Symmetry in Dynamical Systems
Symétrie dans les systèmes dynamiques

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STEPHEN ANCO, Brock University

Symmetry multi-reduction method for partial differential equations with c onservation laws

For partial differential equations (PDEs) that have $n \geq 2$ independent variables and a symmetry algebra of dimension at least $n-1$, an explicit algorithmic method is presented for finding all symmetry-invariant conservation laws that will reduce to first integrals for the ordinary differential equation (ODE) describing symmetry-invariant solutions of the PDE. This significantly generalizes the double reduction method known in the literature. Moreover, the condition of symmetry-invariance of a conservation law is formulated in an improved way by using multipliers, thereby allowing symmetry-invariant conservation laws to be obtained directly, without the need to first find conservation laws and then check their invariance. This cuts down considerably the number and complexity of computational steps involved in the reduction method. If the space of symmetry-invariant conservation laws has dimension $m \geq 1$, then the method yields m first integrals along with a check of which ones are non-trivial via their multipliers. Several examples of interesting symmetry reductions are considered: travelling waves and similarity solutions in $1+1$ dimensions; line travelling waves, line similarity solutions, and similarity travelling waves in $2+1$ dimensions; rotationally symmetric similarity solutions in $n+1$ dimensions.

ROY H. GOODMAN, New Jersey Institute of Technology

Instability of Leapfrogging Vortex Pairs

We investigate the stability of a one-parameter family of periodic solutions of the four-vortex problem known as 'leapfrogging' orbits. These solutions, which consist of two pairs of identical yet oppositely-signed vortices, were known to Gröbli (1877) and Love (1883), and can be parameterized by a dimensionless parameter α related to the geometry of the initial configuration. Simulations by Acheson (2000) and numerical Floquet analysis by Tophøj and Aref (2012) both indicate, to many digits, that the bifurcation occurs when $1/\alpha = \phi^2$, where ϕ is the golden ratio. This study aims to explain the origin of this remarkable value. Using a trick from the gravitational two-body problem, we change variables to render the Floquet problem in an explicit form that is more amenable to analysis. We then implement G. W. Hill's method of harmonic balance to high order using computer algebra to construct a rapidly-converging sequence of asymptotic approximations to the bifurcation value, confirming the value found earlier. We also explain, via careful numerical study, many of the features of the nonlinear dynamics seen in earlier accounts.

CASEY JOHNSON, Claremont Graduate University

Spectrum of different Stieltjes star graphs

The spectrum of a single Stieltjes string, a thread bearing a finite number of point masses, is uniquely determined by the number and size of the masses. In 2002, F.R. Gantmakher and M.G. Krein solved the inverse problem which identified the location and mass of each bead given just the spectrum corresponding to Dirichlet boundary conditions and the spectrum corresponding to Neumann boundary conditions. Joining multiple Stieltjes strings of various lengths together to form a star graph shape which are often symmetric has fascinating implications on the spectrum of the graph. For these new star graphs, is the spectrum still uniquely determined? What can we say about the spectrum?

YURI LATUSHKIN, University of Missouri

The Maslov index and the spectrum of differential operators

We will review some recent results on connections between the Maslov and the Morse indices for differential operators. The Morse index is a spectral quantity defined as the number of negative eigenvalues counting multiplicities while the Maslov index is a geometric characteristic defined as the signed number of intersections of a path in the space of Lagrangian planes with the train of a given plane. The problem of relating these two quantities is rooted in Sturm's Theory and has a long history going back to the classical work by Arnold, Bott and Smale, and has attracted recent attention of several groups of mathematicians.

We will briefly mention how the relation between the two indices helps to prove the fact that a pulse in a gradient system of reaction diffusion equations is unstable. We will also discuss a fairly general theorem relating the indices for a broad class of multidimensional elliptic selfadjoint operators. Connections of the Maslov index and Hadamard-type formulas for the derivative of eigenvalues will be also discussed.

This talk is based on a joint work with M. Beck, G. Cox, C. Jones, P. Howard, R. Marangell, K. McQuighan, A. Sukhtayev, and S. Sukhtaiev.

MARK LEVI, Penn State

Gaussian curvature and spinning tops

I will describe the hidden role of Gaussuan curvature in the dynamics of the Lagrange top and will relate this to the Jacobi fields for geodesic flows. I will also describe the recently discovered phenomenon of "ponderomotive magnetism". This is based on joint work with Oleg Kirillov and Graham Cox.

GEORGE PATRICK, University of Saskatchewan

Geometric mechanics and the lump dynamics of the CP^1 sigma model on the sphere.

Usually the potential energy of a simple mechanical system breaks the larger symmetry of its kinetic energy. However, the CP^1 sigma model is a lagrangian field theory of potential plus kinetic type where the potential has the larger symmetry. In this case equilibria are not isolated but rather occur in a nongeneric way on group orbits of the symmetry of the potential, and the dynamics near such equilibria may be approximated by a geodesic flow. Applied to the CP^1 sigma model [J.M. Speight, Low-energy dynamics of a CP^1 lump on the sphere], this yields a finite dimensional simple mechanical system with the additional feature that its configuration space has nontrivial (constant type) isotropy, thereby providing a specific example of the general situation considered in the 1983 technical report [R. Montgomery, The structure of reduced cotangent phase spaces for non-free group actions].

MANUELE SANTOPRETE, Wilfrid Laurier University

On the Relationship between Two Notions of Compatibility for Bi-Hamiltonian Systems

Bi-Hamiltonian structures are of great importance in the theory of integrable Hamiltonian systems. The notion of compatibility of symplectic structures is a key aspect of bi-Hamiltonian systems. Because of this, a few different notions of compatibility have been introduced. In this talk we show that, under some additional assumptions, compatibility in the sense of Magri implies a notion of compatibility due to Fassò and Ratiu, that we dub bi-affine compatibility. There are at least two proofs of this fact. The first one uses the uniqueness of the connection parallelizing all the Hamiltonian vector fields tangent to the leaves of a Lagrangian foliation. The second proof uses Darboux-Nijenhuis coordinates and symplectic connections. In this talk we will attempt to outline both proofs.

ALEXEY F. SHEVYAKOV, University of Saskatchewan

Symbolic computation of symmetries and first integrals in dynamical systems

While the solution structure of dynamical systems given by ordinary differential equations (ODE) and ODE systems is well understood, and the solution behaviour can be analyzed locally and globally using various analytical methods, for the vast majority of nonlinear systems, it is not possible to write down the *general solution* in a closed form. In many cases, however, it is possible to reduce the differential order of the ODEs. Order can be reduced when one knows a conserved quantity (a *first integral*) that is constant on solutions, and/or a local *symmetry group* that leaves the ODE system invariant. A sufficient number of first integrals and/or local symmetries known for a given ODE model leads to its complete integration.

Lie groups of point, contact, and higher-order symmetries can be systematically computed using the Lie's algorithm, which, however, may require heavy computations.

Similarly, conservation laws of partial differential equations, and as a special case, first integrals of ODEs, can be computed using the direct (multiplier) method and the Euler differential operator that annihilates divergences. Such computations are also algorithmic, but for nontrivial examples, can be computationally demanding.

In this talk, we will demonstrate symbolic computations of symmetries and first integrals of ODEs using the `Maple`-based `GeM` symbolic software package. Examples of dynamical systems based on ODEs and nonlinear PDEs, such as shallow water models and the b-family of peakon equations, will be presented.

MICHAEL WARD, UBC

The Slow Dynamics of Spot Patterns for Reaction-Diffusion Systems on the Sphere

In the singularly perturbed limit corresponding to large a diffusivity ratio between two components in a reaction-diffusion (RD) system, many such systems admit quasi-equilibrium spot patterns, where the solution concentrates at a discrete set of points in the domain. In this context, we derive and study the differential algebraic equation (DAE) that characterizes the slow dynamics for such spot patterns for the Brusselator RD model on the surface of a sphere. Asymptotic and numerical solutions are presented for the system governing the spot strengths, and we describe the complex bifurcation structure and demonstrate the occurrence of imperfection sensitivity due to higher order effects. Localized spot patterns can undergo a fast time instability and we derive the conditions for this phenomena, which depend on the spatial configuration of the spots and the parameters in the system. In the absence of these instabilities, our numerical solutions of the DAE system for $N = 2$ to $N = 10$ spots suggest a large basin of attraction to a small set of possible steady-state configurations. We discuss the connections between our results and the study of point vortices on the sphere, as well as the problem of determining a set of elliptic Fekete points, which correspond to globally minimizing the discrete logarithmic energy for N points on the sphere.