TRISTRAM BOGART, MSRI/San Francisco State University

Few smooth d-polytopes with N lattice points

The AIM workshop "Combinatorial Challenges in Toric Varieties" centered on several questions about lattice polytopes motivated by toric geometry. In particular, a lattice polytope is said to be smooth if it is simple and the primitive rays of each vertex cone comprise a lattice basis. The questions are whether smoothness implies any or all of a list of successively stronger properties including normality, Koszul, and existence of a regular unimodular triangulation.

To gather evidence for these questions, we set out to test the properties systematically in dimension three. In order to do this, we proved a key finiteness property: given a finite list F of rational cones and an integer N, there are only finitely many lattice polytopes whose normal cones are (integrally equivalent to) elements of F and that contain at most N lattice points. In particular this holds if F simply consists of the unimodular cone, the smooth case. The proof is algorithmic and was implemented by Benjamin Lorenz to perform computations that give a positive answer to all of the questions for the case of three-polytopes with at most twelve lattice points. In algebro-geometric language, the strongest result is that if X is a toric three-fold embedded in $\mathbb{P}^{N-1}$ by a complete linear series for $N \leq 12$, then the defining ideal $I(X)$ has a square-free quadratic initial ideal.

This project is joint work with Christian Haase, Milena Hering, Lorenz, Benjamin Nill, Andreas Paffenholz, Francisco Santos, and Hal Schenck.