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Using Lyapunov Functions to Construct Lyapunov Functionals

Many epidemic models are based on ODEs and can be written as

$$x'(t) = f(x(t)).$$

Standard analysis includes calculating the basic reproduction number $\mathcal{R}_0$. Models often exhibit threshold behaviour where there is a disease-free equilibrium that is globally asymptotically stable for $\mathcal{R}_0 < 1$ and an endemic equilibrium that is globally asymptotically stable for $\mathcal{R}_0 > 1$. In the last ten years, the case where $\mathcal{R}_0 > 1$ has been resolved for many ODE models through the use of Lyapunov functions based on the Volterra function

$$g(x) = x - 1 - \ln x.$$

Other models include delay to better describe certain biological processes. These systems can be written as

$$x'(t) = f(x(t), x(t-\tau))$$

or, more generally,

$$x'(t) = f(x_t), \tag{1}$$

where $x_t : [-\tau, 0] \to R^n$ for some $\tau > 0$. Lyapunov functionals based on the Volterra function g have recently been used to resolve the global stability for many delay models for the case $\mathcal{R}_0 > 1$. In reviewing the many examples, it becomes clear that the Lyapunov functional for the delay equation is very strongly related to the Lyapunov function that works for the corresponding ODE.

In this work, we analyze the connection between the Lyapunov functional that works for Equation (1) and the Lyapunov function that works for the corresponding ODE. We obtain a test that allows one to classify, a priori, the terms that can incorporate delay without affecting the global asymptotic behaviour of the system.