Geometric Functional Analysis: Analytic, Discrete, and Probabilistic Aspects
Analyse fonctionnelle géométrique : Aspects analytiques, discrets et probabilistes
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GIOACCHINO ANTONELLI, Courant Institute (NYU)
Nonnegative curvature and existence of isoperimetric sets

The isoperimetric problem is a well-known variational problem in Geometric Analysis. In this talk we will first review the recent literature on the existence of isoperimetric sets in spaces with nonnegative curvature. Then we will show how to construct a convex body C in $\mathbb{R}^3$ such that for every volume $v < 1$ there is no relative isoperimetric set with volume v, and for every volume $v > 1$ there is at least one relative isoperimetric set with volume v.

This is based on recent works with E. Bruè, M. Fogagnolo, F. Glaudo, and M. Pozzetta.

ANDRII ARMAN, University of Manitoba
On some covering problems related to Borsuk's conjecture

Borsuk's number $f(n)$ is the smallest integer such that any set of diameter 1 in n-dimensional Euclidean space can be covered by $f(n)$ sets of a smaller diameter. Exponential upper bound $f(n) \leq \left(\sqrt{\frac{3}{2}} + o(1)\right)^n$ was first obtained by Schramm (1988) and later by Bourgain and Lindenstrauss (1989), while a lower bound $f(n) \geq (1.2 + o(1))^{\sqrt{n}}$ was obtained by Kahn and Kalai (1993).

To obtain an upper bound on $f(n)$, Bourgain and Lindenstrauss provided exponential bounds (both upper and lower) for the Grünbaum's problem – the problem of determining the minimal number of open balls of diameter 1 needed to cover a set of diameter 1.

On the other hand, in order to obtain an upper bound on $f(n)$ Schramm provided an exponential upper bound on the illumination number of n-dimensional bodies of constant width. Kalai (2015) asked for a corresponding lower bound, namely if there exists an n-dimensional convex body of constant width with the illumination number exponential in n.

In this talk I will outline the construction that answers Kalai's question in the affirmative and provide a new lower bound in the Grünbaum's problem. The talk is based on a joint work with Andriy Bondarenko and Andriy Prymak.

ALMUT BURCHARD, University of Toronto
On pointwise (non)-monotonicity of heat kernels for metrics on the two-sphere

I will present recent work with Ángel Martínez, regarding pointwise monotonicity of heat kernels. Previously we had found that Riemannian metrics for which the heat kernel $K_t(x, y)$ decreases as x moves away from y along a minimal geodesic are extremely rare, though there are non-trivial examples (beyond products of standard spheres). Here we show that the only metrics on $\mathbb{S}^2$ with this monotonicity property are the uniform ones. The proof depends on a surprising connection with Hersch's inequality for the principal eigenvalue of the Laplacian.

DMITRY FAIFMAN, Tel Aviv University
Some Whitney extension problems in valuation theory

Consider smooth, translation-invariant valuations on convex bodies. Assume that a collection of valuations is given on a family S of subspaces of $\mathbb{R}^n$. Are they the restrictions of a single valuation? Clearly, compatibility of the given data on intersections is a necessary condition. Is it sufficient? We will discuss several distinct instances of this problem, whence it acquires distinct flavors. When S is the whole k-grassmannian, and the valuations j-homogeneous, we will see that the condition is sufficient, provided $k-j>1$. This can be seen as a dimensional localization of the phase transition from densities to valuations. In another setting, where S consists of pairwise non-intersecting subspaces, we again establish a positive answer. As a corollary, we will deduce a Nash embedding theorem for smooth valuations on manifolds, which in turn has integral-geometric consequences in this setting. Finally, we will consider the setting of finite generic families of subspaces, giving rise to a surprising extension phenomenon. Based on a joint work with Georg Hofstaetter.

JOSHUA FLYNN, CRM/ISM, McGill University
The Isoperimetric Problem and Related Mean Curvature Type Flows

The isoperimetric problem asks to find, among all domains of a given volume, those whose boundaries have minimal surface area. A natural approach to this problem is to consider volume preserving and area decreasing geometric flows. In this talk, we introduce a flow which is a novel modification of the mean curvature-type flow first introduced by Guan and Li, which was later generalized by Guan-Li-Wang and Li-Pan. These flows are defined in terms of conformal Killing vector fields and rely on Minkowski identities to prove volume preservation and area monotonicity. Our results allow one to establish the isoperimetric inequalities in general geometries for a category of surfaces larger than the usual star-shaped or convex categories all previous works were restricted to.

RYAN GIBARA, University of Cincinnati
Traces and extensions of Sobolev functions in metric measure spaces

In smooth Euclidean domains, it is known that Sobolev functions admit traces to the boundary that are in an appropriate Besov class and, vice versa, Besov functions on the boundary admit extensions that are Sobolev in the domain. All of these concepts make sense in the setting of a metric measure space, where the geometry of the space, as manifested in the existence of well behaved curves, plays a key role. In this talk, we will discuss recent work on generalizing the known trace and extension results to unbounded domains. This work, joint with Riikka Korte and Nageswari Shanmugalingam, arose in connection to the study of Dirichlet problems for the p-Laplacian on unbounded uniform domains in metric measure spaces.

JULIAN HADDAD, Universidad de Sevilla
Fiber symmetrization and the Rogers-Brascamp-Lieb-Luttinger inequality

Abstract: We prove a Rogers-Brascamp-Lieb-Luttinger inequality for functions defined in the space of $n \times m$ matrices, using a particular form of fiber-symmetrization. Some applications on symmetrization of matrix norms are given. We also discuss a conjectured inequality by Schneider, on the higher-order difference body.

ALEXANDER KOLDOBSKY, University of Missouri-Columbia
Comparison problems for the Radon transform

Given two non-negative functions f and g such that the Radon transform of f is pointwise smaller than the Radon transform of g, does it follow that the L^p-norm of f is smaller than the L^p-norm of g for a given $p>1$? We consider this problem for the classical and spherical Radon transforms. In both cases we point out classes of functions for which the answer is affirmative, and show that in general the answer is negative if the functions do not belong to these classes. The results are in the spirit of the solution of the Busemann-Petty problem from convex geometry, and the classes of functions that we introduce generalize the class of intersection bodies introduced by Lutwak in 1988. We also deduce slicing inequalities that are related to the well-known Oberlin-Stein type estimates for the Radon transform.

NGUYEN H. LAM, Memorial University of Newfoundland
A new approach to weighted Hardy-Rellich inequalities

In this talk, we present a new way to use the notion of Bessel pair to establish the optimal Hardy-Rellich type inequalities. We also talk about necessary and sufficient conditions on the weights for the Hardy-Rellich inequalities to hold. Symmetry and symmetry breaking properties of the Rellich type and Hardy-Rellich type inequalities will also be discussed. This is joint work with Anh Do and Guozhen Lu.

DYLAN LANGHARST, Institut de mathématiques de Jussieu
On the measures satisfying a monotonicity of the surface area with respect to Minkowski sum

If K and L are convex bodies, then K being a subset of L implies the surface area of K is less than the surface area of L. If A,B and C are also convex bodies, then the Lebesgue measure satisfies the following supermodularity inequality for their Minkowski sums: $|A+B|+|A+C|<|A|+|A+B+C|$. In this talk, we explore weighted analogues of these properties by replacing the Lebesgue measure with a nice Borel measure. Recently, G. Saracco and G. Stefani showed that if a Borel measure with density has the monotonicity property, then it must be a multiple of the Lebesgue measure. We study the case of supermodularity for any Radon measure, and show it is equivalent to a variant of the monotonicity problem. We verify that a Radon measure with the supermodularity property must be the Lebesgue measure. We then consider restricted versions of the problem.

BRAYDEN LETWIN, University of Alberta
On a generalization of Grünbaum's inequality

Grünbaum's inequality gives sharp volume bounds between a convex body and a division of the body by a hyperplane through its centroid. We provide a generalization of this inequality by looking at divisions of the body by a hyperplane that does not necessarily contain the centroid. As an application, we arrive at a sharp inequality that compares the maximal section(s) of a convex body to any section, which builds on work done by Makai and Martini in 1996. This is joint work with Vlad Yaskin.

ALEXANDER LITVAK,
On the minimum of Gaussian variables.

Let $X=(\xi_1,...,\xi_m)$ be a centered Gaussian random vector, such that the variances of each ξ_i equals to 1. Under what assumptions on the covariance matrix is the expectation of $\min_i |\xi_i|$ minimized? We discuss known results and conjectures related to this question.

MOKSHAY MADIMAN, University of Delaware
Submodularity questions in convex geometry

I will present some results, obtained in collaboration with M. Fradelizi, M. Meyer, and A. Zvavitch, on the log-submodularity of the volume of Minkowski sums on different classes of compact convex sets. The underlying question is the following: given a class $\mathcal{G}_n$ of compact convex sets in $\mathbb{R}^n$ closed under Minkowski summation and affine transformations, and $A,B,C\in\mathcal{G}_n$, what is the best constant α in the inequality

$$|A+B+C|\cdot|A|\leq\alpha|A+B|\cdot|A+C|,$$

where $|\cdot|$ denotes volume? This constant, which we may denote $\alpha(\mathcal{G}_n)$, has interesting interpretation, and it is particularly useful to identify cases where it is equal to 1, since in this case, it is intimately connected to inequalities for projections. If $\mathcal{K}_n$ is the collection of compact convex sets and $\mathcal{Z}_n$ is the collection of zonoids in $\mathbb{R}^n$, we show that $\alpha(\mathcal{K}_2)=\alpha(\mathcal{Z}_2)=1$ and $1=\alpha(\mathcal{Z}_3)<\alpha(\mathcal{K}_3)=4/3$. We will also present some estimates and some conjectures for higher dimensions.

KENNETH MOORE, University of British Columbia
Minimal reflective and folding symmetry of convex sets

In this talk, we discuss two generalizations of the Kovner-Besicovitch measure of symmetry. For a convex body in $\mathbb{R}^n$, the k-symmetry is defined as the largest possible ratio of overlap of the body and its reflection through a k-dimensional affine subspace. Chakerian and Stein proved general lower bounds for k-symmetry in 1965, but construction of low symmetry objects appeared to be difficult in dimensions above 2. We present an inductive construction that attains the current best upper bounds on minimal k-symmetry. We also show new upper and lower bounds for another version of symmetry first studied by Lassak, called folding symmetry, in $\mathbb{R}^2$. We will offer a few conjectures and promising directions. This talk is based on a joint work with Ritesh Goenka, Rui Sun, and Ethan White.

ELI PUTTERMAN, Tel Aviv University
Small-ball probabilities for mean widths of random polytopes

The classical theory of random polytopes addresses questions such as computing the expectation or variance of geometric parameters associated to a random polytope (e.g., volume, number of facets, or mean width); more recent theory also aims to obtain concentration of measure for such quantities. The new theory of higher-order projection bodies naturally leads to a question in random polytopes which current theory, surprisingly, does not address: bounding a high negative moment of the mean width of a certain random polytope, which requires bounding the probability that this mean width is a small fraction of its expectation ("small-ball estimates"). These small-ball estimates use different tools from those commonly employed in the field of random polytopes, and it turns out that the behavior of the negative moment demonstrates a phase transition. We will conclude by mentioning some related open problems.

ZHEN SHUANG, Memorial University of Newfoundland
Fractional p-Laplacian and Signal Decomposition

The fractional p-Laplacian $(-\Delta)_p^{\frac{\alpha}{2}}$ can be recovered by a weighted Laplace operator

$$div_{x,\tau}\left(\tau^{1-\alpha p 2^{-1}}\nabla_{x,\tau}u(x,\tau)\right)$$

through a limit of a function in the one-more-dimensional upper space. Hence an evolutionary equation with fractional p-Laplacian can be replaced with another one with the weighted Laplace operator to perform signal decomposition since it takes too much time to approximate the fractional p-Laplace evolutionary equation. The signal decomposition is to decompose a signal into different smoothness degrees.

RUI SUN, University of Alberta
Measure of Axiality for Convex Figures

For a convex body K in R^2 with $\mu(K) = 1$, the Kovner Besicovitch theorem states that one can inscribe a centrally symmetric (symmetric about a point) body of area at least 2/3 inside K. The minimizer is given by the triangle, where the maximal inscribable centrally symmetric body has only area of 2/3. In the spirit of this theorem, we consider the question: what is area of the largest convex body which we can inscribe in K that is symmetric about a line (we call this axiality)? Lassak showed that for any K, this number is at least 2/3. Unlike in the centrally symmetric case, no minimizer was found to verify if this lower bound is sharp. In our paper, we are able to improve this bound to 0.695. On the other hand, the known concrete examples of bodies with low axiality had an axiality of $2\sqrt{2}-2 \approx$ 0.828 which is attained by triangles and a specific parallelogram (and thus giving us an upperbound on the axiality of arbitrary convex bodies). Only recently, Choi discovered a quadrilateral which improved this upper bound to 0.816. We found an example with axiality $\frac{1}{3}(\sqrt{2}+1)$ and believe this to be the minimizer. I will explain in my presentation of how we attain the improvements on the upper and lower bounds of axiality. This talk is based on a joint work with Ritesh Goenka, Kenneth Moore, and Ethan White.

CHENGJUN YUE, Memorial University of Newfoundland
Around Poisson-Bessel potentials of fractional L^1-Hardy-Sobolev spaces

Let $u_\alpha(x,t)$, $\alpha \in (0,2)$ be the solution of the equation

$$\Delta_{x,t}u_\alpha(x,t) + (1-\alpha)t^{-1}\partial_t u_\alpha(x,t) = 0$$

on $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$ subject to $u_\alpha(x,0) = f(x)$ on $\mathbb{R}^n$. As the endpoint of the Poisson-Bessel potential u_α, the potential $u_0(x,t)$ solves the equation

$$\Delta_{x,t}\big(\ln t^{-1})u_0(x,t)\big) + t^{-1}\partial_t\big((\ln t^{-1})u_0(x,t)\big) = 0$$

on $\mathbb{R}^{n+1}_+$ subject to $u_0(x,0) = f(x)$ on $\mathbb{R}^n$. The main goal of this paper is to characterize a nonnegative measure μ on $\mathbb{R}^{n+1}_+$ such that $f(x) \mapsto u_\alpha(x,t)$ induces a bounded embedding from the fractional L^1-Hardy-Sobolev space $H^{\alpha,1}(\mathbb{R}^n)$, $\alpha \in (0,2)$ into the weak Lebesgue space $WL^q_\mu(\mathbb{R}^{n+1}_+)$, $q \in [1,\infty)$ and $f(x) \mapsto u_0(x,t)$ induces a bounded embedding from the Hardy $H^{0,1}(\mathbb{R}^n)$ into the Lebesgue space $L^q_\mu(\mathbb{R}^{n+1}_+)$, $q \in [1,\infty)$.

Based on these trace principles, we propose $(H^{\alpha,1}, L^q)$ model and $(H^{\alpha,1}, \log)$ model for image denoising, which significantly improve the reconstruction from images polluted by Gaussian noise or Poisson noise compared with the famous Rudin-Osher-Fatemi model.

BARTLOMIEJ ZAWALSKI, Kent State University, Department of Mathematical Sciences
On star-convex bodies with rotationally invariant sections

We will outline the proof that an origin-symmetric star-convex body K with sufficiently smooth boundary and such that every hyperplane section of K passing through the origin is a body of affine revolution, is itself a body of affine revolution. This will give a positive answer to the recent question asked by G. Bor, L. Hernández-Lamoneda, V. Jiménez de Santiago, and L. Montejano-Peimbert [DOI:10.2140/gt.2021.25.2621, Remark 2.9], though with slightly different prerequisites. Our argument is built mainly upon the tools of differential geometry and linear algebra, but occasionally we will need to use some more involved facts from other fields like algebraic topology or commutative algebra. The talk is based on the article [DOI:10.1007/s13366-023-00702-1].

ZENGLE ZHANG, Chongqing University of Arts and Sciences
The dual Orlicz-Minkowski problems for log-concave functions

The dual curvature measure of convex bodies is a geometric measure induced by dual quermassintegrals of convex bodies and is a central concept of the dual Brunn-Minkowski theory. This measure was first introduced by Huang-LYZ. Related Minkowski problems have attracted a great deal of attention. In particular, the dual Minkowski problem can be reformulated as a Monge-Ampère equation involving radial functions of convex bodies. Recently, the dual Minkowski problem has been extended to the setting of unbounded convex sets, log-concave functions, and as well as the Orlicz theory.

In this talk, I will discuss the Orlicz moment and the related variational formula in terms of the Asplund sum of log-concave functions. I will talk about the related dual Orlicz curvature measure of log-concave functions and the corresponding Minkowski problem. A solution to this dual Minkowski problem will be presented for even data as well. This talk is based on the joint work with Niufa Fang, Deping Ye, and Yiming Zhao.

YIMING ZHAO, Syracuse University
The Minkowski problem in Gaussian probability space

The classical Minkowski problem, which asks for the characterization of surface area measure in Euclidean space with Lebesgue measure, largely motivated the development of elliptic PDEs throughout the last century. In this talk, we will discuss the corresponding problem in Gaussian probability space and some recent results. This is based on joint works with Yong Huang, Dongmeng Xi, and with Shibing Chen, Shengnan Hu, Weiru Liu.