The Functional Analytic and Representation Theoretic Foundations of Wavelet Theory
Analyse fonctionelle et théorie des représentations: fondements de la théorie des ondelettes
(Org: **Jean-Pierre Gabardo** (McMaster), **Vignon Oussa** (Bridgewater State) and/et **Keith Taylor** (Dalhousie))

TWAREQUE ALI, Concordia University, Montreal
Quaternionic wavelets on quaternionic Hilbert spaces

The standard wavelet group can be identified with the semidirect product of the reals R with R*, the two-dimensional wavelet group with the semidirect product of the complexes C with C*. We look at the semidirect product of the quaternions H with H*. It is interesting to study representations of this group on Hilbert spaces over the complexes and over the quaternions. In this talk we shall discuss some preliminary results in this direction.

ENRICO AU-YEUNG, Pacific Institute for the Mathematical Sciences
Non-Uniform Gabor sampling and Balayage of Fourier transforms

Consider the following two problems; one about Gabor frames and the other about translates of the Poisson kernel. (1) Find a sufficient condition for a sequence of points in the time-frequency domain so that these points generate a Gabor frame in $L^2(R)$. (2) Let $P(t) = 1/(1+t^2)$ be the Poisson kernel. Find a necessary and sufficient condition for a sequence of points x[n], so that the sequence of functions obtained by the translates of the Poisson kernel, namely $f_n(t) = P(t - x[n])$, spans the space $L^1(R)$. In this talk, we provide a unified treatment to these type of problems using the theory of Balayage, which was initially developed by Beurling, in the setting of Fourier frames.

BRADLEY CURREY, Saint Louis University
Cross-sections for multiply generated abelian group actions

Let $\mathfrak{g}$ be the real span of a finite set of commuting $n \times n$ real matrices, and put $G = \exp \mathfrak{g}$. When G satisfies a rationality condition, we show that one of the following holds: either there is a co-null, G-invariant, open subset of $\mathbb{R}^n$ in which every orbit is regular, or there is a co-null, G-invariant, $\mathcal{G}_\delta$ subset of $\mathbb{R}^n$ in which every orbit is not regular. We characterize these two situations in terms of the structure of G. In the regular a.e. case, we present an explicit construction of a Borel cross-section for the orbits. Examples will be provided, and natural questions, motivated by work of Führ, Larson, Schulz, Speegle and Taylor, will be raised. This is joint work with D. Arnal, B. Dali, and V. Oussa.

HARTMUT FÜHR, RWTH Aachen
Wavelet coorbit theory in higher dimensions

Coorbit theory provides a functional-analytic framework for the construction and study of Banach frames arising from the action of an integrable representation. This talk is concerned with existence and basic properties of coorbit spaces associated to wavelet transforms arising from an irreducible, square-integrable representation of a semidirect product of the type $G = \mathbb{R}^d \rtimes H$ acting naturally on $\mathrm{L}^2(\mathbb{R}^d)$. Here H is a suitably chosen, closed matrix group.

The talk provides a unified and rather general approach to a setting that so far has only been studied for very special choices of affine group actions (such as the similitude group, or the shearlet group). It establishes the well-definedness of a scale of Besov-type coorbit spaces, and provides the existence of atomic decompositions for these spaces in terms of suitably chosen band-limited Schwartz functions. Under suitable assumptions on the dual action of H I establish easily verified concrete conditions for frame atoms, in terms of vanishing moments, smoothness and decay. In particular, these results imply the existence of compactly supported smooth atoms.

JEAN-PIERRE GABARDO, McMaster University
Convolution inequalities in locally compact groups and unitary systems

We consider certain convolution inequalities for positive Radon measures on a locally compact group G, also assumed σ-compact. These appear naturally in connection with Bessel or frame inequalities for certain unitary systems U_t, $t \in G$, of operators acting on a Hilbert space $\mathcal{H}$ and associated with a positive Radon measure μ on G and an analyzing vector $\psi \in \mathcal{H}$. Using this approach, we obtain some general results in the form of inequalities relating the Bessel or frame constants to other constants defined in terms of the measure μ and the analyzing vector ψ.

MAHYA GHANDEHARI, University of Saskatchewan

LENKA HAKOVA, Czech Technical University in Prague
Weyl group orbit functions and their remarkable properties

Several families of multivariable special functions, called orbit functions, are defined in the context of Weyl groups of compact simple Lie groups/Lie algebras. They are closely related to the irreducible representations of Lie groups and to Jacobi polynomials. In this talk we will summarize their most significant properties, namely their symmetries with respect to the affine Weyl group and continuous orthogonality. This allows us to define continuous Fourier-like transforms. Moreover, it is shown that each orbit function is an eigenfunction of the Laplace operator and the eigenvalues are know explicitly.

BIN HAN, University of Alberta
Theory and Application of Frequency-based Framelets

Linked with discretization of continuous wavelet transforms, most wavelets and framelets studied in the literature are homogeneous affine (or wavelet) systems generated by square integrable functions. In this talk, we introduce frequency-based nonhomogeneous affine systems and frequency-based dual framelets, which naturally link many aspects of wavelet analysis together. We fully characterize frequency-based dual framelet and provide a natural explanation of the oblique extension principle by showing that every dual framelet filter bank is naturally associated with a pair of frequency-based dual framelets. Based on such characterization, we propose a family of directional tensor product complex tight framelets. Using such directional tight framelets, we shall demonstrate that their performance for image denoising is comparable or even better than several well-known methods such as undecimated wavelet transform and dual tree complex wavelet transform.

PALLE JORGENSEN, University of Iowa
Multiresolutions, multivariable operator theory, and representations.

We offer an operator theoretic approach to multiresolutions. This in turn is motivated by filters from signal processing; with the multiplicity in a multiresolution corresponding to the number of frequency bands in the associated filter. Multiresolutions are important, not only in wavelets, but more generally as well, because (among other things)they offer fast and efficient algorithms; and they encompass a host of other applications, for example numerical analysis and in learning theory. By now, the applications to wavelet offer a proven and sucessful alternative to classical Fourier methods, Fourier series and integrals; applied to analysis and synthesis problems. In general, with multiresolutions, one obtains recursive and computational spectral resolutions which are localized, so better adapted to discontinuities. And they offer better numerical schemes. Multiresolutions are further useful in the study of self-similarity, in the analysis of fractals, and of non-linear dynamical systems. A special case of this is illustrated by the renormalization property for scaling functions from wavelet theory.

CHUN-KIT LAI, McMaster University
Frames of multi-windowed exponentials on subsets of $\mathbb{R}^d$

Given discrete subsets $\Lambda_j \subset \mathbb{R}^d$, $j = 1, \ldots, q$, consider the set of windowed exponentials $\bigcup_{j=1}^{q} \{g_j(x) e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda_j\}$ on $L^2(\Omega)$. We show that a necessary and sufficient condition for the windows g_j to form a frame of windowed exponentials for $L^2(\Omega)$ with some Λ_j is that $m \leq \max_{j \in J} |g_j| \leq M$ almost everywhere on Ω for some subset J of $\{1, \cdots, q\}$. If Ω is unbounded, we show that there is no frame of windowed exponentials if the Lebesgue measure of Ω is infinite. If Ω is unbounded but of finite measure, we give a sufficient condition for the existence of Fourier frames on $L^2(\Omega)$. At the same time, we also construct examples of unbounded sets with finite measure that have no tight exponential frame.

LENKA MOTLOCHOVA, Université de Montréal
Discretization of Weyl group orbit functions

Weyl group orbit functions arise in connection with each simple compact Lie group G. They have several pertinent properties. We will focus on their pairwise discrete orthogonality within each family when summed up over points of a finite fragment of a lattice lying in the fundamental region of the affine Weyl group of G. This allows us to implement discrete Fourier-like transforms which are useful in the processing of multidimensional digital data sampled on lattices of any symmetry.

VIGNON OUSSA, Bridgewater State University
Continuous Wavelets on Nilpotent Lie Groups and Admissibility

Let N be a simply connected, connected non-commutative nilpotent Lie group with Lie algebra $\mathfrak{n}$. Let H be a subgroup of the automorphism group of N. Assume that H is a commutative, simply connected, connected Lie group with Lie algebra $\mathfrak{h}$. Furthermore, let us assume that the linear adjoint action of $\mathfrak{h}$ on $\mathfrak{n}$ is diagonalizable with real eigenvalues. Thus, $N \rtimes H$ is a completely exponential solvable Lie group. We consider the quasiregular representation $\tau = \mathrm{Ind}_H^{N \rtimes H}(1)$ acting in $L^2(N)$ as follows

$$\tau(n,1) f(m) = f\left(n^{-1}m\right), \tau(1,h) f(m) = \left|\det\left(Ad(h)\right)\right|^{-1/2} f\left(h^{-1}mh\right).$$

In our talk, mainly motivated by the admissibility of τ, we will discuss the decomposition of τ into its irreducible components. We will also present the following recent results. If $G = N \rtimes H$ is unimodular, then τ is never admissible, and if G is nonunimodular, τ is admissible if and only if $H \cap \mathrm{Cent}(G)$ is trivial. We will also discuss how these results can be generalized to other type of exponential Lie groups.

MARTIN SCHAEFER, TU Berlin
Alpha-Molecules

In order to efficiently represent multivariate data, which are often governed by anisotropic features – consider images with edges for example – many new representation systems beyond wavelets have been developed over the last decade. Since there is by now a whole zoo of such directional systems – ridgelets, curvelets and shearlets to name just a few – it is desirable to have some common framework, which builds upon their essential similarities. Such a framework would allow to deduce general results for many representation systems simultaneously.

In this talk we want to present the concept of α-*molecules*, which generalizes the recently introduced *parabolic molecules*. Systems of α-molecules feature a characteristic tiling of the frequency plane and an anisotropic scaling law, which is specified by the parameter α. Hence the concept incorporates the main features, which are common to most directional systems, and it is general enough to comprise the classical radial wavelet systems ($\alpha = 1$), ridgelets ($\alpha = 0$) as well as curvelets and shearlets ($\alpha = 1/2$).

As an application of the concept, we analyze the sparse approximation behavior of α-molecules. For this the notion of *sparsity equivalence* is introduced. With the help of this notion, it is possible to identify large classes of α-molecules providing the same sparse approximation results. In view of these results, one natural consequence is that curvelets and shearlets exhibit the same approximation behavior.

This is joint work with P. Grohs (ETH Zürich), S. Keiper (TU Berlin) and G. Kutyniok (TU Berlin).

FRANEK SZAFRANIEC, Jagiellonian University, Krakow
Framings and operators

Framings, as considered in

P. G. Casazza, D. Han, and D. R. Larson, Frames for Banach spaces, in *The functional and harmonic analysis of wavelets and frames* (San Antonio, TX, 1999), *Contemp. Math.* **247** (1999), 149-182,

extract the essence of frames. I intend to show a way of generating frames by means of operators (based on joint work with Dave Larson).

KEITH TAYLOR, Dalhousie University
Wavelets and Representation Theory

Group representation theory connects to the theory of wavelets in several distinct ways. I will review some of these connections and provide more details for select higher dimensional transforms.

ERIC WEBER, Iowa State University
Vector Valued Wavelets and Multiresolution Analyses

In many situations in which wavelets are used to analyze data, the data is inherently vector valued in nature, such as color image data. Typically, scalar valued wavelets and the corresponding scalar valued filter banks are applied to vector valued data component-wise. We will discuss the construction of vector valued wavelets and multiresolution analyses, and the associated filter banks. This is joint work with Brody Johnson.