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Finding solutions with distinct variables to systems of linear equations over $\mathbb{F}_p$

Let us fix a prime p and a homogeneous system of m linear equations $a_{j,1}x_1 + \cdots + a_{j,k}x_k = 0$ for $j = 1, \ldots, m$ with coefficients $a_{j,i} \in \mathbb{F}_p$. Suppose that $k \geq 3m$, that $a_{j,1} + \cdots + a_{j,k} = 0$ for $j = 1, \ldots, m$ and that every $m \times m$ minor of the $m \times k$ matrix $(a_{j,i})_{j,i}$ is non-singular. Then we prove that for any (large) n, any subset $A \subseteq \mathbb{F}_p^n$ of size $|A| > C \cdot \Gamma^n$ contains a solution $(x_1, \ldots, x_k) \in A^k$ to the given system of equations such that the vectors $x_1, \ldots, x_k \in A$ are all distinct. Here, C and Γ are constants only depending on p, m and k such that $\Gamma < p$.

The crucial point here is the condition for the vectors $x_1, \ldots, x_k$ in the solution $(x_1, \ldots, x_k) \in A^k$ to be distinct. If we relax this condition and only demand that $x_1, \ldots, x_k$ are not all equal, then the statement would follow easily from Tao's slice rank polynomial method. However, handling the distinctness condition is much harder, and requires a new approach. While all previous combinatorial applications of the slice rank polynomial method have relied on the slice rank of diagonal tensors, we use a slice rank argument for a non-diagonal tensor in combination with combinatorial and probabilistic arguments.