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Quaternion Algebras and the Burkhardt Quartic

The Burkhardt quartic threefold in $\mathbb{P}^4$ is given by

$$B : f(y_0, y_1, y_2, y_3, y_4) := y_0(y_0^3 + y_1^3 + y_2^3 + y_3^3 + y_4^3) + 3y_1y_2y_3y_4 = 0.$$

This variety has been studied extensively since 1890 (originally by Heinrich Burkhardt), and has several different characterizations. Points on the Burkhardt quartic correspond to the class of curves that admit a model of the form $y^2 = h(x)$ where h is a squarefree polynomial of degree 6, together with 40 decompositions of the form

$$h(x) = G(x)^2 + \lambda H(x)^3.$$

Part of this correspondence involves marking 6 points on a conic C, and in order to obtain 6 corresponding points on $\mathbb{P}^1$ for defining h, it is necessary that C has a k-rational point. The Burkhardt has another, natural symmetric model $B' \subset \mathbb{P}^5$ given by

$$B' : \sigma_1(y_0, ..., y_5) = \sigma_4(y_0, ..., y_5) = 0,$$

where the σ_i are elementary symmetric functions. This model and the original Burkhardt are isomorphic over $\mathbb{C}$ (in fact over $\mathbb{Q}(\zeta_3)$), so they are geometrically equivalent. However, they are not isomorphic over $\mathbb{Q}$. In other words, B' is a nontrivial twist of B. Several properties over $\mathbb{Q}$ change drastically upon twisting the Burkhardt, in particular whether or not the conic C has $\mathbb{Q}$-rational points (for instance when obtained from B it does, while from B' there are local obstructions over $\mathbb{R}$ and $\mathbb{Q}_3$).