Turbulent Flow and its Mathematical Foundations
Turbulence et ses fondations mathématiques
(Org: **John C. Bowman** and/et **Xinwei Yu** (Alberta))

LUKE BLACKBOURN, University of St Andrews
Number of degrees of freedom and energy spectrum of SQG turbulence.

A simple dynamical systems approach is used as an analytic alternative to the traditional phenomenological method, recovering a number of classical predictions for surface quasi-geostrophic turbulence dissipated by the usual molecular viscosity. It is shown that the system's number of degrees of freedom, which is defined in terms of local Lyapunov exponents, scales as $\Re^{3/2}$. Here, $\Re$ is the Reynolds number, defined in terms of the energy dissipation rate, the viscosity, and the domain length scale. This result implies the Kolmogorov $k^{-5/3}$ scaling of the energy inertial range and determines the dissipation wavenumber, which marks the high-wavenumber end of this range.

Support for these analytical results, as well as for the classical idea of a dissipation anomaly, where inviscid singularities lead to nonzero energy dissipation in the inviscid limit, is provided through a series of direct numerical simulations.

JOHN BOWMAN, University of Alberta
Casimir Cascades in Two-Dimensional Turbulence

The nonlinear terms of the 2D incompressible Navier-Stokes equation are well-known to conserve energy and enstrophy. In addition, they also conserve the global integral of any continuously differentiable function of the scalar vorticity field. However, the phenomenological role of these additional inviscid invariants remains unclear: Polyakov's minimal conformal field theory model indicates that high-order Casimir invariants cascade to large scales, while Eyink suggests that they might instead cascade to small scales.

Numerical investigations of this problem are hampered by the fact that pseudospectral simulations, which necessarily truncate the wavenumber domain, do not exactly conserve global integrals of arbitrary powers of the vorticity. Given that the "rugged" quadratic energy and enstrophy invariants (which do survive spectral truncation) play a key role in the Kolmogorov theory of the turbulent cascade, it is natural to ask whether higher-order invariants might also play a role. In this work, well-resolved numerical simulations based on a specially optimized dealiased ternary convolution are used to demonstrate that the fourth power of the vorticity cascades to small scales. Inertial-range pumping of this quantity by the large-scale forcing, as discussed by Falkovich and Lebedev, is also examined.

DAVID DRITSCHEL, St. Andrews

EVELYN LUNASIN, University of Michigan
Global well-posedness for the 2D Boussinesq system without heat diffusion with anisotropic viscosity

In this talk I will discuss global existence and uniqueness theorems for the two-dimensional non-diffusive Boussinesq system with viscosity only in the horizontal direction. In proving the uniqueness result, we have used an alternative approach by writing the transported temperature (density) as $\theta = \Delta\xi$ and adapting the techniques of V. Yudovich for the 2D incompressible Euler equations. This new idea allows us to establish uniqueness results with fewer assumptions on the initial data for the transported quantity θ. Furthermore, this new technique allows us to establish uniqueness results without having to resort to the paraproduct calculus of J. Bony. If time permits I will also discuss the global regularity of an inviscid α-regularization for the two-dimensional inviscid, non-diffusive Boussinesq system of equations, which we call the Boussinesq-Voigt equations and a Voigt-α regularization for the inviscid 3D Boussinesq equations with diffusion.

This is joint work with Adam Larios and Edriss S. Titi

PHIL MORRISON, The University of Texas at Austin
On the zero viscoscity limits of turbulence models

Properties of the zero viscosity limits of various turbulence models will be discussed. Ranging from Spectral Reduction [1] to low degree-of-freedom models that describe the transition to turbulence. Ideas about constructing dissipation-free models appropriate for inertial range dynamics will be described.

[1] J. C. Bowman, B. A. Shadwick, and P. J. Morrison, "Spectral Reduction: A Statistical Description of Turbulence," Physical Review Letters vol. 83, 5491–5494 (1999).

MALCOLM ROBERTS, University of Alberta
Continuum Shell Models of Turbulence

Shell models are ad hoc models which mimic the behaviour of the spectral Navier–Stokes equations, and can be very useful as test-beds for ideas about physical turbulence. The Sabra, GOY, and DN models evolve single complex value $u_n(t)$ which represent an average of all the velocity modes u_k with $k \in (k_0\lambda^n, k_0\lambda^{n+1})$, where λ is the geometric shell spacing parameter. By taking the limit as $\lambda \to 1$, one arrives at a continuum limit. In this talk, we prove that the steady-state of this continuum limit exhibits Kolmogorov-scaling for moments of the velocity and non-zero dissipation in the limit of vanishing viscosity.

CHUONG VAN TRAN, University of St Andrews
A quantitative description of turbulence

In Kolmogorov's theory of turbulence, the fluid velocity remains continuous in the inviscid limit, implying a mild vorticity divergence, though over a nonzero volume of fluid. Furthermore, the velocity associated with the vortices carrying diverging vorticity vanishes. This precludes certain forms of singularities. For example, vortex tubes would not collapse in straightforward ways as to render a dual divergence of velocity and vorticity (or a divergence of vorticity alone with the associated velocity remaining nonzero).

This talk discusses the Kolmogorov picture and explores ways to describe some of its features in quantitative terms.

XINWEI YU, University of Alberta
On The Exact Laws of MHD Turbulence

Exact laws are important properties of turbulent fluids. For MHD turbulence, such laws have been derived by various authors from the statistical point of view. In this talk, we show that these laws hold for weak solutions of the MHD equations, in a distributional sense. As a corollary, we give a simplified proof of the sufficient condition of energy conservation for MHD weak solutions.

ZHICHUN ZHAI, University of Alberta
Analytical results for the Lagrangian averaged Euler equations

3D Lagrangian averaged Euler equation (also known as Euler-alpha equation) has both practical and theoretical significance. On one hand, it can be applied to the study of turbulence as a closure model; on the other hand, it enjoys similar geometrical and analytical structures as that of the 3D Euler equations and thus can be studied as a regularized model of the latter. In this talk, we will discuss some analytical results for 3D Lagrangian averaged Euler equation, such as local well-posedness in Triebel-Lizorkin spaces, a Beale-Kato-Majda type necessary and sufficient condition for global existence involving the stream function, and new sufficient conditions for global existence in terms of mixed Lebesgue norms of the generalized Clebsch variables.

This is joint work with Xinwei Yu.