Applications of Symmetries, Conservation Laws, and Related Algebraic Structures for Nonlinear Partial Differential Equations
Applications des symétries, des lois de conservation et des structures algébriques connexes aux équations aux dérivées partielles non linéaires
(Org: **Stephen Anco** (Brock Univ.), **Kostya Druzhkov** (University of Saskatchewan) and/et **Alexey Shevyakov** (University of Saskatchewan))

STEPHEN ANCO, Brock University
Hidden symmetry groups in classical mechanics and beyond

Integrability and superintegrabitiy are important notions in n-dimensional classical Hamiltonian mechanics. A main example of superintegrable system in 3 dimensions comes from the Kepler potential, which possesses the Laplace-Runge-Lenz (LRL) vector as a global constant of motion in addition to energy and angular momentum. The Poisson bracket algebra of these constants of motion has the structure of the Lie algebra of the group $S0(4)$. This group represents a hidden symmetry structure since it is larger than the kinematic symmetry group $SO(3) \times R$ manifestly given by rotations and time translation.

This talk will explain how a similar hidden symmetry structure exists for all central force systems in n dimensions when local constants of motion are considered. In particular, every such system possesses $2n-1$ local constants of motion, including a generalized LRL vector, plus an integral of motion that depend explicitly on time. The latter quantity will be shown to lead to an enlarged symmetry group structure.A key tool is a version of Noether's theorem holding in the space of configuration variables extended to include time.

Mathematical and physical properties of the generalized LRL vector and the additional integral of motion, along with the associated symmetry group, will be presented.The n-dimensional Kepler potential and isotropic oscillator are used as examples.

GEORGE BLUMAN, UBC Vancouver
Use of the symmetry-based method to construct non-invertible mappings

In this talk we review the recently introduced symmetry-based method to construct a nonlocally related system for any admitted Lie point symmetry of a given PDE. We focus on how to use the symmetry-based method to find non-invertible mappings that relate nonlinear PDEs to linear PDEs; linear PDEs to other linear PDEs.

Two situations arise for the admitted trivial Lie point symmetries of a linear PDE. A third situation arises for an admitted nontrivial Lie point symmetry of a PDE.

STATHIS CHARALAMPIDIS, California Polytechnic State University

JAIDEN DAHLKE, Brock University

KOSTYA DRUZHKOV, University of Saskatchewan

MICHEL GRUNDLAND, CRM, Université de Montréal

WILLY HEREMAN, Colorado School of Mines
Symbolic computation of conservation laws of nonlinear partial differential equations

A direct method will be presented for the symbolic computation of conservation laws of nonlinear PDEs involving multiple space variables and time. Using the scaling symmetries of the PDE, the conserved densities are constructed as linear combinations of scaling homogeneous terms with undetermined coefficients. The variational derivative is used to compute the undetermined coefficients. The homotopy operator is used to invert the divergence operator, leading to an analytic expression of the flux vector.

The method is algorithmic and has been implemented in Mathematica. The software is being used to compute conservation laws of nonlinear PDEs occurring in the applied sciences and engineering. The software package will be demonstrated for PDEs that model, e.g., shallow water waves, ion-acoustic waves in plasmas, sound waves in nonlinear media, transonic gas flow, and stress and displacement in elastic materials. Examples include the Korteweg-de Vries and Zakharov-Kuznetsov equations, and a constitutive equation arising in elasticity.

SHAWN MCADAM, University of Saskatchewan

MAHDIEH MOGHADAM, Brock University

REHANA NAZ, Lahore School of Economics
Lie symmetries, closed-form solutions, and conservation laws

Title: Lie symmetries, closed-form solutions, and conservation laws of a constitutive equation modeling stress in elastic materials and a technology diffusion model

Abstract:

The Lie-point symmetry method is used to find some closed-form solutions for a constitutive equation modeling stress in elastic materials and a technology diffusion model.

The Lie algebra for the governing PDE system for a constitutive equation modeling stress in elastic materials is five-dimensional. Using the optimal system of one-dimensional subalgebras, closed-form solutions for the model are obtained. Based on the scaling symmetry of the PDE and using Euler and homotopy operators, several conservation laws are computed with symbolic software.

A critical component of economic growth is growth in productivity which is dependent on technology adoption. While most technologies are created in developed economies, they diffuse to developing economies through various channels such as trade, migration and knowledge spillovers. The first model that integrates compartmental models with diffusion is developed to analyze technology adoption within a framework of a system of second-order non-linear partial differential equations. A three-dimensional Lie algebra is established for a technology diffusion model. The combinations of Lie symmetries are used to obtain reductions and establish closed-form solution for the technology diffusion model. The closed-form solutions allow for graphical representations of the technology diffusion process over an effective distance and time and show the commonly observed S-curve path of technology diffusion. Furthermore, a sensitivity analysis is performed to develop policy insights into the factors influencing the diffusion of technology.

ALEXEY SHEVYAKOV, University of Saskatchewan
Exact Internal Waves in a Two-Fluid System

Although the Euler and Navier-Stokes fluid dynamics equations have been known for over 150 years, modern science is far from fully understanding their analytical properties. Exact and approximate solutions are available in a limited number of simplified cases, while direct numerical simulations are resource-intensive and often lack precision.

Many simplified models have been derived from Euler and Navier-Stokes equations to describe specific phenomena, such as surface and internal waves. These models aim to reduce the complexity of the original equations while preserving essential features of the phenomena and offering physical insight and computational accuracy. Examples include dimension and symmetry reductions, linearizations, and more general approximations based on asymptotic relationships. Fundamental nonlinear partial differential equations of mathematical physics, including Burgers', Korteweg-de Vries, nonlinear Schrödinger, and Kadomtsev-Petviashvili equations, arise in this context. Reduced models often reveal rich mathematical structures; their exact solutions can closely describe physical phenomena.

I will discuss a model of nonlinear internal waves in a stratified system of two non-mixing fluids with different densities, contained in a horizontal channel. This model, developed by Miyata and then Choi and Camassa, was derived through layer-averaging under the "shallow water" assumption, which assumes a small ratio of channel depth to wavelength, without requiring wave amplitudes to be small. I will introduce a transformation that simplifies the Choi-Camassa model, reducing it to a simpler dimensionless form, and demonstrate that the model admits simple physical exact solutions, including traveling waves, cnoidal waves, and kinks.

SUBHANKAR SIL, University of British Columbia

Non-invertible mappings relating linear PDEs to corresponding nonlinear PDEs through symmetry-based method

We show that the well-known Hopf-Cole transformation mapping the linear heat equation to the nonlinear Burgers' equation naturally extends to the mapping of any linear PDE to a non-invertibly equivalent nonlinear PDE. This map is obtained through the symmetry-based method by using the admitted obvious scaling symmetry in the dependent variable of any linear homogeneous PDE. Moreover, each nontrivial point symmetry of any linear PDE yields a corresponding nonlocally related nonlinear PDE through the symmetry-based method. The mapping relating the linear PDE and the corresponding nonlinear PDE is not one-to-one. We demonstrate interesting examples of how the linear heat equation (one-dimensional and two-dimensional) can be non-invertibly mapped into corresponding nonlocally related nonlinear PDEs through its admitted point symmetries.

RAFAEL DE LA ROSA SILVA, Universidad de Cádiz

The natural extension to PDEs of Lie's reduction of order algorithm for ODEs

In this talk, we further consider the symmetry-based method for constructing nonlocally related systems for PDEs. We look at Lie's reduction of order algorithm from a different point of view to show how a given ODE is nonlocally related to its reduced ODE. We show that the natural extension to PDEs of Lie's reduction of order method for ODEs is simply the symmetry-based method for PDEs.

THOMAS WOLF, Brock University

Towards a classification of evolution equations with Lax pairs over the octonions

The talk reports on a project on classifying integrable polynomial evolutionary equations with operator Lax pairs for an octonion variable. The method uses a scaling ansatz to set up a general polynomial form for the evolution equation and the Lax pair. A condition for linear differential operators to be a Lax pair over octonions is formulated and solved for the unknown coefficients in the polynomials. The talk will also report on computational aspects including improvements of algorithms to solve over-determined non-linear algebraic systems and even corrections of the programming language itself. First results include 3rd and 5th order equations with KdV and mKdV scaling weights.

ZUHAL KUCUKARSLAN YUZBASI, University of British Columbia and Fırat University

New non-invertible mappings of Schrödinger equations to free particle equations

Discovering an equivalent PDE system for a given PDE system that is not invertibly related to the given system is a successful strategy. Such an equivalent PDE system is called a nonlocally related system. Nonlocally related PDE systems, which are obtained through the symmetry-based method and the CL-based method, are important in the analysis of a given PDE system. In this talk, we focus on an application of the symmetry-based method. Particularly, we show how to obtain systematically non-invertible mappings of Schrödinger equations to free particle equations in (1+1) and (2+1) dimensions, respectively.