JON BORWEIN, Dalhousie Faculty of CS

Computationally discovered and proved generating functions

This lecture will describe older and very recent work [2], [4] in which Bailey, Bradley and I hunted for various desired generating functions for zeta functions and then were able to methodically prove our results.

One example is

$$3\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}(k^2-x^2)}\prod_{n=1}^{k-1}\frac{4x^2-n^2}{x^2-n^2}=\sum_{n=1}^{\infty}\frac{1}{n^2-x^2} \tag{1}$$

$$\left[=\sum_{k=0}^{\infty}\zeta(2k+2)\,x^{2\,k}\;=\frac{1-\pi x\cot(\pi x)}{2x^2}\right].$$

The constant term in (1) recovers the well known identity

$$3\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}k^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}=\zeta(2).$$

Equivalently, for each positive integer k one has the *generalized hypergeometric* identity

$${}_3\mathrm{F}_2\left(\left.\begin{matrix}3k,-k,k+1\\2k+1,k+\frac{1}{2}\end{matrix}\right|\frac{1}{4}\right)=\frac{\binom{2k}{k}}{\binom{3k}{k}}. \tag{2}$$

As I hope to show, discovering (1) and then proving (2) formed one of the most satisfying experimental mathematics experiences I have had. I will also describe more recent work to appear in [3, 2008] regarding

$${}_3\mathrm{F}_2\left(\left.\begin{matrix}3k,-k,k+1\\2k+1,k+\frac{1}{2}\end{matrix}\right|1\right). \tag{3}$$

References

[1] D. H. Bailey and J. M. Borwein, *Experimental Mathematics: Examples, Methods and Implications.* Notices Amer. Math. Soc. **52**(2005), 502–514.

[2] David Bailey, Jonathan Borwein and David Bradley, *Experimental Determination of Apéry-type Formulae for* $\zeta(2n+2)$. Experiment. Math. **15**(2006), 281–289. [D-drive Preprint 295]

[3] Jonathan M. Borwein and David H. Bailey, *Mathematics by Experiment: Plausible Reasoning in the 21st Century.* A. K. Peters, Natick, MA, 2004; second extended edition, 2008.

[4] Jonathan M. Borwein, David H. Bailey and Roland Girgensohn, *Experimentation in Mathematics: Computational Paths to Discovery.* A. K. Peters, Natick, MA, 2004.