**SUSAN NIEFIELD**, Union College
*Adjoints and Projectives in Double Categories of Monoids*

It is well known that for a module $M$ over a commutative ring $R$, the endofunctor $-\otimes_R M$ has a left adjoint if and only if $M$ is finitely generated and projective. Following their ring/quantale analogy, Joyal and Tierney (AMS Memoirs 309, 1984) showed this characterization holds for modules over a quantale without the finiteness condition. In a paper with Wood (TAC 32, 2017), we proved a general theorem characterizing the existence of a left adjoint to $-\otimes_R M$ for modules over a monoid $R$ in a suitable symmetric monoidal closed category $\mathcal{V}$, which we applied to obtain corollaries for rings and quantales.

Recently, Paré (Outstanding Contributions to Logic 20, Springer, 2021) considered adjoints and Cauchy completeness in double categories, and showed that an $(R, Q)$-bimodule $M$ has a right adjoint in the double category of commutative rings if and only if it is finitely generated and projective as a left $R$-module. Subsequently (TAC 43, 2025), we incorporated this adjoint result for double categories into a version of the 2017 theorem with Wood, which we then applied to rings and quantales. However, the proofs of the latter were again separate due to the finiteness condition on rings.

In this talk, adding additional conditions on $\mathcal{V}$, we introduce a notion of $projective$ module over a monoid in $\mathcal{V}$ which includes finiteness for rings and, when added to our theorem characterizing adjoints in a double category, gives a single proof of the application to rings and quantales.