Spectral Methods and Singular Integral Equations
Méthodes Spectrales et Équations Intégrales Singulières
(Org: **Sheehan Olver** (Imperial College London) and/et **Richard Slevinsky** (Manitoba))

TRAVIS ASKHAM, New Jersey Institute of Technology
Fast multipole methods for continuous charge distributions

Applications with continuous charge distributions or continuously varying material properties require the calculation of so-called volume integrals involving the Green's function of the governing PDE. These integrals can be both singular and nearly singular and thus require special quadrature. We present a method for generating such quadrature rules that is efficient enough to be done on-the-fly. We also discuss how these quadrature rules are incorporated in a fast multipole method and demonstrate some applications of the scheme to optical scattering problems.

JIM BREMER, University of California, Davis
A fast algorithm for simulating scattering from a radially symmetric potential

Standard solvers for the variable coefficient Helmholtz equation in two spatial dimensions have running times which grow at least as fast as $\mathcal{O}\left(k^2\right)$ in the wavenumber k of the problem. I will describe an algorithm which only applies in the very special case in which the coefficient is radially symmetric, but whose running time is $\mathcal{O}(k\ log(k))$.

MATTHEW COLBROOK, University of Cambridge
A Mathieu function boundary spectral method for acoustic scattering

Many problems in fluid dynamics and acoustics are modelled by singular integral equations with complicated boundary conditions (BCs). This talk considers 2D Helmholtz scattering off (multiple finite) plates, with a focus on BCs ranging from linear models of variable elasticity (fourth-order ODEs), impedance and porosity, to non-linear inertial corrections. A boundary spectral collocation method using Mathieu functions is developed to solve these systems. The method is accurate and flexible for a wide range of frequencies and different BCs, and can robustly compute expansions in tens of thousands of Mathieu functions. As well as discussing numerical analysis aspects, I will demonstrate applications to acoustic black holes, reduction of aerofoil-turbulence interaction noise, and the importance of non-linear corrections for accurately predicting the noise generated by metal foam-like materials. More generally, a goal of this talk is to demonstrate that modern spectral methods can be used in a simple and effective manner for contemporary problems of acoustic scattering, with pointers to ongoing problems.

[1] Colbrook, M.J., Kisil, A.V. "A Mathieu function boundary spectral method for scattering by multiple variable poro-elastic plates, with applications to metamaterials and acoustics." Proceedings of the Royal Society A (2020)

[2] Ayton, L.J., Colbrook, M.J., Geyer, T.F., Paruchuri, C., Sarradj, E. "Reducing aerofoil-turbulence interaction noise through chordwise-varying porosity." JFM (2020)

[3] Colbrook, M.J., Priddin, M.J. "Fast and spectrally accurate numerical methods for perforated screens." IMA Journal of Applied Mathematics (2020)

[4] Colbrook, M.J., Ayton, L.J. "Do we need non-linear corrections? On the boundary Forchheimer equation in acoustic scattering." Submitted

DAN FORTUNATO, Flatiron Institute
The ultraspherical spectral element method

We introduce a novel spectral element method based on the ultraspherical spectral method and the hierarchical Poincaré–Steklov scheme for solving second-order linear partial differential equations on polygonal domains with unstructured quadrilateral or triangular meshes. Properties of the ultraspherical spectral method lead to almost banded linear systems, allowing the element method to be competitive in the high-polynomial regime ($p > 5$). The hierarchical Poincaré–Steklov scheme enables precomputed solution operators to be reused, allowing for fast elliptic solves in implicit and semi-implicit time-steppers. The resulting spectral element method achieves an overall computational complexity of $\mathcal{O}(p^4/h^3)$ for mesh size h and polynomial order p, enabling hp-adaptivity to be efficiently performed. We develop an open-source software system, ultraSEM, for flexible, user-friendly spectral element computations in MATLAB. Joint work with Alex Townsend (Cornell University) and Nicholas Hale (Stellenbosch University).

TIMON GUTLEB, Imperial College London
Computing Equilibrium Measures with Power Law Kernels

Equilibrium measure problems naturally appear in the mathematical description of particle swarms in which particle behavior may be modeled via attractive and repulsive forces, for example ensemble movements in bird flocks, cellular scale organisms and classical particle interactions. Analytic solutions to equilibrium measure problems with power law kernels $K(x) = \frac{|x|^\alpha}{\alpha} - \frac{|x|^\beta}{\beta}$ exist for certain parameter choices but little is known about the behavior of solutions in high non-integer power cases, where discrete particle simulations predict interesting gap formation phenomena as the repulsive power increases. We introduce a banded sparse spectral method for such problems utilizing recurrence relationships in weighted ultraspherical polynomial bases. Numerical experiments agree with known analytic results as well as independent particle swarm simulations. Our method can be used to study solution behavior, uniqueness of solutions and the above-mentioned gap forming phenomenon.

ANDREW HORNING, Cornell University
Twice is enough for dangerous eigenvalues

A popular class of methods for large-scale eigenvalue problems use Cauchy's integral formulas to compute eigenvalues of a large matrix in a target region. We analyze the stability of these methods in the singular limit, i.e., as eigenvalues of the matrix approach the contour. Remarkably, contour-integral eigensolvers that incorporate subspace iterations are stable: the "dangerous eigenvalues" near the contour contribute large round-off errors in the first iteration, but are self-correcting in later iterations. For matrices with orthogonal eigenvectors (e.g., real-symmetric or complex Hermitian), two iterations is enough to reduce round-off errors to the order of the unit-round off. In contrast, contour-integral eigensolvers that construct Krylov subspaces typically fail to converge to unit round-off accuracy when an eigenvalue is close to the contour. However, we suggest a simple new restart strategy that recovers full precision in the target eigenpairs after two iterations.

NILIMA NIGAM, Simon Fraser University
Steklov eigenfunctions: how and why to compute them

In this talk we present a fast and accurate discretization strategy for computing the Steklov eigenpairs of the Laplacian. We'll also present recent result on the use of this method for three distinct problems: the spectral asymptotics of the Steklov eigenvalues on regular polygons, spectral optimization, and the solution of Robin problems.

SHEEHAN OLVER, Imperial College
Sparse spectral methods for singular integral and fractional differential equations

The ultraspherical spectral method originated as an approach for generate sparse, almost banded discretisations for ordinary differential equations. It was subsequently generalised to partial differential equations on simple geometries, singular integral equations with logarithmic kernels, and fractional differential equations. In this talk we review these developments and discuss new generalisations to power-law kernels.

Contains joint work with Timon Gutleb, Mikael Slevinsky and Alex Townsend.

MANAS RACHH, Flatiron Institute
Towards automatically adaptive solvers for Maxwell's equations in three dimensions

The numerical simulation of Maxwell's equations plays a critical role in chip and antenna design, radar cross section determination, biomedical imaging, wireless communications, and the development of new meta-materials and better waveguides to name a few. In order to enable design by simulation for problems arising in these applications, automatically adaptive solvers which resolve the complexity of the geometry and the input data play a critical role. In two dimensions, this has been made possible through the development of high-order integral equation based solvers which rely on well-conditioned integral representations, efficient quadrature formulas, and coupling to fast multipole methods. However, much is still to desired of these solvers in three dimensions (both in terms of their efficiency and accuracy), particularly in the context of enabling automatic adaptivity in complex geometries. In this talk, I will present an efficient high-order solver for solving Maxwell's equations in complex three dimensional geometries with focus on the efficient quadrature methods for computing singular integrals.

RICHARD MIKAEL SLEVINSKY, University of Manitoba
Fast associated classical orthogonal polynomial transforms

We discuss a fast approximate solution to the associated classical – classical orthogonal polynomial connection problem. We first show that associated classical orthogonal polynomials are solutions to a fourth-order quadratic eigenvalue problem with polynomial coefficients such that the differential operator is degree-preserving. Upon linearization, the discretization of this quadratic eigenvalue problem is block upper-triangular and banded. After a perfect shuffle, we extend a divide-and-conquer approach to the upper-triangular and banded generalized eigenvalue problem to the blocked case, which may be accelerated by one of a few different algorithms. Associated orthogonal polynomials arise from iterated Stieltjes transforms of orthogonal polynomials; hence, fast approximate conversion to classical cases combined with fast discrete sine and cosine transforms provides a modular mechanism for synthesis of singular integral transforms of classical orthogonal polynomial expansions.

ALEX TOWNSEND, Cornell University
Computing the spectra of differential operators

Spectral methods for solving differential eigenproblems usually follow the "discretize-then-solve" paradigm. Discretize first, and then solve the matrix eigenproblem. The discretize-then-solve paradigm can be tricky for differential eigenproblems as the spectrum of matrix discretizations may not converge to the spectrum of the differential operator. Moreover, it is impossible to fully capture the continuous part of the spectrum with a finite-sized matrix eigenproblem. In this talk, we will discuss an alternative "solve-then-discretize" paradigm for differential eigenproblems. To compute the discrete spectrum, we will discuss a continuous analogue of FEAST by approximating the action of the resolvent operator. For the continuous spectra, we will use a Cauchy-like integral to calculate a smoothed version of the so-called spectral measure. This is joint work with Matthew Colbrook and Andrew Horning.

TOM TROGDON, University of Washington
On arbitrary-precision enabled inverse scattering for the 1-dimensional Schrödinger operator

There is renewed interest in singularity dynamics of integrable systems in the complex x-plane. This was originally studied by Kruskal, Kruskal and Thickstun, and Bona and Weissler and more recently by Weideman and Ankiewicz, Clarkson and Akhmediev. One approach to study this is to perform numerical analytic continuation of the solution for real x. This motivates us to study methods to approximate solutions of the Korteweg-de Vries equation with high precision. As a first step, we consider the small time evaluation of a very special class of solutions by solving Riemann–Hilbert problems (i.e., singular integral equations) with arbitrary precision.