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The analytical solution to the multi-term time-fractional diffusion-wave equation

Applying the inverse operator method and the multivariate Mittag-Leffler function, we derive the analytic solution for the following multi-term time-fractional diffusion-wave equation in the Caputo fractional derivative sense:

$$\begin{cases} \frac{{}_c\partial^\rho}{\partial t^\rho} M(t,\sigma) + \sum_{j=1}^{m} \lambda_j \, \frac{{}_c\partial^{\rho_j}}{\partial t^{\rho_j}} M(t,\sigma) = \triangle M(t,\sigma) + g(t,\sigma), \\ M(0,\sigma) = \theta(\sigma), \quad M_t'(0,\sigma) = \beta(\sigma), \end{cases}$$

where $\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial \sigma_i^2}$, all λ_j for $j = 1, 2, \cdots, m$ are arbitrary constants, and $1 < \rho_1 < \rho_2 < \cdots < \rho_m < \rho \leq 2$. In particular if $\lambda_1 = \cdots = \lambda_m = 0$ and $\rho = 2$, then the above equation turns out to be the non-homogeneous wave equation in $\mathbb{R}^n$:

$$\begin{cases} \frac{\partial^2}{\partial t^2} M(t,\sigma) = \triangle M(t,\sigma) + g(t,\sigma), \\ M(0,\sigma) = \theta(\sigma), \quad M_t'(0,\sigma) = \beta(\sigma), \end{cases}$$

which has the uniform solution for all $n \geq 1$

$$M(t,\sigma) = \sum_{s=0}^{\infty} I_t^{2s+2} \triangle^s g(t,\sigma) + \sum_{s=0}^{\infty} \frac{t^{2s}}{(2s)!} \triangle^s \theta(\sigma) + \sum_{s=0}^{\infty} \frac{t^{2s+1}}{(2s+1)!} \triangle^s \beta(\sigma),$$

where I_t^{2s+2} is the Riemann–Liouville fractional partial integral operator. We further show that the solution given above coincides with the classical results, such as d'Alembert and Kirchoff's formulas, with an example demonstrating its power and simplicity.