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*On real numbers generated by finite automata*

The  $b$ -adic expansion of any rational number is eventually periodic, but how regular or random (depending on the viewpoint) is the  $b$ -adic expansion of an irrational algebraic number? It seems that this natural question has been first addressed by Borel in 1950, who made the conjecture that such an expansion should satisfy some of the laws shared by almost all real numbers. Algebraic irrationals are thus expected to be “rather complex” in a probabilistic framework.

In 1965, Hartmanis and Stearns proposed an alternative approach of the notion of complexity of real numbers, by emphasizing the quantitative aspect of the notion of calculability introduced by Turing. According to them, a real number is said to be computable in time  $T(n)$  if there exists a multitape Turing machine which gives the first  $n$ -th terms of its binary expansion in (at most)  $T(n)$  operations. The “simplest” real numbers in that sense, that is, for which one can choose  $T(n) = O(n)$ , are said to be computable in real time. The problem of Hartmanis and Stearns, to which a negative answer is expected, is the following: do there exist irrational algebraic numbers which are computable in real time?

In 1968, Cobham suggested to restrict this problem to a particular class of Turing machines, namely to the case of finite automata. After several attempts, Loxton and van der Poorten finally claimed to have completely solved the restricted problem in 1988. Unfortunately, their proof, which rests on a method introduced by Mahler, contains a rather serious gap, as it has been observed by Becker in 1993.

In this talk, I will report on the recent proof of the Cobham–Loxton–van der Poorten conjecture mentioned above. This result has been obtained in a joint paper with Yann Bugeaud and Florian Luca. A positive answer to a conjecture of Shallit, concerning the Diophantine properties of real numbers generated by finite automata, will also be given. This last result has been obtained in a joint work with Julien Cassaigne.