Analytic and Combinatorial Aspects in Convex and Discrete Geometry
Aspects analytiques et combinatoires en géométrie convexe et discrète
(Org: **Károly Bezdek** (University of Calgary, Canada), **Fodor Ferenc** (University of Szeged, Hungary and University of Calgary, Canada) and/et **Alina Stancu** (Concordia University, Canada))

MARIA ALFONSECA, North Dakota State University
On planar bodies of variable density that float in equilibrium in all orientations

In 1935, Ulam asked if a convex body of uniform density which floats in equilibrium in every orientation must be a sphere. The answer has been shown to be negative for bodies of constant density 1/2. In this talk we show that any star body may be made to float in equilibrium in all orientations if it has a specific variable density, which we determine explicitly. We also consider the case, in two dimensions, of a non-compactly supported, non-radial density that achieves the same result. This is joint work with Stephanie Juntunen.

GERGELY AMBRUS, University of Szeged, and Alfréd Rényi Institute of Mathematics
Softness properties of tilings

We study various softness properties of tilings of $\mathbb{R}^d$ that are homeomorphic images of polyhedral tilings. Given a topological cell C of $\mathbb{R}^d$, a boundary point $x \in \partial C$ is called k-*soft* if ∂C locally contains a k-dimensional smooth manifold through x, while it is k-*sharp* otherwise. The main question we study is the following: for given $2 \leq d$ and $1 \leq k \leq d-1$, what is the infimum of the average number of k-sharp vertices per cell in homeomorphic images of sufficiently regular polyhedral tilings of $\mathbb{R}^d$? Domokos, G. Horváth and Regős showed that in dimension two, every such tiling has at least two 1-sharp vertices per cell on average, and this bound is sharp. In $\mathbb{R}^3$, the answer depends on the value of k. On the one hand, we prove that every normal, locally polyhedral tiling of $\mathbb{R}^3$ can be transformed into a *completely* 1-*soft* tiling. On the other hand, we show that the average number of 2-sharp vertices per cell must be at least $10/3$, and there exists a sequence of periodic examples approaching this bound. Our proof combines variants of Euler's formula for polyhedral complexes with the signature theorem of Izmestiev and Schlenker. The method also implies a sharp weak lower bound for the *harmonic degree* of the tilings, that was introduced by Domokos and Lángi.

This is a joint work with Dorottya Dancsó (University of Szeged).

KÁROLY BEZDEK, University of Calgary
Notes on uniform contractions of congruent balls

The talk is centered around stronger results on uniform contractions of intersections and unions of congruent balls, thereby extending previous partial confirmations of the Kneser-Poulsen conjecture in these settings. In addition, we establish Blaschke-Santalo-type inequalities for r-ball bodies.

KÁROLY J. BÖRÖCZKY, Alfred Renyi Institute of Mathematics
Keith Ball's conjecture on an L2 version of the Blaschke-Santalo inequality

The proof of Keith Ball's four decades old conjecture on an L2 version of the Blaschke-Santalo inequality is presented. Joint work with Christos Saroglou and Konstantinos Patsalos.

DMITRY FAIFMAN, Université de Montréal
The B-theorem for the first Dirichlet eigenvalue

The B-conjecture, first proved by Brascamp and Lieb, asserts the log-concavity of the Gaussian measure of a centrally-symmetric convex body under linear transformations. Using Brownian motion and the Kac formula, we deduce a corresponding log-convexity property of the first Dirichlet eigenvalue of convex domains. Notably, this log-convexity persists even without central symmetry. We then deduce the uniqueness of the Faber-Krahn position minimizing the first eigenvalue, thus answering a question of Schmuckenschlager and obtaining a spectral analogue of the John position. We also discuss some other related inequalities. Joint work with Iosif Polterovich.

FERENC FODOR, University of Szeged, Hungary
Random points in ball-convex bodies: Sylvester's problem and beyond

We study Sylvester's classical four-point problem for ball-convex bodies, where convex hulls are replaced by intersections of closed balls of fixed radius R. We show that the Efron–Buchta identities extend to this setting.

We determine the corresponding Sylvester probabilities for three and four uniform random points in an arbitrary planar ball-convex body. We also extend Groemer's inequality, showing that the expected area of the ball-convex hull is minimised by the disc of equal area.

For the unit disc, we obtain formulas involving dilogarithmic functions in terms of R, recover the classical Sylvester probabilities as $R \to \infty$, and derive the distribution function of the circumradius of three uniform random points on $[1, \infty)$.

Joint work with A. Bakó–Szabó (University of Szeged, Hungary) and F. Besau (Technical University of Vienna, Austria).

KYLE FRIDBERG, Cornell University
Density bounds for connected components on the square grid

Consider the square grid ($\mathbb{Z}^2$) under the two natural notions of nearest-neighbor connectivity (4 and 8-connectivity). For any natural number k, what fraction of the grid can be covered by connected components of size at most k? We give a simple exact answer in the 8-connected case, along with a natural construction that achieves it. In the 4-connected case, the situation is far more subtle; we construct a bound that is tight for at least $1/2$ of all natural numbers k. Our results further generalize to handle any prescribed set of connected components and rectangular subsets of the square grid. This improves upon prior work of Gravier and Payan (2001). Finally, we discuss a continuous analogue related to the Honeycomb theorem.

BALÁZS GRÜNFELDER, University of Szeged, Hungary
Intrinsic volume moments of random beta-prime polytopes

We consider random beta-prime polytopes, which are convex hulls of i.i.d. beta-prime distributed random points in $\mathbb{R}^d$. After a suitable scaling, these polytopes converge in distribution to the convex hulls of Poisson point processes with power-law intensity density functions. We prove convergence of moments for the intrinsic volumes of random beta-prime polytopes.

We also consider a related spherical model in the upper open hemisphere of S^d. We choose i.i.d. random points in S^d_+ according to the uniform distribution or certain other rotationally symmetric distributions around the north pole of the hemisphere. The spherical convex hull of the random points is a spherical random polytope. After gnomonic projection, the push-forwards of such polytopes are beta-prime polytopes. We prove moment convergence of the spherical volume of the spherical random polytopes.

Joint work with Ferenc Fodor and Péter Kevei (both from the University of Szeged).

JULIÁN HADDAD, Universidad de Sevilla
Approximate counterexamples to the Barker–Larman problem in dimension 4

Consider a convex body $K \subseteq \mathbb{R}^n$ containing the unit Euclidean ball B, and the hyperplanes H tangent to B.

The Barker–Larman problem asks whether K must be a Euclidean ball if the sections by all these hyperplanes H have constant $n-1$ dimensional volume.

The problem for bodies of revolution in dimension $n = 4$ has a very special structure, and the answer is very likely NO.

We show a way to construct approximate counterexamples as solutions to a bifurcation equation in the space of periodic functions.

PAVLOS KALANTZOPOULOS, University of Waterloo

Commutativity in a Poincaré-Type Inequality

Can a geometric test reveal whether two matrices commute? Motivated by the log-Brunn–Minkowski conjecture, we study the Poincaré-type inequality

$$\mathrm{Var}_{\mu_{K,\Sigma}}\left(\langle A\Sigma^{-1}y, y\rangle\right) \leq \int_{\mathbb{R}^n} \left\langle \left(A^2\Sigma^{-1} + A\Sigma^{-1}A\right) y, y\right\rangle \, d\mu_{K,\Sigma}(y).$$

Here A is symmetric, and $\mu_{K,\Sigma}$ is the centered Gaussian measure with positive-definite covariance Σ, restricted and normalized on an origin-symmetric convex body K. We show that the inequality holds for every origin-symmetric orthogonal box K exactly when A and Σ commute. However, the inequality holds without commutativity if K is unconditional with respect to an orthonormal eigenbasis of Σ.

BERNARDO GONZÁLEZ MERINO, Universidad de Murcia

Sharp stability estimates near the simplex and applications in Convex Geometry

Stability estimates quantify how close a convex body must be to an extremal configuration when it nearly attains equality in a geometric inequality. In this talk, we establish a sharp, dimension-independent stability estimate for the Minkowski asymmetry near its maximal value, attained precisely by simplices.

More precisely, we show that if an n-dimensional convex body K satisfies $s(K) \geq n - \varepsilon$, for some $\varepsilon \in [0, 1)$, then its Banach–Mazur distance to an n-simplex S^n satisfies

$$d_{\mathrm{BM}}(K, S^n) \leq 1 + \varepsilon + \frac{\varepsilon^2}{2(1-\varepsilon)}.$$

This improves previous estimates both in the admissible range of the error and in the strength of the bound, and is optimal to linear order in ε.

We then explore several applications of this stability principle. First, we obtain a sharp stability result for the maximal Banach–Mazur distance to the Euclidean ball, together with a new relation between this distance and the Minkowski asymmetry. Second, we establish a common generalization of Schneider's higher-order Rogers–Shephard inequality and the L_p-Rogers–Shephard inequality, along with stability estimates of optimal linear order. Finally, we derive improved upper bounds for the diameter of the Banach–Mazur compactum in fixed dimensions.

This is a joint work with René brandenberg and Florian Grundbacher.

STEPHANIE MUI, Georgia Institute of Technology

On the Minkowski problem with respect to a mixed Euclidean-Gaussian density

I will discuss a new class of Minkowski problems corresponding to the mixed Euclidean-Gaussian volume, inspired by the isoperimetric problem studied by Fusco-Maggi-Pratelli (2011). Using the flow method, we prove the existence of smooth normalized solutions for the corresponding Monge-Ampere-type equation in the symmetric case. By approximation, we obtain the existence of origin-symmetric solutions to the mixed Euclidean-Gaussian Minkowski problem. This is joint work with Lei Qin and Sinan Wang.

SERHII MYROSHNYCHENKO, University of the Fraser Valley

Two Centres of Mass, One Sharp Bound

In their book "Unsolved Problems in Geometry", Croft, Falconer, and Guy asked how far apart the centroids of a planar convex body and its boundary can be relative to the body's width or diameter (Problem A25). They also raised the question in higher dimensions. Nazarov, Ryabogin, and Yaskin settled the planar case and conjectured the optimal bound in all dimensions.

We prove this conjecture. For every convex body $K \subset \mathbb{R}^n$, $n \geq 2$, and every unit vector u, the volume centroid $c(K)$ and the surface-area centroid $c(\partial K)$ satisfy

$$\left|\langle c(\partial K) - c(K), u\rangle\right| \leq D_n \, w_K(u),$$

where $w_K(u)$ is the width of K in direction u. The constant D_n is optimal in both this inequality and the resulting diameter bound. The same sharp constant appears in the analogous width bound for the distance between the orthogonal projection of $c(K)$ onto a hyperplane and the centroid of the projected body.

The proof compares uniform and length-weighted averages of the midpoints of parallel chords. The Kannan–Lovász–Simonovits localization lemma reduces the comparison to a one-dimensional inequality. A family of polytopes converging to a line segment establishes sharpness.

This is joint work with Dmitry Ryabogin, Kateryna Tatarko, and Vladyslav Yaskin.

BRANDON OLIVA, Case Western Reserve University
Illumination bodies for ball-convex bodies.

We introduce illumination bodies in the class of n-dimensional R-ball convex bodies. We will prove that the illumination bodies are convex, and that a left derivative of volume gives rise to a surface area measure for ball-convex bodies, the relative surface area measure. This talk is based on a paper titled "Illumination bodies for ball-convex bodies" by Elisabeth Werner, Diliya Yalikun, and Brandon Oliva.

LEI QIN, University of Montreal
A counterexample to quasi-concavity for the Gaussian p-Capacity in convex rings.

In classical theory, the Capacitary potential of a convex ring has convex level sets. In this talk, we show that the analogous quasi-concavity property fails for Gaussian p-Capacity. More precisely, we construct a convex ring whose inner and outer boundaries are both strictly convex, but whose Gaussian p-capacitary potential possesses a nonconvex intermediate level set.

ADAM SAGMEISTER, Bolyai Institute, University of Szeged & Alfréd Rényi Institute of Mathematics
Inequalities for ball convex bodies in planes of constant curvature

Bodies of constant width and reduced convex bodies have been studied broadly, as many of the extremum problems in convex geometry have their solutions within these classes of convex bodies. In this talk, we connect these concepts through ball convexity, and introduce the notion of r-ball reducedness. We discuss a ball convex extension of both Pál's classical isominwidth inequality and the Blaschke–Lebesgue inequality as well as its optimal stability. We also characterize r-ball reduced polygons, and prove several inequalities regarding their areas, perimeter, diameter, inradius and circumradius. The talk is based on joint works with Ferenc Fodor and Nathan Robock.

ETHAN SAUNDERS, University of Calgary
Colouring Line Tilings

Let L be a collection of finitely many straight lines in general position in $\mathbb{E}^2$. L naturally partitions the plane into 2-dimensional faces, 1-dimensional faces, and 0-dimensional faces and we call this partition a *line tiling* of $\mathbb{E}^2$. We consider the problem of colouring the 1-faces of a line tiling such that 1-faces that are incident at a 0-face are assigned different colours. We show that four colours are always nessecary and sufficient to colour the 1-faces of a line tiling.

EGON SCHULTE, Northeastern University
Symmetric Point-Circle Configurations in Space

A point-circle configuration in Euclidean 3-space is said to be point-circle transitive if any two incident point-circle pairs are equivalent under the symmetry group of the configuration. The talk presents progress on the classification of all point-circle transitive circle configurations in space. This is joint work with Jieying Jin and Deborah Oliveros.

VIKTOR VÍGH, University of Szeged
On Kuperberg's quadrilateral conjecture

Kuperberg conjectured in 1983 that every convex body K in the plane is contained in a quadrilateral of area at most $\frac{3}{\sqrt{5}}\,|K|$, the extremal bodies being the affine-regular pentagons. On the basis of numerical experiments we propose a stronger conjecture of a purely polygonal nature: every convex polygon P is contained in a quadrilateral of area at most $\frac{3}{\sqrt{5}}\,|P|$ whose sides are parallel to sides or diagonals of P. This chord conjecture implies the inequality conjectured by Kuperberg. For pentagons it is a theorem of Hong, Ismailescu, Kwak and Park, of which we give a short proof by area identities. We also show that the classical bound $\sqrt{2}$ remains valid for quadrilaterals with sides parallel to chords; the proof is similar to earlier proofs of this bound, but it starts from an inscribed quadrilateral of maximal area instead of a circumscribed quadrilateral of minimal area. We close with a discussion of the difficulties in reaching the constant $\frac{3}{\sqrt{5}}$.

TINGZHOU YU, University of Alberta
On the rank of random symmetric matrices

How likely is an $n \times n$ random symmetric matrix to have rank at most $n - k$? For $k = 1$, Campos, Jenssen, Michelen and Sahasrabudhe proved in a breakthrough paper that the singularity probability for symmetric Rademacher matrices is exponentially small. More recently, Han went beyond singularity by proving a large deviation inequality for the rank: in the symmetric subgaussian model, the probability of rank at most $n - k$ is bounded above by e^{-cnk} for $1 \le k \le c'\sqrt{n}$.

In this talk, I will present an extension of Han's result. Under weaker assumptions on the entry distributions, we obtain an upper bound of $2e^{-cnk}$ for the full range $1 \le k \le n$. For the adjacency matrix of the Erdős–Rényi random graph $G(n, p)$ with $p = p(n)$ uniformly bounded away from 0 and 1, the case $k = 1$ confirms a conjecture of Costello and Vu.

This is joint work with Huaijin Liang and Kexin Yu.

SHIRA ZERBIB, Iowa State University
Piercing convex sets with hyperplanes

We discuss the number of hyperplanes needed to pierce a finite family of convex sets in $\mathbb{R}^d$ with a given matching number or other local intersection properties.