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Two categorical approaches to differentiation

In the past decade, we (coauthors Rick Blute, Robin Cockett and I) have formulated two different abstract categorical approaches to differential calculus, based on the structure of linear logic (an idea of Ehrhard and Regnier). The basic idea has two types of maps ("analytic" or "smooth", and "linear"), a comonad S (a "coalgebra modality"), somewhat like the **!** of linear logic, and a differentiation operator. In our first approach (**monoidal differential categories**), the coKleisli category (the category of cofree coalgebras) of S consists of smooth maps, and differentiation operates on coKleisli maps to smoothly produce linear maps. Our second approach (**Cartesian differential categories**) reversed this orientation, directly characterizing the smooth maps and situating the linear maps as a subcategory. If S is a "storage modality", meaning essentially that the "exponential isomorphisms" from linear logic ($S(X \times Y) \simeq S(X) \otimes S(Y)$ and $S(1) \simeq S(\top)$) hold, we get a tight connection between these approaches in the Cartesian (monoidal) closed cases: the linear maps of a Cartesian closed differential storage category form a monoidal closed differential storage category, and the coKleisli category of a monoidal closed differential storage category is a Cartesian closed differential storage category. Two technical aides in proving these results are the development of a graphical calculus as well as a term calculus for the maps of these categories. With the term calculus, one can construct arguments using a language similar to that of ordinary undergraduate calculus.