Category Theory
Théorie des catégories

(Org: **Salja Deni**, **Robert Morisette** and/et **Dorette Pronk** (Dalhousie University))

ALEXANDRE CLÉMENT, Inria, LMF, Université Paris-Saclay
Complete Equational Theories for Quantum circuits

This talk synthesises recent work on complete sets of relations for quantum circuits. We introduce the first complete equational theory for quantum circuits, thereby answering a question which had been open for two decades. A complete equational theory is a set of equations between circuits that enables one to transform any two equivalent circuits into one another, by successive local rewriting. We obtain this result by means of a back-and-forth encoding between quantum circuits and a graphical language for linear optical circuits, for which we had previously obtained a simple complete equational theory. We show that the set of equations given by this method can be simplified into one made of a few simple and intuitive equations, which we prove to be minimal, in the sense that none of the equations can be derived from the others. One of the equations, although trivial regarding its semantics, acts on an arbitrary number of qubits, and we show that this cannot be avoided in the framework of ancilla-free quantum circuits. Finally, we extend the completeness result to circuits with ancillae and/or qubit discarding, and we show that in this framework, the equation on an arbitrary number of qubits can be removed, leading to an equational theory made of equations acting on at most 3 qubits. Additionally, we also obtain a completeness result for a finitely generated, universal fragment of quantum circuits. This is joint work with Noé Delorme, Nicolas Heurtel, Shane Mansfield, Simon Perdrix, Benoît Valiron, and Renaud Vilmart.

AMÉLIE COMTOIS, University of Ottawa
Notions of Smallness and Completeness for V-Graded Categories

Categories graded by a monoidal category $\mathcal{V}$, which we call $\mathcal{V}$-graded categories, generalize both $\mathcal{V}$-enriched categories and $\mathcal{V}$-actegories. In previous work, we studied a notion of weighted limits for $\mathcal{V}$-graded categories that specializes to recover the usual notion of $\mathcal{V}$-enriched weighted limit. These $\mathcal{V}$-graded weighted limits are equivalently $[\mathcal{V}^{op}, SET]$-enriched weighted limits with representable-valued weights.

In this talk, we continue to explore $\mathcal{V}$-graded weighted limits by first defining a notion of smallness in the $\mathcal{V}$-graded context then identifying equivalent conditions for the existence of all small $\mathcal{V}$-graded weighted limits. Given that $\mathcal{V}$-graded categories are $[\mathcal{V}^{op}, SET]$-categories and that $[\mathcal{V}^{op}, SET]$ is not locally small, there are some complications with the usual notion of smallness. We therefore define our own notions of smallness and local smallness of $\mathcal{V}$-graded categories by requiring that the hom-presheaves be small presheaves. A $\mathcal{V}$-graded weighted limit is small if the shape of its diagram is a small $\mathcal{V}$-graded category. This $\mathcal{V}$-graded notion of smallness specializes to recover the familiar notion of smallness for $\mathcal{V}$-enriched categories. It is well known that a $\mathcal{V}$-category has all small weighted limits if it has all powers and small conical limits. We extend this result to provide equivalent conditions for a $\mathcal{V}$-graded category to admit all small $\mathcal{V}$-graded weighted limits. This talk is based on joint work with Richard Blute and Rory Lucyshyn-Wright.

DARIEN DEWOLF, St. Francis Xavier University
Crossed Modules of Inverse Semigroups as Internal Categories

(*This is joint work with Dorette Pronk and David Zeidler.*)

A classical theorem of Spencer and Brown (1976) gives an equivalence

$$\mathsf{XMod/Group} \simeq \mathsf{Cat(Group)},$$

identifying crossed modules of groups with categories internal to Group (i.e., 2-groups). This equivalence underlies the 2-dimensional Seifert-van Kampen theorem and stands as one of the earliest uses of higher-dimensional algebra.

In this talk, we extend the Spencer-Brown equivalence to *inverse semigroups*: semigroups in which every element has a unique partial inverse, modelling local or partial symmetries. The proof of Spencer-Brown rests on a tight correspondence between split epimorphisms and semidirect products of groups, and neither construction behaves well on the nose in InvSemiGrp. After explaining how Billhardt's *full restricted semidirect product* and *split Billhardt congruences* together recover this correspondence, we introduce *Billhardt categories internal to inverse semigroups* (internal categories in InvSemiGrp whose source map is idempotent-splitting) and prove that

$$\mathsf{XMod}/\mathsf{InvSemiGrp} \simeq \mathsf{BCat}(\mathsf{InvSemiGrp}).$$

The crossed module axioms must be reformulated to compensate for the failure of full invertibility, but the equivalence restricts to Spencer-Brown's on the full subcategory Group $\subset$ InvSemiGrp.

AARON FAIRBANKS, Dalhousie University
Comonads as spaces

Comonads on Set generalize both categories and topological spaces. We characterize topological spaces as precisely the density comonads of diagrams of subsets of some set, i.e. topological subbases. Following Garner's work on ionads, we develop aspects of the theory of topological spaces for arbitrary comonads on arbitrary categories, including notions of basis, subbasis, continuous map, and specialization. Continuous maps and ordinary comonad morphisms form a double category, which in the case of comonads on Set corresponding to categories recovers the double category of functors and retrofunctors of Clarke and Di Meglio. Joint work with Kevin Carlson and David Spivak.

MARTIN FRANKLAND, University of Regina
An invitation to n-angulated categories

Triangulated categories arise in topology and in algebra, to capture the structure of cofiber sequences. Examples include the stable homotopy category of spaces and the derived category of a ring. Geiss, Keller, and Oppermann introduced n-angulated categories to capture the structure found in certain cluster tilting subcategories in quiver representation theory. This talk will provide an introduction to n-angulated categories, highlighting some similarities and differences with triangulated categories (the case n=3). I will briefly advertise joint work with Sebastian Martensen and Marius Thaule on Toda brackets in n-angulated categories.

ALI HAMAD, University of Ottawa
Ultracategories: Past and Future

Ultracategories were introduced by Makkai to axiomatise the idea of a category with ultraproduct functors, and to show the conceptual completeness theorem, linking the syntax and semantics of coherent logic. We give an outline of the historical development of the theory of ultracategories, starting with the initial work of Makkai and the subsequent work of Marmolejo and Zawadowski, passing by the new framework of Lurie which redefined ultracategories more concisely. We then explore new research in ultracategories which sprung in three directions: realising ultracategories as certain algebras for a pseudo-monad; using ultracategories to study the idea of continuous families of models; and defining generalisations of ultracategories aimed at showing a more general version of conceptual completeness. We end by exploring new research directions.

NATHAN HAYDON, University of Waterloo
Compositional First-Order Logic

We discuss a recent string-diagrammatic calculus that is complete for first-order logic. The calculus achieves its results by combining cartesian and linear bicategories. We present the development of the calculus, motivate the inference rules, and suggest future directions.

JACK JIA, UW/PI
Introduction to Deligne Categories

A symmetric tensor category is (roughly speaking) an abelian k-linear rigid symmetric monoidal category. The classic examples of such categories take the form Rep(G) where G is a group. By a remarkable theorem of Deligne, if the base field k has characteristic 0, then all symmetric tensor categories satisfying some dimensional constraints are in fact equivalent to a category of representations (of a supergroup).

But what happens when these constraints are lifted? To explore the case where objects are allowed to have non-integer dimensions, Deligne constructed symmetric tensor categories $\underline{\text{Rep}}(S_t)$, $\underline{\text{Rep}}(GL_t)$, where t is an arbitrary complex number. In this talk, I will introduce these categories and show why they can be viewed as the result of "interpolating" the classical representation categories Rep(S_n), Rep(GL_n).

CAMERON KRULEWSKI, Dalhousie University
Dagger Categories and Higher Spin Statistics

Dagger categories, which generalize the category of Hilbert spaces equipped with the dagger operation, may be used in functorial field theory to model the physical property of unitarity: one requires unitary functorial field theories to be functors of dagger categories. In the case of invertible, fully-extended functorial field theories, we construct an action of the orthogonal group extending the reflection and spin flip actions on manifolds as well as the complex conjugation and fermion parity operations on super Hilbert spaces. We show that for this subclass of theories, unitarity imposes an equivariance condition for the O-action, which we interpret as a higher version of the spin-statistics theorem in quantum field theory.

This talk is based on joint work with L. Müller and L. Stehouwer.

Cameron Krulewski, Lukas Müller, and Luuk Stehouwer. "A Higher Spin-Statistics Theorem for Invertible Quantum Field Theories." Commun. Math. Phys. 406, 230 (2025).

ROSE KUDZMAN-BLAIS, Kyoto University
Linear and Cyclic $*$-Autonomous Proarrow Equipments

There is often more than one type of arrow between objects. Frequently, there are maps which strictly preserve structure and more relaxed morphisms which behave like objects themselves, e.g. functions versus relations. A paradigm to study both simultaneously is given by double categories, introduced by Ehresmann, and pseudo double categories, defined by Grandis and Paré. In particular, these double categories often arise from proarrow equipments, as developed by Wood.

On the other hand, with the advent of linear logic by Girard, and the introduction of cyclic $*$-autonomous and linear bicategories by Cockett, Koslowski and Seely, it has become clear that certain bicategories have two linked compositions: tensor and par. Indeed, besides standard relational composition, the bicategory Rel is equipped with a relational sum composition. Further examples were then developed by Blute, K-B and Niefield. Crucially, the tensor structures of most examples are canonically proarrow equipments, inducing well-studied double categories. It is then natural to ask how these stricter maps interact with par.

In this talk, we will discuss joint work with Robert Morissette and Dorette Pronk exploring how a proarrow equipment can be compatible with the par of linear bicategories and the linear negation of cyclic $*$-autonomous bicategories. This compatibility induces new double categories with par as loose composition. We will demonstrate how most examples fit within this paradigm, moreover how piercean bicategories are captured by this new framework, as introduced by Bonchi, Di Giorgio, Trotta to study the cartesian structure of the linear bicategory Rel.

MICHAEL LAMBERT, Norwich University
Twisted Double Functors

A twisted double functor between strict double categories is a lax double functor on the transpose of the source double category. This, together with our canonical examples, inspires a new definition of a "twisted double functor" between double categories that interchanges loose and tight structures as it makes its assignments from source to target. This talk will introduce this new notion and illustrate it by examples. Particularly, a profunctor-valued "twisted hom" functor can associated to any double category. This packages the loose structure of the represented double category and is naturally twisted in our sense. By precomposition with a point, any such twisted hom functor gives rise the further example of a canonical twisted representable. Finally, we will also discuss examples of adjunction-valued twisted double functors arising from doctrines and bi-indexed categories. This is a report on joint work with Evan Patterson and David Jaz Myers.

RORY LUCYSHYN-WRIGHT, Brandon University
Infinitary cartesian theories: A logic of locally presentable categories

In 1977, Coste defined a system of restricted first-order logic that was later called *cartesian logic* by Johnstone (cf. Freyd), with finite conjunction and provably unique existential quantification. Coste established a complete deductive system of cartesian logic and showed that the categories of models of cartesian theories are equivalently locally finitely presentable categories.

In this talk on joint work with Andrew Krenz, we define a deductive system of *α-ary cartesian logic* for a regular cardinal α, and we prove a completeness theorem for α-ary cartesian logic. We define a notion of α-ary cartesian theory, and we show that the categories of models of α-ary cartesian theories are equivalently locally α-presentable categories, noting that "limit theories" in the sense of Adámek and Rosický provide special examples of α-ary cartesian theories. Moreover, building on joint work with Jason Parker, we prove a sharper result for a fixed set of sorts S, namely that the categories of models of S-sorted α-ary cartesian theories are equivalently S-sorted locally α-presentable categories in a suitable sense.

ROBERT MORISSETTE, Dalhousie University
Oppositizing Cells in Equipments

Given an arbitrary double category $\mathbb{D}$, one can define the double category $\mathbb{D}^{con}$ with the same underlying structure as $\mathbb{D}$, but the directions of the cells formally reversed. For equipments (sometimes called framed bicategories or fibrant double categories), that is, double categories for which every tight arrow f has a representative loose arrow (called a companion of f) with a loose right adjoint (called a conjoint of f), the situation becomes more subtle. It is well-known that, in an equipment, an arbitrary cell can be equivalently represented (as a globular cell) in four different ways. One representation requires only the presence of companions, another only conjoints, and the remaining two require both. However, taking opposites of cells is not well-defined on these equivalence classes. Moreover, for the first two types of these representatives, opposites of each of these kinds of cells naturally fit back together into double categories (these are the double categories of retrocells and coretrocells of Paré), while the other two do not in general.

In this talk, we will examine this idea of oppositizing cells through the lens of the formal theory of fibrations, give some conditions for when each of the four kinds of opposite cells fit back together into double categories, and, time permitting, speak on connections to logic and how we believe this connects two different versions of "double categories of relations".

HAYATO NASU, Dalhousie University
On the decomposition of a strong epimorphism into regular epimorphisms

One of the most fundamental theorems in algebra is the fundamental theorem of homomorphisms, which states that $\mathrm{Im}(f) \cong A/\mathrm{Ker}(f)$ for any homomorphism $f\colon A \to B$. In categorical terms, this states that strong epimorphisms and regular epimorphisms are equivalent in categories of algebraic structures such as groups, rings, and so on. Although this equivalence fails in general, it is known that in any locally presentable category, every strong epimorphism can be decomposed into a transfinite composite of regular epimorphisms. In this talk, I will discuss how many regular epimorphisms are needed in such a decomposition, which measures the extent to which the fundamental theorem of homomorphisms holds for general algebra-like structures such as n-categories.

This talk is based on joint work with Yuto Kawase.

Yuto Kawase, Hayato Nasu. "On the decomposition of a strong epimorphism into regular epimorphisms." arXiv:2604.05744.

ROBERT PARÉ, Dalhousie University
A topos for categories

The unruly behaviour of coequalizers in $\mathbf{Cat}$ creates problems for double categories. To address these, we embed it into a topos in a way that preserves the desirable features of $\mathbf{Cat}$ and "fixes" the quotients. We treat the objects of this topos as if they were actual categories and go on to investigate what category theory would be like without composition. In this talk we concentrate on adjoints, the cornerstone of category theory.

EVAN PATTERSON, Topos Institute
Double-categorical logic for unbiased categorical structures

Double categories are increasingly recognized as a powerful tool for formal category theory. The speaker and collaborators have developed a notion of double-categorical theory whose models are categorical structures, categorifying the familiar story from categorical logic that (one-dimensional) theories have set-theoretical structures as models. A cartesian double theory given by finite presentation describes the operations of a categorical structure in biased terms, categorifying the situation for algebraic theories; for example, a monoidal category is axiomatized in terms binary and nullary products, rather than n-ary products for each natural number n. However, some categorical structures, like multicategories and PROPs, are natively defined in unbiased form, whereas others, like monoidal categories and cartesian categories, admit unbiased axiomatizations that can be more convenient than the standard biased ones. In this talk, we propose a new notion of double theory able to give finitary axiomatizations of unbiased categorical structures, including all those mentioned above. The intended semantics for such a theory is a double monad on Span generalizing the familiar list monad on Set. We argue that this approach lends itself well to type-theoretic and computer implementation, sketching ongoing research to uniformly specify the internal languages of models of such theories and ultimately to implement those languages in the software system CatColab.

LAURA SCULL, Fort Lewis College
(Lack of) Model Structure for Homotopy of Graphs

Since the introduction of discrete homotopy theories for graphs, researchers have been interested in constructing model structures for these homotopy theories. The results along this line have largely been negative and include results of [GS] for $\times$-homotopy and [CKK] for A-homotopy. I will discuss these results and their generalizations, and what we can learn from these results about the nature of discrete homotopy theories and their resistance to model structures.

This talk is based on joint work with M. Youssef, with support from the rest of the Adjoint School 2025 Homotopy of Graphs Team: R. Hardeman, M. Ramos Huila, J. Nickel, N. Samadzelkava.

[GS] Goyal, S., Santhanam, R. "(Lack of) Model Structures on the Category of Graphs". Appl Categor Struct 29, 671–683 (2021)

[CKK] Daniel Carranza, Krzysztof Kapulkin, and Jinho Kim. "Nonexistence of Colimits in Naive Discrete Homotopy Theory". Applied Categorical Structures (2023).

DANIEL TEIXEIRA, Dalhousie University
(∞,∞)-categories with adjoints are a bit like spaces

Recent developments have been reinforcing the idea that (∞,∞)-categories are directed analogues of spaces (=∞-groupoids) (=homotopy types) (=anima). Some examples are the directed analogues of products, cylinders, cones, loop spaces, suspensions, joins, spectra, fibers, fibrations, and so on. Despite this happy observation, in many ways, (∞,∞)-categories have no hope to behave like spaces precisely because of their directionality. For instance, unlike spaces, the "homotopy monoid" of an (∞,∞)-category depends tremendously on the choice basepoint.

However, if we restrict ourselves do (∞,∞)-categories with adjoints (at all levels), the theory starts to resembles the theory of spaces much more closely (essentially because then we can leave the basepoint and come back). We will explain that by examples, focusing in particular in a Grothendieck construction involving such (∞,∞)-categories with adjoints.

JEAN-BAPTISTE VIENNEY, University of Ottawa
Non-uniqueness of differentiation in differential categories

Differential categories are a categorical setting for differentiation based on symmetric monoidal categories. Lemay asked whether the differentiation operator is unique in a differential category. We will present a counterexample answering this question in the negative, that is, a differential category with more than one differentiation operator (actually, infinitely many).

SCOTT WESLEY, Dalhousie University
Parameterized Quantum Circuit Semantics Through Enriched Categories

It is well-known that combinatorial circuits are modeled mathematically by string diagrams in a monoidal category. Given a gate set Σ, the circuits over Σ can be thought of as string diagrams in the free monoidal category generated by Σ. In this model, circuit semantics are then given by monoidal functors out of this free category. For quantum circuits, this functor is often valued in the category of unitary matrices. This model suffices for concrete quantum circuits, but fails to describe parameterized families of quantum circuits, such as those which arise in the analysis of ansatz circuits. In this talk, we introduce an approach to parameterized circuit semantics, which is based on enriched category theory. This framework subsumes many constructions from quantum information, such as parameterized rotations and conditional operations. We show that many of the properties exhibited by these constructions arise from more fundamental categorical concepts, such as enrichment over Cartesian and monoidal closed categories.