}{c^4}\right)$$

subject to

$$\frac{h^2}{a^2} + \frac{k^2}{b^2} + \frac{l^2}{c^2} = 1 .$$

We can assume, by symmetry of the ellipsoid, that $h, k, l \geq 0$. Now letting

$$\left(\frac{h^2}{a^2}, \frac{k^2}{b^2}, \frac{l^2}{c^2}\right) = (x,y,z) ,$$

we want to maximize

$$(a^2x + b^2y + c^2z)\left(\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2}\right) \tag{1}$$

subject to

$$x + y + z = 1 , \qquad x, y, z \geq 0.$$

Expanding out (1) and subtracting $(x + y + z)^2 = 1$, we equivalently want to maximize

$$\alpha^2yz + \beta^2zx + \gamma^2xy \tag{2}$$

where

$$\alpha = \left|\frac{b}{c} - \frac{c}{b}\right| , \qquad \beta = \left|\frac{c}{a} - \frac{a}{c}\right| , \qquad \gamma = \left|\frac{a}{b} - \frac{b}{a}\right| .$$

We first show that if α, β, γ are the sides of a non–obtuse triangle, there is a nice geometric interpretation other than the original one the problem came from. Using the polar moment of inertia inequality (see [1989: 28]) for a triangle of sides α, β, γ and circumradius R, with weights $w_1 = x$, $w_2 = y$, $w_3 = z$ at the vertices, and with P being the circumcenter O, we have (using $x + y + z = 1$) that

$$R^2 \geq \alpha^2yz + \beta^2zx + \gamma^2xy , \tag{3}$$

with equality if and only if the centroid of the weighted vertices coincides with the circumcenter. This equality condition, which maximizes (2), gives a system of linear equations for the weights x, y, z.

If the triangle is right–angled with hypotenuse γ then the maximum of (2) is obtained for $x = y = 1/2$, since the circumcenter lies at the midpoint of the hypotenuse. If the triangle is obtuse, one of the weights will have to be negative since O lies outside the triangle. For this case, as well as the case that α, β, γ do not form a triangle, we cannot use (3). However, the solution will still correspond to $x = y = 1/2$. To see this, let

$$\gamma^2 = \alpha^2 + \beta^2 + e^2.$$

Then

$$\alpha^2yz + \beta^2zx + \gamma^2xy = (1 - x - y)(\alpha^2y + \beta^2x) + (\alpha^2 + \beta^2 + e^2)xy$$
$$= \alpha^2y(1 - y) + \beta^2x(1 - x) + e^2xy$$

and this last expression clearly takes on its maximum for $x = y = 1/2$.

If we remove the condition that x, y, z are non-negative, then our solution (3) is still valid as long as α, β, γ are sides of a triangle. However, if they do not form a triangle, say $\gamma = \alpha + \beta + e$ with $e \geq 0$, then there is no maximum value for (2). For in this case

$$\alpha^2yz + \beta^2zx + \gamma^2xy = \alpha^2y + \beta^2x + (e^2 + 2e(\alpha + \beta))xy - (\alpha y - \beta x)^2,$$

and this can be made arbitrarily large by taking x, y arbitrarily large with $\alpha y = \beta x$.

For higher dimensions, $n > 3$, the solution becomes more involved. Here we want to maximize

$$\sum \alpha_{ij}^2 x_i x_j \qquad (4)$$

where the sum is over all $1 \leq i < j \leq n$, the α_{ij}'s are given, $\sum x_i = 1$, and $x_i \geq 0$ for all i. Again by the polar moment of inertia inequality we obtain

$$R^2 \geq \sum \alpha_{ij}^2 x_i x_j \ .$$

Thus again the square of the circumradius is the maximum value of (4) provided the simplex whose corresponding edges are the α_{ij}'s is "acute", i.e. the circumcenter does not lie outside the simplex. In case the simplex is non-acute or the α_{ij}'s are not possible edges of a simplex, we have to consider endpoint extrema as before. This reduces the problem to the same one in dimension $n - 1$. In particular, for $n = 4$, we first consider each of the four problems arising from setting one of x_1, x_2, x_3, x_4 equal to 0. Then we consider setting each pair of the x_i's equal to 0. Finally we take the maximum value of (4) over all these 10 cases.

Also solved by JORDI DOU, Barcelona, Spain; C. FESTRAETS-HAMOIR, Brussels, Belgium; H. FUKAGAWA, Aichi, Japan; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta (two solutions); KEE-WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; DAVID VAUGHAN, Wilfrid Laurier University, Waterloo, Ontario; C. WILDHAGEN, Breda, The Netherlands; and the proposer. There was one incorrect solution sent in.

Several solvers pointed out more generally that if the ratio of the lengths of the axes of an ellipse is $\lambda > 1$ then the minimum angle between the tangent and radius vector is

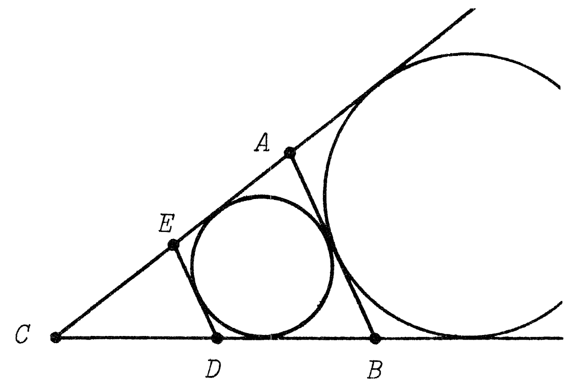
$$\tan^{-1}\left(\frac{2\lambda}{\lambda^2 - 1}\right) = \cos^{-1}\left(\frac{\lambda^2 - 1}{\lambda^2 + 1}\right) .$$

* * *

1342. [1988: 140] *Proposed by J.T. Groenman, Arnhem, The Netherlands.*

Let ABC be a triangle and let D and E be the midpoints of BC and AC respectively. Suppose that DE is tangent to the incircle of ΔABC. Prove that $r_c = 2r$, where r is the inradius of ΔABC and r_c is the exradius to AB.

I. *Solution by Maria Ascensión Lopez Chamorro, I.B. Leopoldo Cano, Valladolid, Spain.*



The triangles ABC and CDE are homothetic with ratio 1/2. The incircle of ΔABC is the excircle to DE of ΔCDE, and therefore, in the homothety with center C and ratio 2, the incircle of ΔABC, with radius r, transforms into the excircle of ΔABC, with radius r_c. Thus $r_c = 2r$.

II. *Solution by Svetoslav Jor. Bilchev, Technical University, Russe, Bulgaria.*

Since DE is tangent to the incircle of ΔABC, $AB + DE = BD + AE$, i.e.

$$c + \frac{c}{2} = \frac{a}{2} + \frac{b}{2} ,$$

or

$$c = \frac{s}{2} ,$$

where s is the semiperimeter of ΔABC. Further, it is well-known that

$$r_c = \frac{F}{s - c}$$

where F is the area of ΔABC, and hence

$$r_c = \frac{F}{s - c} = \frac{F}{s/2} = \frac{2F}{s} = 2r.$$

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; JORDI DOU, Barcelona, Spain; C. FESTRAETS–HAMOIR, Brussels, Belgium; JACK GARFUNKEL, Flushing, N.Y.; L.J. HUT, Groningen, The Netherlands; KEE–WAI LAU, Hong Kong; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; P. PENNING, Delft, The Netherlands; D.J. SMEENK, Zaltbommel, The Netherlands; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer.

The neat first solution (I) was also found by Dou, Penning, and Smeenk. All other solutions (including a second solution by Smeenk) were similar to solution II.

Hut showed more generally that if $CD = CA/n$ and $CE = CB/n$ and DE is tangent to the incircle of ΔABC then $c = (1 - 1/n)s$ and $r_c = nr$. This can be proved by the above methods.

The proposer also showed that the angle C of the problem satisfies

$$C \le \cos^{-1}(7/9) \approx 38.9^\circ.$$

* * *

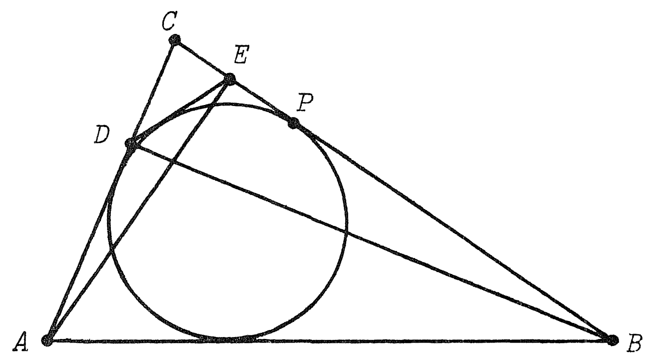
1343. [1988: 140] *Proposed by D.J. Smeenk, Zaltbommel, The Netherlands.*

ABC is an acute triangle and D, E are the feet of the altitudes to BC, AC respectively. Suppose DE is tangent to the incircle. Show that $r_c = 2R$, where R is the circumradius and r_c is the exradius to AB.

Solution by P. Penning, Delft, The Netherlands.

ΔCDE is similar to ΔABC, because

$$\frac{CE}{CA} = \frac{CD}{CB} = \cos\ C.$$



Note that the incircle of ΔABC is the excircle to DE of ΔCDE. Thus, with s the semiperimeter of ΔABC, the semiperimeter of ΔCDE is equal to

$$s\cos C = CP = s - c.$$

Eliminating c with $c = 2R\sin C$, one obtains

$$R = \frac{s(1 - \cos\ C)}{2\ \sin\ C} = \frac{s}{2}\tan\left(\frac{C}{2}\right) .$$

The radius r_c of the excircle to AB is equal to $s\ \tan(C/2)$, so

$$r_c = 2R.$$

Remark. It can be shown that, for this type of triangle,

$$1 + \cos\ C = \cos\ A + \cos\ B$$

and

$$76.345^\circ \approx 2\ \sin^{-1}\left(\frac{\sqrt{5} - 1}{2}\right) \le C \le 90^\circ.$$

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, and MARIA ASCENSIÓN LOPEZ CHAMORRO, I.B. Leopoldo Cano, Valladolid, Spain; SVETOSLAV JOR. BILCHEV, Technical University, Russe, Bulgaria; JORDI DOU, *Barcelona, Spain; C. FESTRAETS–HAMOIR, Brussels, Belgium; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE–WAI LAU, Hong Kong; T. SEIMIYA, Kawasaki, Japan; DAN SOKOLOWSKY, Williamsburg, Virginia; and the proposer.*

Dou points out that his problem Crux *1321 [1989: 116] also gave a condition equivalent to* $r_c = 2R$.

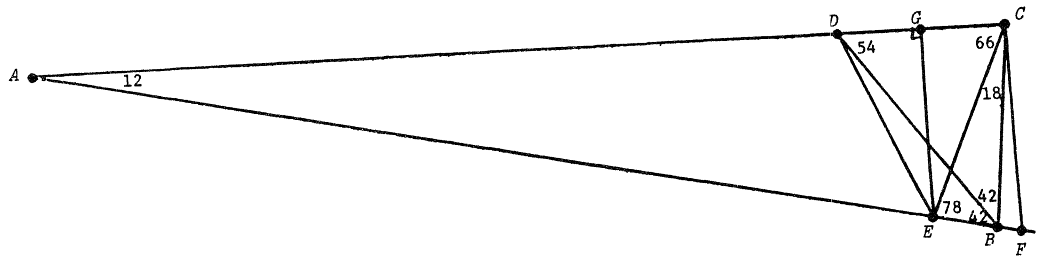
* * *

1346. [1988: 141] *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let ABC be an isosceles triangle with $AB = AC$ and $\angle A = 12^\circ$. Let D on AC and E on AB be such that $\angle CBD = 42^\circ$ and $\angle BCE = 18^\circ$. Prove that $\angle EDB = 12^\circ$. (This problem came to me via a student; I don't know the source.)

Solution by D.J. Smeenk, Zaltbommel, The Netherlands.

The following angles are easily found.



Assume $AB = AC = 1$, so $BC = 2\sin 6^\circ$. Let $F \in AB$ be such that $CF \perp AC$; then

$$\angle CFE = 78^\circ = \angle CEF,$$

so

$$CE = CF = \tan 12^\circ.$$

Since BD is the angle bisector of $\angle ABC$,

$$CD = \frac{CD}{AC} = \frac{BC}{AB + BC} = \frac{2\sin 6^\circ}{1 + 2\sin 6^\circ}$$

$$= \frac{\sin 6^\circ}{\sin 30^\circ + \sin 6^\circ} = \frac{\sin 6^\circ}{2\sin 18^\circ \cos 12^\circ}.$$

Let $G \in AC$ so that $EG \perp AC$. We claim that $GC = GD$, or equivalently

$$\tan 12^\circ \cos 66^\circ = CE\cos 66^\circ = CG = \frac{CD}{2} = \frac{\sin 6^\circ}{4\sin 18^\circ \cos 12^\circ},$$

$$4\sin 12^\circ \sin 18^\circ \sin 24^\circ = \sin 6^\circ,$$

$$8 \cos 6^\circ \sin 18^\circ \sin 24^\circ = 1,$$
$$4 \sin 18^\circ(2 \sin 24^\circ \cos 6^\circ) = 1,$$
$$4 \sin 18^\circ(\sin 30^\circ + \sin 18^\circ) = 1,$$

and finally

$$2 \sin 18^\circ + 4 \sin^2 18^\circ = 1,$$

and this holds, as

$$\sin 18^\circ = \frac{-1 + \sqrt{5}}{4}.$$

So $CG = GD$, which implies $\angle EDG = \angle ECG = 66^\circ$ and thus

$$\angle EDB = 66^\circ - 54^\circ = 12^\circ.$$

Also solved by FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; C. FESTRAETS–HAMOIR, Brussels, Belgium; JACK GARFUNKEL, Flushing, N.Y.; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; L.J. HUT, Groningen, The Netherlands; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; KEE–WAI LAU, Hong Kong; P. PENNING, Delft, The Netherlands; T. SEIMIYA, Kawasaki, Japan; and the proposer.

An article by Colin Tripp on "adventitious angles" and two follow–up reports (Mathematical Gazette *Vol. 59 (1975) 98–106, Vol. 61 (1977) 55–58, Vol. 62 (1978) 174–183) give all solutions in integers for the four given angles when the triangle is isosceles. This type of problem seems to have first appeared as a problem in the* Mathematical Gazette *in 1922 and has occurred in many places since. Related* Crux *problems are* Crux *134 [1976: 151, 173, 222; 1977: 12, 44] and* Crux *255 [1978: 52]. Thanks go to Bellot Rosado and Seimiya for kindly sending in this information.*

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1347. [1988: 141] *Proposed by Lanny Semenko, Erewhon, Alberta.*

The positive integer 275 has the property that

$$275^\circ\mathrm{C} = 527^\circ\mathrm{F},$$

where 527 is obtained by moving the rightmost digit of 275 to the left end. Find another positive integer with this property.

Solution by Kenneth M. Wilke, Topeka, Kansas.

Let $F = 10^n a + b$ for integers a and b with $0 < a < 10$. Then $C = 10b + a$. But

$$F = \frac{9}{5}C + 32$$

or

$$5F = 9C + 160. \tag{1}$$

Substituting the expressions for F and C into (1) we get

$$5\cdot 10^n a + 5b = 90b + 9a + 160$$

or

$$a(5\cdot 10^n - 9) = 85b + 160. \tag{2}$$

From (2), 5 must divide $a(5\cdot 10^n - 9)$; hence $a = 5$. Then

$$5\cdot 10^n - 41 = 17b,$$

$$5\cdot 10^n \equiv 41 \equiv 7 \bmod 17,$$

and hence

$$10^n \equiv 15 \bmod 17$$

or

$$10^{n-2} \equiv 1 \bmod 17. \tag{3}$$

The first solution occurs for $n = 2$, whence $F = 527$ and $C = 275$. Since 10 is a primitive root mod 17, the next solution occurs for $n = 18$ whence

$$F = 5294117647058823527,$$

$$C = 2941176470588235275.$$

An infinite number of such pairs can be found by noting that all solutions of (3) are given by $n = 2 + 16t$ for some nonnegative integer t.

Also solved by FRANK P. BATTLES and LAURA L. KELLEHER, Massachusetts Maritime Academy, Buzzards Bay, Massachusetts; C. FESTRAETS–HAMOIR, Brussels, Belgium; J.T. GROENMAN, Arnhem, The Netherlands; RICHARD I. HESS, Rancho Palos Verdes, California; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; FRIEND H. KIERSTEAD JR., Cuyahoga Falls, Ohio; MURRAY S. KLAMKIN, University of Alberta; SIDNEY KRAVITZ, Dover, New Jersey; KEE–WAI LAU, Hong Kong; SAM MALTBY, student, Calgary; J.A. MCCALLUM, Medicine Hat, Alberta; DAVE MCDONALD, Crimson Elk, Alberta; P. PENNING, Delft, The Netherlands; H.J. MICHIEL WIJERS, Helmond, The Netherlands; C. WILDHAGEN, Breda, The Netherlands; and the proposer.

Hess, Kierstead, and Wijers pointed out that the same problem was proposed by Brian Barwell in 1987 as problem 1573 of the Journal of Recreational Mathematics *(solution in Vol. 20 no. 3 (1988), pp. 234–235). The editor thanks these readers for their information, and will try to keep a closer watch on the "competition" in future!*

* * *

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