ALEXANDRU NICA, University of Waterloo

Hopf algebras and the logarithm of the S-transform in free probability

I will present a joint paper with M. Mastnak (arXiv:0807.4169), where Hopf algebra methods are used in order to study the operation of multiplying freely independent k-tuples of noncommutative random variables with unit mean. This operation is naturally encoded by a group structure $(G_k, \boxtimes)$, where G_k is a suitable set of noncommutative distributions and $\boxtimes$ is the operation of free multiplicative convolution on G_k. We identify $(G_k, \boxtimes)$ as the group of characters of a certain Hopf algebra $Y^{(k)}$. Then, by using the log map from characters to infinitesimal characters of $Y^{(k)}$, we introduce a transform LS_μ for distributions $\mu \in G_k$. Combinatorially, the coefficients of the series LS_μ are obtained from the free cumulants of μ via an explicit summation formula, involving chains in lattices of non-crossing partitions. The LS-transform has the 'linearizing' property that $LS_{\mu \boxtimes \nu} = LS_\mu + LS_\nu$ for μ, ν in G_k such that $\mu \boxtimes \nu = \nu \boxtimes \mu$.

In the particular case $k = 1$, $Y^{(1)}$ is naturally isomorphic to the Hopf algebra Sym of symmetric functions, and the LS-transform is very closely related to the S-transform of Voiculescu, by the formula $LS(z) = -z \log S(z)$. In this case the group G_1 can be identified as the group of characters of Sym in such a way that the S-transform, its reciprocal $1/S$ and its logarithm $\log S$ relate in a natural sense to the sequences of complete, elementary and respectively power sum symmetric functions.