Models, Methods, and Solutions: New Developments in Nonlinear Partial Differential Equations and Stochastic Differential Equations

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STEPHEN ANCO, Brock University

Exact solitary wave solutions for a coupled gKdV-NLS system

We study a coupled gKdV-NLS system $u_t + \alpha u^p u_x + \beta u_{xxx} = \gamma(|\psi|^2)_x$ and $i\psi_t + \kappa\psi_{xx} = \sigma u\psi$ with a general nonlinearity power $p > 0$, which has been introduced in the literature to model energy transport in anharmonic crystal materials. There is a strong interest in obtaining exact solutions describing frequency-modulated solitary waves $u = U(x - ct)$, $\psi = e^{i\omega t}\Psi(x - ct)$, where c is the wave speed, and ω is the modulation frequency. For the KdV case $p = 1$, some solutions are known, while for the mKdV case $p = 2$, no exact solutions have been found to-date, and nothing has been done for higher nonlinearities $p \geq 3$.

In the present work, we derive exact solutions for $p = 1, 2, 3, 4$, starting from the travelling wave ODE system satisfied by U and Ψ. The method is new: (i) obtain first integrals by use of multi-reduction symmetry theory; (ii) apply a hodograph transformation which leads to triangular (decoupled) system; (iii) introduce an ansatz for polynomial solutions of the base ODE; (iv) characterize conditions under which solutions yield solitary waves; (v) solve an algebraic system for the coefficients in the ansatz under those conditions. The resulting solitary waves exhibit a wide range of features: bright and dark peaks; single peaked and multi-peaked; zero and non-zero backgrounds.

ALEX CHERNYAVSKY, State University of New York at Buffalo

Whitham modulation theory for the Zakharov-Kuznetsov equation and stability analysis of its periodic traveling wave solutions

We derive the Whitham modulation equations for the Zakharov-Kuznetsov equation via a multiple scales expansion and averaging two conservation laws over one oscillation period of its periodic traveling wave solutions. We then use the Whitham modulation equations to study the transverse stability of the periodic traveling wave solutions. We find that all periodic solutions traveling along the first spatial coordinate are linearly unstable with respect to purely transversal perturbations, and we obtain an explicit expression for the growth rate of perturbations in the long wave limit. We validate these predictions by linearizing the equation around its periodic solutions and solving the resulting eigenvalue problem numerically. We also calculate the growth rate of the solitary waves analytically. The predictions of Whitham modulation theory are in excellent agreement with both of these approaches.Finally, we generalize the stability analysis to periodic waves traveling in arbitrary directions and to perturbations that are not purely transversal, and we determine the resulting domains of stability and instability.

ADILBEK KAIRZHAN, University of Toronto

A Hamiltonian Dysthe equation for deep-water gravity waves with constant vorticity

In this talk I present a study of the water wave problem in a two-dimensional domain of infinite depth in the presence of nonzero constant vorticity. A goal is to describe the effects of uniform shear flow on the modulation of weakly nonlinear quasi-monochromatic surface gravity waves. Starting from the Hamiltonian formulation of this problem and using techniques from Hamiltonian transformation theory, we derive a Hamiltonian Dysthe equation for the time evolution of the wave envelope. Consistent with previous studies, we observe that the uniform shear flow tends to enhance or weaken the modulational instability of Stokes waves depending on its direction and strength. Our method also provides a non-perturbative procedure to reconstruct the surface elevation from the wave envelope, based on the Birkhoff normal form transformation to eliminate all non-resonant triads. This model is tested against direct numerical simulations of the full Euler equations and against a related Dysthe equation derived by Curtis, Carter and Kalisch (J. Fluid Mech., vol. 855, 2018, pp. 322–350) in the context of constant vorticity. Very good agreement is found for a range of values of the vorticity.

THEODORE KOLOKOLNIKOV, Dalhousie
Recurrent and chaotic outbreaks in SIR model

We examine several extensions to the basic SIR model, which are able to induce recurrent outbreaks (the basic SIR model by itself does not exhibit recurrent outbreaks). We first analyze how slow seasonal variations can destabilize the endemic equilibrium, leading to recurrent outbreaks. In the limit of slow immunity loss, we derive asymptotic thresholds that characterize this transition. In the outbreak regime, we use asymptotic matching to obtain a two-dimensional discrete map which describes outbreak times and strength. We then analyse the resulting map using linear stability and numerics. As the frequency of forcing is increased, the map exhibits a period-doubling route to chaos which alternates with periodic outbreaks of increasing frequency. Other extensions that can lead to recurrent outbreaks include addition of noise, state-dependent variation and fine-graining of model classes.

GREG LEWIS, Ontario Tech University
Numerical continuation for sheared annular electroconvection

We use numerical continuation methods to investigate the flows that occur in a mathematical model of the sheared annular electroconvection experiment. In particular, we study the flow of a liquid crystal film suspended between two annular electrodes, and subjected to an electric potential difference and a radial shear. Due to the Smectic A nature of the liquid crystal, the fluid can be considered two-dimensional, and its motion can be effectively modelled using the 2-D incompressible Navier-Stokes equations coupled with an equation for charge continuity. This system is a close analogue of some laboratory-scale geophysical flow experiments (e.g. the differentially-heated rotating annulus).

For small values of the applied electric potential, a steady axisymmetric flow is observed. As this parameter is increased, a transition to rotating waves, followed by a secondary transition to amplitude-modulated rotating waves, occurs. Subsequently, period-doubling and symmetry breaking bifurcations lead to more complicated flows. Further increase in the parameter leads to flows that resemble a rotating wave with one or two isolated vortices.

We investigate the bifurcation structure of solutions that connects these various types of flow using numerical continuation based on time-integration. In particular, a Newton-Krylov method, which exploits the rotating nature of the flows, is implemented for the continuation of rotating waves and modulated rotating waves, and linear stability analysis of a flow map is used to identify the flow transitions that result due to changes in the model parameters.

This is joint work with Mary Pugh and Stephen Morris (University of Toronto).

ALAN LINDSAY, University of Notre Dame
Inferring the source of diffusive sources through extreme statistics.

Title: Inferring the source of diffusive sources through extreme statistics.

Abstract: A common inverse problem is to recover the source of diffusing molecules from noisy arrivals to small reactive sites. In this talk I will present a perspective on this important problem via extreme statistics. The central premise is that when a single stochastic process exhibits large variability (unreliable), the extrema of multiple processes has a remarkably tight distribution (reliable). In this talk I will present some background on extreme statistics followed by specific applications to directional sensing - the process of acquiring the direction of diffusive sources. We find that extreme statistics provide new insights and corroborate real world observations.

RAPHAEL MADOU, McGill University
Strong solutions on SDEs with singular (form-bounded) drifts via Rockner-Zhao approach.

In this talk, we use the Rockner-Zhao approach to establish strong well-posedness result for SDEs featuring singular drift term, subject to certain minimal assumptions.

The talk is based on joint work with Damir Kinzebulatov.

YANA NEC, Thompson Rivers University

Weak solutions to diffusion equation with piecewise constant diffusivity

A wide class of weak solutions to the steady inhomogeneous diffusion equation is constructed in three sets of coordinates: polar, spherical and elliptic. This framework is relevant in applications involving fluid flow in porous media, but is also very interesting mathematically. Their existence is shown to depend on the geometric layout of the domain, i.e. the particular division into sub-domains, as well as the diffusivity assigned to each sub-domain. The existence hinges on the null and column spaces of a set of matrices, intriguingly identical in all three systems of coordinates. A fixed point of a new type – half stable - half unstable node – is identified with the aid of this class of weak solutions. A variety of flow patterns associated with these solutions allows to explain certain modelling difficulties encountered in large scale environmental applications, such as aquifer sparging wells, natural and landfill gas wells, as well as petroleum and hydraulic wells. One of the prominent properties of these weak solutions is that with the relaxation of the constraint of axial symmetry – that is obviously unrealistic for the foregoing applications – the locus of zero normal flux comprises separatrices connecting isolated stagnation points. The area enclosed is shown to be realistic in stark contrast to the result obtained with an axially symmetric solution, where the locus comprises a curve of stagnation points. The new class of solutions augments known exact solutions to Laplaces's equation in settings where it is separable.

MARIA NTEKOUME, Concordia University

Symplectic non-squeezing for integrable PDEs: the KdV equation on the line

Gromov's symplectic non-squeezing theorem asserts that a smooth symplectomorphism cannot map a ball wholly inside a thinner cylinder. In this talk we will review methods to obtain infinite-dimensional analogues of this theorem for Hamiltonian PDEs. In particular, we will prove that the KdV flow on the line cannot squeeze a ball in $\dot{H}-\frac{1}{2}(\mathbb{R})$ into a cylinder of lesser radius. If time permits, further applications of this method to other completely integrable PDEs will be discussed.

ELKIN RAMÍREZ, McMaster University

SYSTEMATIC SEARCH FOR EXTREME BEHAVIOUR IN 3D NAVIER-STOKES EQUATIONS BASED ON THE LADYZHENSKAYA-PRODI-SERRIN CONDITIONS

This investigation concerns a systematic search for potential singularities in 3D Navier-Stokes flows. It is based on the Ladyzhenskaya-Prodi-Serrin conditions, which assert that if the quantity $\int_0^T \|\mathbf{u}(t)\|^p_{L^q(\Omega)}\,dt$ remains bounded, given that $2/p+3/q \leq 1$ and $q > 3$, then the solution $\mathbf{u}(t)$ of the Navier-Stokes system remains smooth within the interval $[0,T]$. Hence, should a singularity arise at any instant within the interval $[0,T]$, we would anticipate an unbounded growth of this quantity.

We examine these conditions by solving numerically a set of variational optimization problems. These problems aim to determine initial conditions $\mathbf{u}_0$ such tat the corresponding flow maximizes $\int_0^T \|\mathbf{u}(t)\|^p_{L^q(\Omega)}\,dt$ for different values of T while satisfying specific constraints. We address these problems computationally, employing a large-scale adjoint-based gradient approach in Sobolev and Lebesgue spaces.

We extend earlier work by considering various values of q, and different types of gradients to discretize gradient flows. We also studied the limiting case $q = 3$ where the regularity condition is slightly different.

ALEXEY SHEVYAKOV, University of Saskatchewan

New exact plasma equilibria with axial and helical symmetry

Abstract: Exact closed-form solutions of magnetohydrodynamics equations, with and without dynamics, are derived under axial and helical symmetry assumptions. For each symmetry, two distinct families of solutions arise that correspond to different pressure profiles. One profile models plasmas supported by external pressure, and is suitable for the description of plasma configurations in a medium such as atmosphere. The second profile features higher pressure inside the plasma domain and models plasmas residing in a vacuum. Examples of solutions bounded and unbounded in the radial direction, including solutions with boundary current sheets, are presented and discussed. This work is joint with Jason Keller.

WEI SUN, Concordia University
Periodic solutions of some SDEs and SPDEs

We investigate periodic solutions of stochastic dynamical systems induced by some SDEs and SPDEs. We start with different definitions of periodic solutions. Then, we explain how to use the strong Feller property and irreducibility of time-inhomogeneous semigroups to study uniqueness of periodic solutions. Concrete examples are presented to illustrate the results. This talk is based on joint papers with Zuo-Huan Zheng, Xiao-Xia Guo and Chun Ho Lau.

RYAN THIESSEN, University of Alberta
Travelling Wave Solutions in a Novel Glioma Invasion Model.

In a recent paper, Osswald and collaborators presented a detailed study of in-vivo glioma invasion patterns in the healthy brain tissue of living mice. This paper showed that specialized cancer cells build a network much like a healthy brain neuronal network. Working jointly with Thomas Hillen, Kevin Painter, and Nadia Loy, we aim to incorporate this discovery of network formation into previous Glioma blastoma models. Our model is based on the kinetic model framework, where we can quickly introduce new reaction dynamics for the network formation. We can arrive at coupled non-cooperative reaction-diffusion equations by making quasi-equilibrium assumptions and taking the diffusion limit. From this system, we will show the existence of Traveling waves with a minimal spreading speed.

REIHANEH VAFADAR, Laval university
Weak well-posedness of SDEs with divergence-free drifts

We discuss results on weak well-posedness of SDEs with time-inhomogeneous divergence-free singular drifts. These drifts belong to a large class of (form bounded-type) drifts containing e.g. the largest possible Morrey class, which brings us close to the minimal assumptions on the drift such that the corresponding parabolic equation still admits a regularity theory. Our proofs use De Giorgi-Moser's method supplemented with a new iteration procedure, earlier results proved using Nash's method. This is a joint work in progress with D. Kinzebulatov.

THOMAS WOLF, Brock University
Radial compressible fluid flow in $n > 1$ dimensions and their conserved integrals, invariants, symmetries and Casimirs

In this joint work with Stephen Anco conserved integrals and invariants (advected scalars) are studied for the equations of radial compressible fluid flow in $n > 1$ dimensions. Three invariants of up to first order had been found apart from the known entropy.

A recursion operator on invariants is presented, producing two hierarchies of higher-order invariants. One of them consist of Hamiltonian Casimirs. The other one holding non-Casimirs holds only for an entropic equation of state (EOS).

The Hamiltonian structure of the radial fluid flow equations in combination with these non-Casimir invariants provides a corresponding hierarchy of generalized symmetries. The Lie algebra of the first-order symmetries is non-abelian.

For the special cases of barotropic EOS and entropic EOS two new kinematic conserved integrals yield additional first-order generalized symmetries. These provide an explicit transformation group acting on solutions of the fluid equations.

XIAOWEN ZHOU, Concordia University
Speed of explosion for continuous-state branching processes with nonlinear branching mechanism

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Continuous-state branching process (CSBP) with nonlinear branching mechanism is the unique nonnegative solution to certain stochastic differential equation driven by Brownian motion and (or) Poisson random measure. It can also be obtained from spectrally positive Lévy processes by a generalized Lamperti transform. These generalized CSBPs allow rich asymptotic behaviors such as extinction, explosion and coming down from infinity. The explosion behaviors for nonlinear CSBPs have been studied by Li and Zhou (2021) when the big jumps of the process have a finite first moment. In this talk we further consider the explosion behaviors for processes with jumps of infinite first moment. In particular, we identify the speed of explosion when the associated Laplace exponent and the rate function are both regularly varying. This talk is based on joint work with Clement Foucart and Bo Li.