DR. KUMAR MURTY, Fields Institute and University of Toronto

$\zeta(3)$, $\log 2$, and π

The Riemann zeta function $\zeta(s)$ is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for $Re(s) > 1$. Special values of the Riemann zeta function at integer points have been studied for centuries. In particular, Euler showed that at positive even integers $2k$, we have $\frac{\zeta(2k)}{\pi^{2k}}$ is rational. For example, $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$ and so on.

The values at positive odd integers have remained a mystery. In a posthumous 1785 paper, Euler conjectured that there are rational numbers $\alpha,\ \beta$ so that

$$\zeta(3) = \alpha(\log 2)^3 + \beta\pi^2 \log 2.$$

After reviewing the fascinating story of how Euler came to this conclusion, we describe joint work with Payman Eskandari to show that this conjecture is inconsistent with conjectures in modern algebraic geometry. In particular, we use a conjecture of Grothendieck on periods of mixed motives to show that these three numbers $\zeta(3)$, $\log 2$, and π are in fact algebraically independent!

Thus, the value $\zeta(3)$ remains very enigmatic!