Calculus of Variations and its Applications
Calcul des variations et ses applications
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BLAISE BOURDIN, McMaster University
Phase-field approximation of diffusion-driven fracture

This talk focusses on a class of problems where crack propagation is driven by a diffusion process. This general framework encompasses a broad range of phenomena including thermal and dessication cracks, fracture in fluid-saturated porous media, or fracture of materials undergoing phase-change.

The main difficulty in building a rigorous phase-field model of such problems is the different time scales involved in the fracture and diffusion processes. The former is often assumed to remain in an equilibrium state at all time, whereas the later is inherently driven by its non-equilibrium nature. Γ-convergence, commonly used to derive the convergence of phase-field models to their sharp interface counterparts do not provide any insight on out-of-equilibrium evolutions. Instead, we propose to reformulate the diffusion problem in terms of the minimizing motion of an energy, and the coupled problem as a PDE-constrained optimization problem. We then propose compatible phase-field approximations of the fracture and diffusion process can be derived, and the convergence of the constrained minimization problem can be proved.

DENIS BRAZKE, University of Heidelberg
Γ–limit for a sharp interface model related to pattern formation on biomembranes

We derive a macroscopic limit for a sharp interface version of a model proposed by Komura, Shimokawa and Andelman to investigate pattern formation in biomembranes due to competition of chemical and mechanical forces. We identify sub- and supercrital parameter regimes and show with the introduction of the autocorrelation function that the ground state energy leads to the isoperimetric problem in the subcritical regime, which is interpreted to not form fine scale patterns

This is joint work with Hans Knüpfer and Anna Marciniak–Czochra.

ALMUT BURCHARD, University of Toronto
Symmetry-breaking in isodiametric capacitor problems

A classical theorem of Szegö says that balls maximize electrostatic capacity among sets of given diameter (while minimizing capacity among sets of given volume). On the other hand, balls do not maximize Riesz-capacity (with a Riesz-potential $|x|^{-\lambda}$, for fixed $\lambda > 0$) among sets of given diameter in high dimensions. Thus symmetry-breaking occurs as the Riesz kernel transitions from the Newtonian case ($\lambda = n - 2$) to the logarithmic case (corresponding to $\lambda = 0$), once λ is small enough relative to n. (Joint work with R. Choksi, E. Hess-Childs, and A. Martìnez.)

CARRIE CLARK, University of Illinois Urbana-Champaign
Droplet formation in a nonlocal aggregation model

The study of aggregation in the physical sciences has produced a rich class of nonlocal shape optimization problems. In this talk, we will discuss droplet formation in energy minimizing configurations for a family of interaction kernels which have a "well-barrier" type shape. Short distance attraction, combined with mid distance repulsion, and long distance neutrality drives the separation into droplets.

ANDREW COLINET, McMaster University
Zeroth Order Limiting Behaviour of the Ginzburg-Landau Functional

For $\Omega \subseteq \mathbb{R}^2$ there is an extensive literature concerning the limiting behaviour of the Ginzburg-Landau energy,

$$E_\varepsilon(u) = \int_\Omega \Big\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(|u|^2 - 1)^2 \Big\},$$

as $\varepsilon \to 0^+$. In such works, it is shown that if a sequence of functions has vorticity concentrate, as $\varepsilon \to 0^+$, about a finite collection of interior points of Ω then the Ginzburg-Landau energy converges, after renormalizing, to the total variation of a measure supported over the same interior points. However, much less is known when the vorticity of solutions is permitted to concentrate about points along the boundary.

We consider this question for a connected open set $\Omega \subseteq \mathbb{R}^2$ with $C^{2,1}$ boundary and we prove that similar conclusions to the interior case remain true up to the boundary provided the functions we consider satisfy suitable boundary restrictions. In addition, we also show that there are necessary topological restrictions on the vorticity.

ROBERT HASLHOFER, University of Toronto

Classification of compact ancient noncollapsed flows in R^4

To capture singularities under mean curvature flow one wants to understand all ancient solutions. In addition to shrinkers and translators one also encounters ancient ovals, namely compact noncollapsed solutions that are not self-similar. In this talk, I will explain that any bubble-sheet oval for the mean curvature flow in R^4, up to scaling and rigid motion, either is the $O(2)\times O(2)$-symmetric ancient oval constructed by White, or belongs to the one-parameter family of $Z_2^2 \times O(2)$-symmetric ancient ovals constructed by Du and myself. In particular, this seems to be the first instance of a classification result for geometric flows that are neither cohomogeneity-one nor selfsimilar. This is joint work with Beomjun Choi, Toti Daskalopoulos, Wenkui Du and Natasa Sesum. I will also briefly mention the noncompact case, which is joint work with Kyeongsu Choi and Or Hershkovits.

NICHOLAS KEVLAHAN, McMaster University

Data assimilation for bathymetry in the nonlinear shallow water equations

The shallow water equations (SWE) are a widely used model for the propagation of surface waves in oceans, lakes and rivers. Common applications include modelling the propagation of tsunami waves, storm surges and flooding. We consider the problem of determining under which conditions a space-time variational data assimilation approach based on observations of the free surface is able to reconstruct the bathymetry to a given accuracy (e.g. sufficient for modelling wave propagation). We then use density-based global sensitivity analysis (GSA) to assess the sensitivity of the surface wave and reconstruction error to model parameters and second order adjoint analysis (SOA) to analyze the sensitivity of the surface wave error, given the reconstructed bathymetry, to perturbations in the observations.

This is joint work with Bartek Protas (McMaster University) and Ramsha Khan (University of Stockholm)

SULLIVAN MACDONALD, McMaster University

Degenerate Ellipticity and Hypoellipticity for Divergence Operators

We discuss differential operators of the form $L = -\mathrm{div}(Q\nabla\cdot)$, where Q is a non-negative definite symmetric matrix-valued function. Even though L fails to be elliptic at points where Q is singular, in some settings we can recover regularity results for weak solutions to the Dirichlet problem

$$\begin{aligned} Lu &= f \quad x \in \Omega, \\ u &= 0 \quad x \in \partial\Omega. \end{aligned}$$

In this talk, I will present some recent joint work with S. Rodney concerning sufficient conditions for a priori boundedness of weak solutions to these problems. We show that even if a very weak Sobolev inequality with gain on Orlicz scale holds, one can still recover boundedness under appropriate hypotheses on the data f.

On the other hand, if rapid degeneracy of Q forbids a Sobolev inequality with gain on Orlicz scale from holding, it would still be useful to understand qualitative behaviours of the operator L. To this end, I also discuss recent work at finding sufficient conditions for hypoellipticity of second-order divergence operators by using decompositions of Holder continuous functions into sums of squares. This work complements some recent results by Sawyer and Korobenko, which show that $L = -\mathrm{div}(Q\nabla\cdot)$ is hypoelliptic when the matrix Q admits a suitable decomposition.

DMITRY PELINOVKY, McMaster University

BARTEK PROTAS, McMaster University
Searching for Singularities in Navier-Stokes Flows Using Variational Optimization Methods

This investigation concerns a systematic computational search for potentially singular behavior in 3D Navier-Stokes flows. Enstrophy $\mathcal{E}(t)$ serves as a convenient indicator of the regularity of solutions — as long as this quantity remains finite, the solutions are guaranteed to be smooth and satisfy the equations in the classical sense. Another well-known conditional regularity result are the Ladyzhenskaya-Prodi-Serrin conditions asserting that the quantity $\mathcal{L}_{q,p} := \int_0^T \|\mathbf{u}(t)\|^p_{L^q(\Omega)}\,dt$, where $2/p + 3/q \leq 1$, $q > 3$, must remain bounded if the solution is smooth on the interval $[0,T]$. However, there are no finite a priori bounds available for these quantities and hence the regularity problem for the 3D Navier-Stokes system remains open. To quantify the maximum possible growth of $\mathcal{E}(T)$ and $\mathcal{L}_{q,p}$, we consider families of variational PDE optimization problems in which initial conditions are sought subject to certain constraints so that these quantities in the resulting Navier-Stokes flows are maximized. These problems are solved computationally using a large-scale adjoint-based gradient approach. By solving these problems for a broad range of parameter values we demonstrate that the maximum growth of $\mathcal{E}(T)$ and $\mathcal{L}_{q,p}$ appears finite. Thus, in the worst-case scenarios the two quantities remain bounded for all times and there is no evidence for singularity formation in finite time.

[Joint work with Dongfang Yun and Di Kang]

IVAN SALGADO, University of Toronto
Approximate Solutions to the Superconducting Interface Model

The superconducting interface model is a semilinear hyperbolic system of PDEs introduced in 2016 by Kyle Thompson. It proposes a more tractable, yet closely related alternative to a 1984 model of Edward Witten for cosmic strings carrying a superconducting current. In the superconducting interface model, we consider the system

$$\begin{cases} \epsilon^2(\partial_t^2\varphi - \Delta_x\varphi) + \lambda_\varphi(\varphi^2 - 1)\varphi + \beta\varphi|\sigma|^2 = 0 \\ \epsilon^2(\partial_t^2\sigma - \Delta_x\sigma) + \lambda_\sigma(|\sigma|^2 - m_\sigma^2)\sigma + \beta\varphi^2\sigma = 0 \end{cases},$$

where $0 < \epsilon \ll 1$, $(\lambda_\varphi, \lambda_\sigma, m_\sigma, \beta) \in (0,\infty)^4$ are parameters, and

$$\varphi : [0,T] \times \mathbb{R}^n \to \mathbb{R} \text{ and } \sigma : [0,T] \times \mathbb{R}^n \to \mathbb{C},$$

for some $T > 0$ and $n \in \mathbb{N}$ with $n \geq 2$. We are interested in solutions (φ, σ) such that

$$\varphi \approx \begin{cases} +1 & \text{in } \mathcal{O}^+ \\ -1 & \text{in } \mathcal{O}^- \end{cases},$$

where $\mathcal{O}^+$ and $\mathcal{O}^-$ are disjoint open subsets of $[0,T] \times \mathbb{R}^n$ separated by an "interface" with thickness of order ϵ, and the current-carrying field σ decays exponentially away from the interface. The problem is of particular interest when σ showcases a clear interaction with the geometry of the interface.

The purpose of this talk is to present a methodology for finding solutions to the superconducting interface model by first constructing approximate solutions, and then linearizing the system of PDEs around these approximations. We will talk about some of the results regarding the construction of the approximate solutions and the laws of motion which represent the coupling between the current supported around the interface and the geometry of the interface.

DOMINIK STANTEJSKY, McMaster University
A finite element approach for minimizing line and surface energies arising in the study of singularities in liquid crystals

I will present an algorithm designed to calculate minimizers T of a geometric energy arising in the theory of liquid crystal colloids. The energy involves the two dimensional area of T outside an obstacle, a contribution from T on the obstacle surface, and the length of the boundary ∂T reduced by a prescribed curve to make the problem nontrivial. It can be seen as a generalization of both the obstacle and Plateau problem. We discretize the energy by a finite element method and apply an ADMM scheme to carry out the minimization. We validate our algorithm in the case of a spherical obstacle and give examples of minimizing configurations in the case of a peanut- and croissant-shaped obstacle.

ZHICHAO WANG, The University of British Columbia
Min-max minimal hypersurfaces with higher multiplicity

Recently, X. Zhou proved that the Almgren-Pitts min-max solution has multiplicity one for bumpy metrics (Multiplicity One Theorem). In this talk, we exhibit the first set of examples of non-bumpy metrics on the $(n+1)$-sphere $(2 \leq n \leq 6)$ in which the varifold associated with the two-parameter min-max construction must be a multiplicity-two minimal n-sphere. This is proved by a new area-and-separation estimate for certain minimal hypersurfaces with Morse index two inspired by an early work of Colding-Minicozzi. This is a joint work with X. Zhou.