Symmetry Methods and Analytical Techniques for Nonlinear Partial Differential Equations
Méthodes de symétrie et techniques analytiques pour les équations différentielles partielles non linéaires
(Org: **Stephen Anco** (Brock University), **Jean-Francois Ganghoffer** (Université de Lorraine) and/et **Alexey Shevyakov** (University of Saskatchewan))

STEPHEN ANCO, Brock University
General symmetry multi-reduction method for partial differential equations with conservation laws

A powerful application of symmetries is finding symmetry-invariant solutions of nonlinear partial differential equations (PDEs). For a given symmetry, these solutions satisfy a reduced differential equation with one fewer independent variable. It is well known that a double reduction occurs whenever the starting nonlinear PDE possesses a conservation law that is invariant with respect to the symmetry.

Recent work has developed a broad generalization of the double-reduction method by considering the space of invariant conservation laws with respect to a given symmetry. In its simplest formulation, the generalization is able to reduce a nonlinear PDE in 2 variables to an ODE with m first integrals where m is the dimension of the space of invariant conservation laws. Nonlinear PDEs in 3 or more variables can be reduced to an ODE similarly by using an algebra of given symmetries. Importantly, the algebra does not need to be solvable.

The general method employs multipliers and is fully algorithmic. In particular, no a priori knowledge of conservation laws of the nonlinear PDE is necessary, and the multi-reduction is carried out in one step.

In this talk, a summary of the general multi-reduction method will be presented for obtaining invariant solutions of physically interesting PDEs. Examples will be shown for quadruple reduction from a single symmetry; complete integration from a solvable algebra in one step; reduction via a non-solvable algebra.

NICOLETA BILA, Fayetteville State University
Symmetry Reduction Operators for Monge-Ampère Equations

In this talk, reduction operators related to two-dimensional Monge-Ampère equations are discussed. A degenerated case that occurs while applying the nonclassical method (due to Bluman and Cole) to these types of nonlinear partial differential equations is studied. It is shown that specific Monge-Ampère equations may be reduced to systems of first-order partial differential equations, and, additionally, their solutions are related to Monge and Bateman nonlinear partial differential equations. The connection of these results with the direct method (by Clarkson and Kruskal) is also presented.

GEORGE BLUMAN, University of British Columbia, Vancouver
The natural extension of Lie's reduction of order algorithm for ODES to PDEs

In the 19th Century, Sophus Lie initiated his study of continuous groups (Lie groups) to put order to the hodgepodge of heuristic techniques for solving ODEs. Lie's algorithm showed how the invariance of an ODE under a one-parameter Lie group of point transformations (point symmetry) leads systematically to its reduction of order. By looking at Lie's algorithm from a different point of view, it is shown how it extends naturally to PDEs.

ALEXANDR CHERNYAVSKIY, SUNY Buffalo
Dark-bright soliton perturbation theory for the Manakov system

A direct perturbation method for studying dynamics of dark-bright solitons of the Manakov system in the presence of perturbations is presented. We combine multiscale expansion method, perturbed conservation laws, and a boundary layer approach, which breaks the problem into an inner region, where the bulk of the soliton resides, and an outer region, which evolves independently of the soliton. We show that a shelf develops around the dark soliton component, with speed of the shelf proportional to the background intensity. Conservation laws of the Manakov system are used to determine the properties of the shelf and perturbed solutions. Our analytical predictions are corroborated by numerical simulations.

KOSTYA DRUZHKOV, University of Saskatchewan
Stationary-action principle and the intrinsic geometry of PDEs

One usually formulates the principle of stationary action in terms of Lagrangians on jet bundles. However, at least in classical mechanics, the Lagrangian formalism can be described using only the intrinsic geometry of equations of motion (and the result is the Hamiltonian formalism). We will show that in the general case, the situation is similar. Each Lagrangian of a system of differential equations generates a unique element of the cohomology of some cochain complex produced by the intrinsic geometry of the system. Such cohomology elements can be considered variational principles. Using a non-covariant approach (the spatial part of a space+time decomposition), one can interpret variational principles of this type as a direct reformulation of the Hamiltonian formalism in terms of the intrinsic geometry of variational equations.

MATTHEW FARKAS, Washington

THOMAS HILLEN, University of Alberta
Symmetries in Non-local Adhesion Models

Non-local Models for Cellular Adhesion

Cellular adhesion is one of the most important interaction forces between cells and other tissue components. In 2006, Armstrong, Painter and Sherratt introduced a non-local PDE model for cellular adhesion, which was able to describe known experimental results on cell sorting and cancer growth. The analysis becomes challenging through non-local cell-cell interaction and interactions with boundaries. In this talk I will use symmetry methods to analyse aggregations and pattern formation of the non-local adhesion model. (joint work with A. Buttenschoen).

CHRISTOPHER KENNEDY, Queen's University
Interaction between long internal waves and free surface waves in deep water

We present a study of the two-dimensional water wave problem consisting of a density-stratified fluid composed of two immiscible layers separated by a sharp interface. A goal is to describe the interaction between long, larger amplitude, nonlinear waves on the interface and modulated, smaller amplitude, free wave packets on the surface when the lower fluid is infinitely deep. In the first part, starting from the Hamiltonian formulation of this problem and using techniques from Hamiltonian transformation theory, we describe the resonant interaction of the waves by a system of equations where the internal wave solves a high-order Benjamin-Ono equation coupled to a linear Schroedinger equation for the time evolution of the wave envelope of the free surface. In the second part, we establish a local well-posedness result for the BO-Schroedinger system in the physical regime where the densities of the two fluid layers are close. Neglecting the higher-order coupling terms, we perform a gauge transformation to eliminate the higher-order non-linear terms and reformulate our BO equation, from which our proof follows by a fixed-point argument. This is a joint work with A. Kairzhan and C. Sulem.

PHILIC LAM, MSc Mathematics and Statistics ,Brock University
A search for integrable evolution equations with Lax pairs over the octonions

This talk reports on work searching for octonion evolution equations of KdV type and mKdV type that have a Lax pair.

We consider $u(t,x)$ as an octonion variable in evolution equations $u_t = F(u, u_x, u_{xx}, u_{xxx})$, and we aim to find a Lax pair $L_t = [M, L]$ where L and M are linear differential operators in terms of ∂_x with coefficients involving u and x-derivatives of u. For F, we assume it is homogeneous under a scaling of t, x, u which is either the scaling in the KdV equation or the mKdV equation. This gives a polynomial ansatz with undetermined (real) coefficients. Similarly, for L and M, we assume they are scaling homogeneous, where the scaling weight of M is the same as that of ∂_t while the scaling weight of L can be chosen freely.

The determining condition is $\left(L_t - [M, L]\right)\big|_{u_t=F} = 0$. We split this condition in the jet space of u, and do a further splitting with respect to a real basis (8-dimensional) for the octonions. This gives a large overdetermined system in the undetermined (real) coefficients in ansatz for F, L, M. We use Maple to do the splittings, and depending on the complexity of the system, we solve it using 'rifsimp' in Maple or a package called 'Crack' in Reduce.

As a main result, we obtain a single KdV octonion equation, three mKdV octonion equations, and also a single potential-KdV octonion equation, each of which has more than one Lax pair.

SOUGATA MANDAL, Indian Institute of Technology

JASKARAN MANN, Brock University
mKdV Loop Travelling Waves and Interactions of Loop Solitons

The modified Korteweg-de Vries(mKdV) equation is an integrable non-linear evolution equation which has applications in modeling various physical phenomena. It also describes the curvature of curve which undergoes a certain non-stretching geometrical evolution in the Euclidean plane. This curve motion finds applications in various areas, such as describing the dynamics of inelastic rope, modeling the evolution of the boundary of vortex patch(swirling region) in thin, sheet-like layer of incompressible fluid, and understanding the behavior of electrons quantized in thin-layered materials by studying the boundaries of electron cloud densities under strong electromagnetic fields. This talk focuses on mKdV curve motions called loop solutions. One class arise from soliton, heavy-tail(rational), and periodic solutions of the mKdV equation. These loop solutions exhibit intriguing symmetrical shapes: the soliton and heavy-tail cases describe a single loop which is open, and asymptotically straight or circular, respectively; the periodic case describes both open and closed loops which can have multiple crossings. Additionally, a class of colliding loop solutions are obtained from the 2-soliton solution of the mKdV equation. The collisions show interesting interaction patterns. A summary of different types of patterns will be given by categorizing the various shapes that occur during the interaction, which depend on the speed ratio of the initial two loops. Analytical and numerical methods are employed to determine the loop solutions for both classes, as well as the conditions determining interaction type in the case of collision. These findings contribute to a deeper understanding of the mKdV equation and solitons

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SHAWN MCADAM, University of Saskatchewan
Symmetry and numerical analysis of nonlinear Love wave model

Love waves are horizontally polarized transverse waves (SH waves) that form from the constructive interference of SH waves reflecting off the surface of the Earth. In this talk, I motivate a nonlinear model of SH waves and apply it to numerically study the qualitative behaviour of Love waves near the focus of an earthquake. I then compare these numerical solutions to solutions (and approximate solutions) obtained via symmetry methods.

ALEXEY SHEVYAKOV, University of Saskatchewan
Exact spherical vortex solutions in fluid and plasma dynamics

We revisit Hill's solution, which characterizes a self-propelling spherical vortex within nested toroidal pressure surfaces, confined by a spherical boundary in an ideal Eulerian fluid. The re-derivation employs Galilei symmetry alongside the Bragg-Hawthorne equation in spherical coordinates. Using the equivalence between the equilibrium Euler equations in fluid dynamics and the static magnetohydrodynamic equations, we derive a generalized type of vortex solution applicable to both dynamic fluid equilibria and static plasma equilibria with a nonzero azimuthal vector field component, while satisfying physical boundary conditions. By applying the separation of variables to the Bragg-Hawthorne equation in spherical coordinates, we develop new fluid and plasma equilibria characterized by nested toroidal flux surfaces and boundary vorticity sheets and current sheets, respectively. Additionally, we analytically demonstrate and numerically illustrate the instability of the original Hill's vortex when subjected to certain radial perturbations of the spherical flux surface.

SUBHANKAR SIL, University of British Columbia

Revisit of differential invariant method for finding nonlocal symmetries of nonlinear partial differential equations

In this talk, we consider examples using the symmetry-based differential invariant method for finding nonlocally related systems of DEs. In particular, through the DI method, we obtain nonlocal symmetries for nonlinear wave equations, telegraph equations, diffusion-convection equations and reaction-diffusion equations. We recover previously known nonlocal symmetries of these equations obtained by the conservation-law based method. (Previous work showed that the CL-based method is a special case of the DI method).

CRISTINA STOICA, Wlifrid Laurier University

Super-integrable systems with stochastic perturbations

In the physical world, many systems are subject to stochastic perturbations. The latter are often responsible for removing the symmetries of the ideal unperturbed systems. However, within the class of Hamiltonian systems, the framework of J-M. Bismut (“Mecanique Aleatoire", Springer 1981) permits stochastic inputs while preserving the Hamiltonian structure and, under appropriate constraints, some or even all of the symmetries of the deterministic system. Developed further in the last decades, the theory of stochastic geometric mechanics displays many interesting features and open questions. In this talk I will present the case study of two super-integrable systems: the two-dimensional harmonic oscillator and the Kepler problem.

This work is joint with Archishman Saha (University of Ottawa).

THOMAS WOLF, Brock