Functional Differential Equations and Applications
Équations différentielles fonctionnelles et applications
(Org: **Victor Leblanc** (University of Ottawa))

JACQUES BÉLAIR, Université de Montréal
Stability analysis in hematopoietic models

Time-delayed differential equations occur naturally in the representation of population models in general, and of stage-structured cell populations in particular. We present a number of variously sophisticated models for different aspects of hematopoiesis (production of blood cells) and analyse the stability of their equilibrium solutions. Characteristic equations obtained by linearisation around these equilibria will be shown to naturally contain multiple delays as well as state-dependant delays as a consequence of well-identified physiological hypotheses. The complexity of the characteristic equations and the associated stability charts, and bifurcation diagrams, will be shown not to be a monotonically increasing function of the complexity of the models.

JASON BRAMBURGER, University of Ottawa
Steady-state/Hopf interactions in the van der Pol oscillator with delayed feedback

In this talk we consider the traditional Van der Pol Oscillator with a forcing dependent on a delay in feedback. The delay is taken to be a nonlinear function of both position and velocity which gives rise to many different types of bifurcations. In particular, we study the Zero-Hopf bifurcation that takes place at certain parameter values using methods of centre manifold reduction of DDEs and normal form theory. We present numerical simulations that have been accurately predicted by the phase portraits in the Zero-Hopf bifurcation to confirm our numerical results and provide a physical understanding of the oscillator with the delay in feedback.

ELENA BRAVERMAN, University of Calgary
On stability of equations with a distributed delay

We study delay-independent stability in nonlinear models with a distributed delay and one or several positive equilibrium points which occur in population dynamics and other applications. It is assumed that the distributed delay is incorporated into the production term only. In particular, for models with one positive equilibrium we construct a relevant difference equation such that its stability implies stability of the equation with a distributed delay and a finite memory. This result is, generally speaking, incorrect for systems with infinite memory. If the relevant difference equation is unstable, we describe the general delay-independent lower and upper solution bounds and also demonstrate that the equation with a distributed delay is stable for small enough delays.

In the case when the production function incorporating the delay is monotone increasing, the dynamics is very similar to that of the ordinary differential equation. The qualitative behaviour of such equations can be comprehensively described also in the case of multiple positive equilibrium points.

SUE-ANN CAMPBELL, University of Waterloo
A Plankton Model with Time Delay

We consider a three compartment (nutrient-phytoplankton-zooplankton) model with nutrient recycling. When there is no time delay the model has a conservation law and may be reduced to an equivalent two dimensional model. We consider how the conservation law is affected by the presence of a state dependent time delay in the model. We study the stability and bifurcations of equilibria when the total nutrient in the system is used as the bifurcation parameter. This is joint work with Matt Kloosterman and Francis Poulin.

YUMING CHEN, Wilfrid Laurier University
Global dynamics of delayed reaction-diffusion equations in unbounded domains

We consider a nonlocal delayed reaction–diffusion equation in an unbounded domain that includes some special cases arising from population dynamics. Due to the non-compactness of the spatial domain, the solution semiflow is not compact. We first show that, with respect to the compact open topology for the natural phase space, the solutions induce a compact and continuous semiflow Φ on a bounded and positively invariant set Y in $C_+ = C([-1,0], X_+)$ that attracts every solution of the equation, where X_+ is the set of all bounded and uniformly continuous functions from $\mathbb{R}$ to $[0,\infty)$. Then, to overcome the difficulty in describing the global dynamics, we establish a priori estimate for nontrivial solutions after describing the delicate asymptotic properties of the nonlocal delayed effect and the diffusion operator. The estimate enables us to show the permanence of the equation with respect to the compact open topology. With the help of the permanence, we can employ standard dynamical system theoretical arguments to establish the global attractivity of the nontrivial equilibrium. The main results are illustrated with the diffusive Mackey–Glass equation. This is a joint work with Profs. Taishan Yi and Jianhong Wu.

NEMANJA KOSOVALIC, York University
Algebraic-Delay Differential Systems: Age Structured Population Modeling and C^0-Extendable Submanifolds

Consider a population of individuals occupying some habitat, which is structured by age. Suppose there are two distinct and non-competing life stages, the immature stage and the mature stage. A natural question is "What determines the age of maturity?". In many biological contexts, the age of maturity is determined by whether or not the resource concentration density, which depends on the immature population, reaches a prescribed threshold. Mathematically, this situation takes the form of a first order nonlinear transport equation coupled to an algebraic-delay term. This system gives rise to a solution semiflow, and in this talk we discuss the problem of the differentiability of this semiflow with respect to initial data. The main challenge is finding the right phase space and the right type of differentiability to recover the desired result. This is joint work with Jianhong Wu and Yuming Chen.

VICTOR LEBLANC, University of Ottawa
Nilpotencies and realizability in delayed nonlinear oscillators

The effects of delayed feedback terms on nonlinear oscillators have been extensively studied, and have important applications in many areas of science and engineering. We study a particular class of second-order delay-differential equations near a point of triple-zero nilpotent bifurcation. Using center manifold and normal form reduction, we show that the three-dimensional nonlinear normal form for the triple-zero bifurcation can be fully realized at any given order for appropriate choices of nonlinearities in the original delay-differential equation.

JÉRÉMIE LEFEBVR, University of Geneva
Non-autonomous stability in non-linear delayed systems

Delayed non-linear systems have a widespread use across the engineering, physical and biological sciences, displaying a wide variety of complex dynamics. Non-autonomous perturbations, such as noise, greatly impact the dynamics of such systems and further alter their stability. I will here present an approach to handle that problem using center manifold theory formulated for delay equations, where an explicit time-dependence is taken into account. Using the characteristic time scale separation emerging in the vicinity of instabilities, this approach makes possible to capture the effect of forcing on the stability of a delayed non-linear system in the vicinity of a bifurcation, and serves as new strategy to highlight novel non-linear phenomena in the harmonically driven and/or stochastic regimes. Using this method, we propose to first expose the effect of additive random fluctuations on the stability of a particular scalar delayed non-linear system near a pitchfork bifurcation and demonstrate how noise might be used to prevent the system from entering bistable domains. We extend this result to the delay-induced Hopf bifurcation by showing how noise suppresses ongoing oscillatory dynamics by shifting the instability threshold in parameter space, as revealed by the dynamics of the ensemble average solution.

CONNELL MCCLUSKEY, Wilfrid Laurier University
Using Lyapunov Functions to Construct Lyapunov Functionals

Many epidemic models are based on ODEs and can be written as

$$x'(t) = f(x(t)).$$

Standard analysis includes calculating the basic reproduction number $\mathcal{R}_0$. Models often exhibit threshold behaviour where there is a disease-free equilibrium that is globally asymptotically stable for $\mathcal{R}_0 < 1$ and an endemic equilibrium that is globally asymptotically stable for $\mathcal{R}_0 > 1$. In the last ten years, the case where $\mathcal{R}_0 > 1$ has been resolved for many ODE models through the use of Lyapunov functions based on the Volterra function

$$g(x) = x - 1 - \ln x.$$

Other models include delay to better describe certain biological processes. These systems can be written as

$$x'(t) = f(x(t), x(t-\tau))$$

or, more generally,

$$x'(t) = f(x_t), \tag{1}$$

where $x_t : [-\tau, 0] \to R^n$ for some $\tau > 0$. Lyapunov functionals based on the Volterra function g have recently been used to resolve the global stability for many delay models for the case $\mathcal{R}_0 > 1$. In reviewing the many examples, it becomes clear that the Lyapunov functional for the delay equation is very strongly related to the Lyapunov function that works for the corresponding ODE.

In this work, we analyze the connection between the Lyapunov functional that works for Equation (1) and the Lyapunov function that works for the corresponding ODE. We obtain a test that allows one to classify, a priori, the terms that can incorporate delay without affecting the global asymptotic behaviour of the system.

MING MEI, Champlain College St.-Lambert and McGill University
Stability of oscillating wavefronts for time-delayed reaction-diffusion equation

This talk is concerned with Nicholson's blowflies equation. It is known that, when the ratio of birth rate coefficient and death rate coefficient satisfies $1 < \frac{p}{d} \leq e$, the equation is monotone, and possesses monotone traveling wavefronts, which are intensively studied in the previous research. However, when $\frac{p}{d} > e$, the equation losses its monotonicity, and its traveling waves are oscillatory when the time-delay r or the wave speed c is large, which causes the study of stability of these non-monotone traveling waves to be challenging. In this talk, we present that, when $e < \frac{p}{d} \leq e^2$, all traveling waves $\phi(x+ct)$ with $c \geq c_* > 0$, including the critical waves and non-critical waves, are asymptotically stable, and the non-critical waves are exponentially stable, where $c_* > 0$ is the minimum wave speed. In this case, we allow all traveling waves to be any, monotone or non-monotone with any speed $c \geq c_*$, and any size of the time-delay $r > 0$; While, when $\frac{p}{d} > e^2$ with a small time-delay $r < \left[\pi - \arctan\sqrt{\ln\frac{p}{d}(\ln\frac{p}{d} - 2)}\ \right]/d\sqrt{\ln\frac{p}{d}(\ln\frac{p}{d} - 2)}$, then all traveling waves $\phi(x+ct)$ with $c \geq c_* > 0$ are stable, too, and the non-critical waves are exponentially stable. A new technique with a nonlinear Halanay's inequality is introduced to establish weighted L^2 energy estimates. As a corollary, we also prove the uniqueness of traveling waves in the case of $\frac{p}{d} > e^2$, which was open as we know. Some numerical simulations in different cases are also carried out, which either confirm and support our theoretical results, or open up some new phenomena for future research.

JIANHONG WU, York University, Centre for Disease Modelling
Spatial dynamics of Asia Clam invasion

Asian clam (Corbicula fluminea) is one of the most important nonnative aquatic invasive species in freshwater ecosystem, which can rapidly spread in lakes, canals, streams, and rivers throughout many parts of the world. This species has remarkably distinct mobility pattern in different phases of its life cycle. The adult clams are hermaphroditic, making the dynamic model of Asian clams a coupled system of delay differential equations with nonlocal response. We formulate a novel mathematical model to calculate the invasive speed, describe the characteristic of the speed, and show that the invasion speed coincides with the minimal wave speed of traveling wave. This is based on joint work with Jian Fang, Kunquan Lan and Gunog Seo.

PEI YU, Western University
Computation of normal forms of delay differential equations

Computing normal forms of differential equations (DE) is important in the study of dynamical behaviors such as stability and bifurcation, which is particularly difficult and tedious for delay differential equations (DDE). The center manifold reduction (CMR) method is usually applied to compute normal forms of DDEs. Recently, the multiple time scales (MTS) method has been directly applied to compute normal forms of DDEs and shown to be simpler than the CMR method. In this talk, we will show the equivalence of the MTS method and the CMR method in computing the normal forms of DDEs. The delays involved in the DDEs can be discrete or distributed. Particular attention is focused on dynamics associated with the semisimple singularity. The two methods are proved to be equivalent for ordinary differential equations up to any order, while up to third order for DDEs under certain conditions, which can be satisifed in most of practical problems. The equivalence of the two methods can be extended to other types of DDEs, including neutral functional differential equations (NFDE) (or neutral delay differential equations (NDDE)), and partial functional differential equations (PFDE). Several practical systems with delays are presented to demonstrate the application of the methods, associated with Hopf, Hopf-zero, and double-Hopf singularities.

XIAOQIANG ZHAO, Memorial University of Newfoundland
Global Dynamics of A Time-Delayed Dengue Transmission Model

In this talk, I will present an age-structured dengue model with time delays for the cross infection between mosquitos and human individuals. We first introduce the basic reproduction number R0 for this model and then show that the disease persists if R0 is greater than one, and the disease dies out if R0 is less than one, provided that the invasion intensity is not strong. We further establish a set of sufficient conditions for the global attractivity of the endemic equilibrium by using the method of fluctuations and the theory of chain transitive sets. This talk is based on a joint work with Zhen Wang.

HUAIPING ZHU, York
Oscillation and driving mechanism in compartmental models for mosquito-borne diseases with time delay

West Nile virus, malaria and dengue are typical mosquito-borne diseases which are transmitted to humans through the bite of vector-mosquitoes. Vectors like mosquitoes play a critical role in the transmission and spread of the diseases. To investigate the role of vectors and the transmission dynamics of mosquito-borne diseases, we formulate a system of delay differential equations involving vectors and hosts to study the impact of average temperature. In the model, we choose the standard incidence rate to model the interaction between vectors and amplification hosts. Bifurcation analysis of the system shows that the mosquito population can force the system to oscillate, yet the usual incidence between the vector and host can not. This result indicates that vector population is the driving factor for the oscillation in disease transmission under the impact of temperature. This talk is based on the joint work with Dr. Guihong Fan.

XINGFU ZOU, University of Western Ontario
On the basins of attraction for a class of of delay differential equations with non-monotone bistable nonlinearities

We consider the delay differential equations (DDE) $\dot{x}(t) = -g(x(t)) + f(x(t-\tau))$ which share the same equilibria with the corresponding ordinary differential equation (ODE) $\dot{x}(t) = -g(x(t)) + f(x(t))$. For the bistable case, both the DDE and ODE share three equilibria $x_0 = 0 < x_1 < x_2$ with x_0 and x_2 being stable and x_1 being unstable for the ODE. We are concerned with stability of these equilibria for the DDE and the basins of attraction of x_0 and x_2 when they are asymptotically stable for the DDE. Combining the idea of relating the dynamics of a map to the dynamics of a DDE with invariance arguments for the solution semiflow, we are able to characterize some subsets of basins of attraction of these equilibria for the DDE. In addition, existence of heteroclinic orbits are also explored. The general results are applied to a particular model equation describing the matured population of some species demonstrating the Alee effect.