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Uniqueness of optimal plans for multi-marginal mass transport problems via a reduction argument

In this paper, by introducing a reduction argument, we investigate the relation between an optimal mass transport problem with N-marginals and its associated lower dimensional problems that consist of $k-$marginal problems for $k \in \mathcal{N} = \{1, \dots, N\}$. Namely, for a family of probability spaces $\{(X_k, \mathcal{B}_{X_k}, \mu_k)\}_{k=1}^N$ and a cost function $c : X_1 \times \cdots \times X_N \to \mathbb{R}$ we consider the Monge-Kantorovich problem

$$\inf_{\lambda \in \Pi(\mu_1, \dots, \mu_N)} \int_{\prod_{k=1}^N X_k} c \, d\lambda. \tag{MKP}$$

Then for each ordered subset $\mathcal{P} = \{i_1, \dots, i_p\} \subsetneq \mathcal{N}$ we create a new cost function $c_{\mathcal{P}}$ corresponding to the original cost function c defined on $\prod_{k=1}^p X_{i_k}$. This new cost function $c_{\mathcal{P}}$ enjoys many of the features of the original cost c while it has the property that any optimal plan λ of (MKP) restricted to $\prod_{k=1}^p X_{i_k}$ is also an optimal plan to the problem

$$\inf_{\tau \in \Pi(\mu_{i_1}, \dots \mu_{i_p})} \int_{\prod_{k=1}^p X_{i_k}} c_{\mathcal{P}} \, d\tau. \tag{RMKP}$$

Then, for appropriate choices of index set $\mathcal{P}$, we show that one can recover the optimal plans of (MKP) from (RMKP). Particularly, we determine situations in which the problem (MKP) admits a unique solution depending on the uniqueness of the solution to (RMKP). This allows us to prove many uniqueness results for multi-marginal problems when the unique optimal plan is not necessarily induced by a map. To this end, we extensively benefit from disintegration theorems and the c-extremality notion. Moreover, by employing the reduction method, besides recovering many standard results on the subject including the pioneering work of Gangbo-Swiech, several new applications will be demonstrated to evince the applicability of this method.